Siberian Mathematical Journal, vol. 60, *no.* 2, *pp.* 319–324, 2019

RESIDUAL SEPARABILITY OF SUBGROUPS IN FREE PRODUCTS WITH AMALGAMATED SUBGROUP OF FINITE INDEX

A. A. Kryazheva
UDC 512.543

Abstract: Let P be the free product of groups A and B with amalgamated subgroup H, where H is a proper subgroup of finite index in A and B. We assume that the groups A and B satisfy a nontrivial identity and for each natural n the number of all subgroups of index n in A and B is finite. We prove that all cyclic subgroups in P are residually separable if and only if P is residually finite and all cyclic subgroups in H are residually separable; and all finitely generated subgroups in P are residually separable if and only if P is residually finite and all subgroups that are the intersections of H with finitely generated subgroups of P are finitely separable in H.

DOI: 10.1134/S0037446619020125

Keywords: residually separable subgroup, residually finite group, free product, split extension

1. Introduction

Recall that a group G is called *residually finite* if for each nonidentity element a of G there exists a homomorphism of G onto some finite group under which the image of a differs from the identity [1]. It is known that the free product of two residually finite groups is a residually finite group (see [2]). For generalized free products (i.e., for free products with amalgamated subgroup), this assertion already fails. The residual finiteness of the free product P of two groups A and B with amalgamated subgroup H is studied under some additional constrains imposed on the free factors A and B and on the amalgamated subgroup H. For example, in 1963 Baumslag proved in [3] that if A and B are residually finite and H is finite then P is residually finite. Another natural constraint is the finiteness of the indices of the amalgamated subgroup H in A and B. Under this constraint, Azarov established the following criterion for the residual finiteness of P in [4]:

Theorem 1. *Suppose that A and B are residually finite groups with nontrivial identities and for each natural n the number of all subgroups of A and B of index n is finite. Let $P = (A*B, H)$ be the free product of A and B with amalgamated subgroup H, where H is a proper subgroup of finite index in A and B.*

The group P is residually finite if and only if H has a subgroup L of finite index normal in P.

Here we consider the question of the residual separability of subgroups in the free product of Theorem 1. Recall that a subgroup H in a group G is called *residually separable* if for each element g of G not belonging to H there exists a homomorphism of G onto some finite group under which the image of g does not belong to the image of H (see [5]). Note that the group G is residually finite if and only if its identity subgroup is finitely separable.

The main result of this article is as follows:

Theorem 2. *Let P be the free product of groups A and B with amalgamated subgroup H, where H is a proper subgroup of finite index in A and B. Suppose that A and B satisfy a nontrivial identity and for each natural n the number of all subgroups of index n in A and B is finite.*

(1) *All cyclic subgroups in P are residually separable if and only if P is residually finite and all cyclic subgroups in H are residually separable.*

Ivanovo. Translated from *Sibirskii Matematicheskii Zhurnal*, vol. 60, no. 2, pp. 411–418, March–April, 2019; DOI: 10.17377/smzh.2019.60.212. Original article submitted July 19, 2018; revised July 19, 2018; accepted October 17, 2018.

(2) *All finitely generated subgroups in P are residually separable if and only if P is residually finite and all subgroups that are the intersections of H with finitely generated subgroups of P are finitely separable in H.*

In Theorems 1 and 2, the condition is imposed on the free factors A and B that the number of subgroups of finite index in them is finite. This condition is rather general since it is satisfied, for example, by all finitely generated groups (see [6, Chapter 10, § 38]).

Since, in every polycyclic group, all subgroups are residually separable (see, for example, [7, Subsection 1.3.10]), Theorem 2 readily implies

Corollary. *Let P be the free product of groups A and B with amalgamated subgroup H, where H is a proper subgroup of finite index in A and B. Then the following are equivalent:*

(1) *P is residually finite.*

(2) *All cyclic subgroups in P are residually separable.*

(3) *All finite subgroups in P are residually separable.*

Note that the free products mentioned in the corollary to Theorem 2 need not be a residually finite group. The corresponding example was constructed in [8].

2. Auxiliary Assertions

Lemma 1. *Let G be a group and let H be a subgroup of G. If a subgroup M of H is residually separable in G then it is also residually separable in H.*

PROOF. Suppose that $h \in H$ but $h \notin M$. Since M is a subgroup of G; by hypothesis, M is residually separable in G. Therefore, there exists a homomorphism φ of G onto a finite group such that $h\varphi \notin M\varphi$. Let φ_1 be the restriction of φ to H. Then φ_1 is a homomorphism of H onto a finite group and $h\varphi_1 \notin M\varphi_1$. Thus, the subgroup M is residually separable in H. The lemma is proved.

Lemma 2. *Let Ω be a group class closed under subgroups of finite index and let H be a subgroup of finite index in G. If all subgroups of class Ω in H are residually separable then also all subgroups of class Ω in G are residually separable.*

PROOF. Suppose that a subgroup H of G has finite index in G and all subgroups of class Ω are residually separable in H. Then H has a normal subgroup H_1 of G with finite index in G, where each Ω-subgroup of H_1 is residually separable in H by hypothesis and hence residually separable in H_1 (see Lemma 1). Therefore, we will assume without loss of generality that the subgroup H is normal in G.

Suppose that A is an arbitrary subgroup of G of class Ω and an element g of G does not belong to A. Using this fact that all subgroups of class Ω in H are residually separable, show that A is residually separable in G, i.e., G has a normal subgroup N having finite index in G and such that g does not belong to AN.

If $g \notin AH$ then the subgroup H itself acts as N.

Suppose that $g \in AH$, i.e., $g = ah$ for some $a \in A$ and $h \in H$. Putting $A_1 = A \cap H$, we obtain a subgroup A_1 in H that obviously does not contain the element h of H. Since the intersection of a subgroup of finite index of a group with any subgroup of this group is a subgroup of finite index, the index of the subgroup A_1 in A is finite. Since $A \in \Omega$ and Ω is closed under subgroups of finite index, A_1 also belongs to Ω. Thus, A_1 is residually separable in H. Therefore, there exists a normal subgroup K of finite index in H such that $h \notin A_1K$. Since the index of H in G is finite, K is a subgroup of finite index in G and hence contains a subgroup N normal in G and having finite index in G. Since $h \notin A_1K$ and N is a subgroup in K, we have $h \notin A_1N$.

Show that $g \notin AN$; i.e., the subgroup N is a desired one. Suppose on the contrary that $g \in AN$, i.e., $g = a_1n$ for some $a_1 \in A$, $n \in N$. Since also $g = ah$, we have the equality $ah = a_1n$, i.e., $a^{-1}a_1 = hn^{-1}$. Since $a^{-1}a_1 \in A$, $hn^{-1} \in H$, we obtain the membership $a^{-1}a_1 \in A_1$. Hence, $h = (a^{-1}a_1)n \in A_1N$, which is impossible. Thus, the lemma is proved.

Lemma 3. *Let G be a group and let A and H be subgroups of G. If H is residually separable in G then $A \cap H$ is residually separable in A.*

PROOF. Suppose that H is residually separable in G. Prove that $A \cap H$ is residually separable in A. Take an element g in A such that $g \notin A \cap H$. Then $g \notin H$. The subgroup H is residually separable in G; therefore, there exists a homomorphism φ of G onto a finite group such that $g\varphi \notin H\varphi$. Consequently, $g\varphi \notin (A \cap H)\varphi$. Let φ_1 be the restriction of φ to A. Then φ_1 is a homomorphism of A onto a finite group and $g\varphi_1 \notin (A \cap H)\varphi_1$. Thus, we conclude that $A \cap H$ is residually separable in A.

Lemma 4. *Suppose that H is a subgroup in a group G and the index of H in G is finite and equal to n. The number of all subgroups of index n in G is finite. Then there is a characteristic subgroup N of finite index in G such that $N \subseteq H$.*

PROOF. Since the number of all subgroups of index n in G is finite, we may write down all subgroups of index n in G, say, $H_1, H_2, \dots, H_s$. Take $\varphi \in \mathrm{Aut}\ G$. Since an automorphism of G maps a subgroup of index n to a subgroup of index n, φ permutes the subgroups $H_1, H_2, \dots, H_s$. Therefore,

$$\{H_1, H_2, \dots, H_s\} = \{H_1\varphi, H_2\varphi, \dots, H_s\varphi\},$$

and so

$$\left(\bigcap_{i=1}^{s} H_i\right)\varphi = \bigcap_{i=1}^{s} H_i\varphi = \bigcap_{i=1}^{s} H_i;$$

i.e., if we put $N = \bigcap_{i=1}^{s} H_i$ then $N\varphi = N$. Therefore, N is a characteristic subgroup in G. Since, by Poincaré's theorem, the intersection of finitely many subgroups of finite index in G is a subgroup of finite index in G (see, for example, [6, Chapter 3, § 8]), N is a subgroup of finite index in G. Since one of the subgroups H_i coincides with H, we have $N \subseteq H$. Consequently, N is a desired subgroup. The lemma is proved.

3. On Split Extensions

Let us formulate and prove several auxiliary assertions for split extensions. We will need them in proving Theorem 2.

Recall that a group G is called a *split extension* of a group A by a group B if A is a normal subgroup of G and B is a subgroup of G, while $G = AB$ and $A \cap B = 1$.

Lemma 5. *Let G be a split extension of a finite group A by a group B and let Ω be a group class closed under subgroups of finite index. If all subgroups of class Ω in B are residually separable then also all subgroups of class Ω in G are residually separable.*

PROOF. Obviously, B is a subgroup of finite index in G and the index $[G : B]$ coincides with the order of A. Therefore, the lemma is straightforward from Lemma 2.

Proposition 1. *Suppose that G is a split extension of a group A by a group B and for each natural n the number of all subgroups of index n in A is finite. Let Ω be a group class closed under factorization and subgroups of finite index. Then the following are equivalent:*

(1) *All Ω-subgroups in G are residually separable.*

(2) *All Ω-subgroups in A and B are residually separable and, moreover, all subgroups in A that are intersections of A and Ω-subgroups of G are residually separable in A.*

An obvious example of a group class closed under factorization and subgroups of finite index is the class of finitely generated groups.

Let us formulate one more assertion obtained from Proposition 1 under stronger constraints on Ω. We strengthen the requirement of the closedness of Ω under subgroups of finite index, imposed in Proposition 1, to the requirement of the closedness of Ω under arbitrary subgroups.

Proposition 2. *Let G be a split extension of a group A by a group B and for each natural n the number of all subgroups of index n in A is finite. Let Ω be a class of groups closed under factorization and subgroups. Then the following are equivalent:*

(1) *All Ω-subgroups in G are residually separable.*

(2) *All Ω-subgroups in A and B are residually separable.*

Note that Proposition 2 can be easily proved with the use of Proposition 1. Indeed, if Ω is a group class closed under subgroups then Proposition 1(2) is equivalent to Proposition 2(2).

Obvious examples of group classes closed under subgroups and factorization are given by the classes of all groups, all cyclic groups, all polycyclic groups, and any variety of groups.

Turn to proving Proposition 1. Let us first prove sufficiency. Let G be a split extension of a group A by a group B and for each natural n the number of all subgroups of index n in A is finite. Let Ω be a group class closed under factorization and subgroups of finite index. Suppose that all subgroups of class Ω are residually separable in A and B and all subgroups that are intersections of A with all subgroups of class Ω in G are residually separable in A.

Take an arbitrary subgroup H of class Ω in G and an element g belonging to G but not lying in H.

Let us first consider the case when $g \notin HA$. In this case $gA \notin HA/A$. The quotient group HA/A is a subgroup in G/A. Since $G/A = BA/A \cong B/B \cap A \cong B$ and all subgroups of class Ω are residually separable in B, we conclude that in G/A also all subgroups of class Ω are residually separable. Moreover, the subgroup HA/A of G/A belongs to Ω since Ω is closed under factorization. The last two circumstances imply that the subgroup HA/A is residually separable in G/A. Consequently, there is a homomorphism φ from the quotient group G/A onto a finite group such that $(gA)\varphi \notin (HA/A)\varphi$. Let ψ be a natural homomorphism of G onto the quotient group G/A; then $H\psi = HA/A$. We infer that $g\psi\varphi \notin H\psi\varphi$. Then $\psi\varphi$ is a desired homomorphism.

Consider the case when $g \in HA$, i.e., $g = ha$, where $h \in H$ and $a \in A$. Note that the element a does not belong to the subgroup $H_1 = H \cap A$. By hypothesis, H_1 is residually separable in A. Therefore, there exists a normal subgroup N of finite index in A such that $a \notin H_1N$. Since for each natural n the number of all subgroups of index n in A is finite, by Lemma 4 any subgroup of finite index in A contains a characteristic subgroup of finite index of A. Thus, without loss of generality, we may assume that N is characteristic in A. Since the subgroup A is normal in G, N is also normal in G. Show that $g \notin HN$.

If, on the contrary, an element g belongs to the subgroup HN then g is representable as $g = h_1x$, where $h_1 \in H$ and $x \in N$. On the other hand, $g = ha$, where $h \in H$ and $a \in A$. Then $ha = h_1x$. So, we obtain the equality $h^{-1}h_1 = ax^{-1}$, where $h^{-1}h_1 \in H$ and $ax^{-1} \in A$. Hence, $h^{-1}h_1 \in H_1$. Therefore, the element $a = h^{-1}h_1x$ belongs to H_1N, which contradicts the choice of the subgroup N. Thus, $g \notin HN$.

Consider the quotient group G/N and the natural homomorphism φ of G onto G/N. Since g does not belong to HN, we have $gN \notin HN/N$, i.e., $g\varphi \notin H\varphi$.

Since G is a split extension of A by B and N is contained in A, it is easy to see that G/N is a split extension of A/N by BN/N; moreover, A/N is finite since N has finite index in A. Since the group BN/N is isomorphic to B, by hypothesis, all subgroups of class Ω in BN/N are residually separable. Therefore, by Lemma 5, all subgroups of class Ω in G/N are residually separable.

Since $H \in \Omega$, HN/N belongs to Ω. This and the fact that all subgroups of class Ω in G/N are residually separable imply that HN/N is residually separable in G/N, i.e., $H\varphi$ is residually separable in G/N. Since $g\varphi \notin H\varphi$, there exists a homomorphism ψ of G/N onto a finite group such that $(g\varphi)\psi \notin (H\varphi)\psi$. Thus, H is residually separable in G. Thus, sufficiency in Proposition 1 is proved.

Necessity in Proposition 1 is guaranteed by Lemmas 1 and 3. Proposition 1 is proved.

4. Proof of Theorem 2

Let P be the free product of groups A and B with amalgamated subgroup H, where H is a proper subgroup of finite index in A and B. Suppose that A and B satisfy a nontrivial identity and for each natural n the number of all subgroups of index n in A and B is finite.

(1): Show that all cyclic subgroups in P are residually separable if and only if P is residually finite and all cyclic subgroups in H are residually separable.

Necessity: Suppose that all cyclic subgroups in P are residually separable. In particular, the identity subgroup of P is residually separable. Therefore, P is residually finite.

Since all cyclic subgroups in P are residually separable and H is a subgroup of P, all cyclic subgroups in H are residually separable (see Lemma 1).

Sufficiency: Suppose that P is residually finite and all cyclic subgroups in H are residually separable. Show that all cyclic subgroups in P are residually separable. Since P is residually finite and A and B are subgroups in P, the groups A and B are residually finite. Then, by Theorem 1, H has a subgroup L of finite index normal in P.

Obviously, the quotient group P/L is the free product of the finite groups A/L and B/L with amalgamated subgroup H/L. Therefore, the group P/L has a free subgroup G/L of finite index. Prove that all cyclic subgroups in G are residually separable. The group G is an extension of L by a free group. It is known that such an extension is split, i.e., G is a split extension of L by a free group. It is known that all cyclic subgroups in a free group are residually separable (see [9]). By the hypothesis of the theorem, all cyclic subgroups in H are residually separable. Since L is a subgroup in H, all cyclic subgroups in H are residually separable. Since L is a subgroup of finite index in H and H is a subgroup of finite index in A and B, it follows that L is a subgroup of finite index in A and B. Since the groups A and B enjoy the condition that for each natural n the number of all subgroups of index n in A and B is finite, L satisfies this condition. Note also that the class Ω of all cyclic groups is closed under factorization and subgroups. Thus, the group G of class Ω satisfies all requirements of Proposition 2. Therefore, by Proposition 2, all cyclic subgroups in G are residually separable.

Since G is a subgroup of finite index in P, by Lemma 2, all cyclic subgroups in P are residually separable. Thus, item 1 is proved.

(2): Show that all finitely generated subgroups in P are residually separable if and only if P is residually finite and all subgroups that are intersections of finitely generated subgroups of P with H are residually separable in H.

Necessity: Suppose that all finitely generated subgroups in P are residually separable. In particular, the identity subgroup of P is residually separable. Therefore, P is residually finite.

Consider an arbitrary finitely generated subgroup F in P. Since all finitely generated subgroups in P are residually separable, F is residually separable in P. Then, by Lemma 3, $F \cap H$ is residually separable in H, i.e., all subgroups that are intersections of finitely generated subgroups of P with H are residually separable in H. Thus, necessity is proved.

Sufficiency: Suppose that P is residually finite and all subgroups that are intersections of finitely generated subgroups of P with H are residually separable in H. Show that all finitely generated subgroups in P are residually separable. As in the proof of item 1, it is readily checked that P has a subgroup G of finite index that is the split extension of a group L by a free group, where L is a subgroup of finite index in H normal in P. In accordance with Lemma 2, for proving the residual separability of all finitely generated subgroups in P, it suffices to demonstrate that all finitely generated subgroups in G are residually separable.

Since A and B have finitely many subgroups of each finite index and L is a subgroup of finite index in A and B, the number of subgroups in L of each finite index is also finite. Obviously, the class of all finitely generated subgroups is closed under subgroups of finite index and factorization. Observe also that all finitely generated subgroups in a free group are residually separable (see [9]). In view of Definition 1, it remains to prove that all subgroups that are intersections of finitely generated subgroups of G with L are residually separable in L (in particular, all finitely generated subgroups in L are residually separable).

Consider an arbitrary finitely generated subgroup F in G. By hypothesis, the subgroup $F_1 = F \cap H$ is residually separable in H. Then, by Lemma 3, the subgroup $F_2 = F_1 \cap L$ is residually separable in L. Since $F_2 = F \cap H \cap L = F \cap L$, the subgroup $F \cap L$ is residually separable in L. In other words, all subgroups that are intersections of finitely generated subgroups of G with L are residually separable in L. Thus, item 2 of the theorem is proved, and the theorem is completely proved.

The author is grateful to D. N. Azarov for help in writing this article.

References

1. Malcev A. I., "On isomorphic matrix representations of infinite groups," Mat. Sb., vol. 8, no. 3, 405–422 (1940).
2. Gruenberg K. W., "Residual properties of infinite soluble groups," Proc. London Math. Soc., vol. 7, 29–62 (1957).
3. Baumslag G., "On the residual finiteness of generalized free products of nilpotent groups," Trans. Amer. Math. Soc., vol. 106, no. 2, 193–209 (1963).
4. Azarov D. N., "On the residual finiteness of the HNN-extensions and generalized free products of finite rank groups," Sib. Math. J., vol. 54, no. 6, 959–967 (2013).
5. Maltsev A. I., "On homomorphisms onto finite groups," Ivanovo Gos. Ped. Inst. Uchen. Zap., vol. 18, no. 5, 49–60 (1958).
6. Kurosh A. G., *The Theory of Groups*, Chelsea, New York (1960).
7. Lennox J. and Robinson D., *The Theory of Infinite Soluble Groups*, Clarendon press, Oxford (2004).
8. Azarov D. N., "On the residual finiteness of generalized free products of finite rank groups," Sib. Math. J., vol. 54, no. 3, 379–387 (2013).
9. Hall M., "Coset representations of free groups," Trans. Amer. Math. Soc., vol. 67, 431–451 (1949).

A. A. Kryazheva
Ivanovo State University, Ivanovo, Russia
E-mail address: Stasia.07.10@mail.ru