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THE PARTIAL CLONE OF LINEAR TREE LANGUAGES

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Abstract: A term, also called a tree, is said to be linear, if each variable occurs in the term only once. The linear terms and sets of linear terms, the so-called linear tree languages, play some role in automata theory and in the theory of formal languages in connection with recognizability. We define a partial superposition operation on sets of linear trees of a given type τ and study the properties of some many-sorted partial clones that have sets of linear trees as elements and partial superposition operations as fundamental operations. The endomorphisms of those algebras correspond to nondeterministic linear hypersubstitutions.

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1. Preliminaries

Let $\{f_i \mid i \in I\}$ be an indexed set of operation symbols of type $\tau = (n_i)_{i\in I}$, where f_i is n_i-ary for $n_i \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$. Let $W_\tau(X_n)$ be the set of all n-ary terms of type τ, defined inductively by the following steps:

(i) Each variable $x_j \in X_n$ is an n-ary term of type τ.

(ii) If $t_1, \dots, t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1, \dots, t_{n_i})$ is an n-ary term of type τ.

Let $(W_\tau(X_n))_{n\in\mathbb{N}^+}$ be the many-sorted set of terms of type τ, i.e. the infinite sequence $(W_\tau(X_1), W_\tau(X_2), \dots, W_\tau(X_n), \dots)$. The sorts are the sets of n-ary terms of type τ for all $n \in \mathbb{N}^+$.

The linear terms of type τ are defined in a similar way with the difference that instead of (ii) we define

(ii′) If $t_1, \dots, t_{n_i}$ are n-ary linear terms of type τ and if $\operatorname{var}(t_j) \cap \operatorname{var}(t_k) = \varnothing$ for all $1 \le j < k \le n_i$, then $f_i(t_1, \dots, t_{n_i})$ is an n-ary linear term of type τ, where $\operatorname{var}(t_j)$ denotes the set of variables occurring in t_j.

Let $W_\tau^{\mathrm{lin}}(X_n)$ be the set of all n-ary linear terms of type τ and let $(W_\tau^{\mathrm{lin}}(X_n))_{n\in\mathbb{N}^+}$ be the many-sorted set of all linear terms of type τ.

$(W_\tau(X_n))_{n\in\mathbb{N}^+}$ is closed under the superposition operations

$$S^n_m : W_\tau(X_n) \times (W_\tau(X_m))^n \to W_\tau(X_m)$$

that are inductively defined for $m, n \in \mathbb{N}^+$ as follows:

(i) $S^n_m(x_j, t_1, \dots, t_n) := t_j$ for $1 \le j \le n$ and

(ii) $S^n_m(f_i(s_1, \dots, s_{n_i}), t_1, \dots, t_n) = f_i\big(S^n_m(s_1, t_1, \dots, t_n), \dots, S^n_m(s_{n_i}, t_1, \dots, t_n)\big)$.

The many-sorted algebra

$$\operatorname{clone}\tau := \big((W_\tau(X_n))_{n\in\mathbb{N}^+}; (S^n_m)_{m,n\in\mathbb{N}^+}, (x_i)_{i\le n\in\mathbb{N}^+}\big),$$

the clone of all terms of type τ, satisfies the three equalities:

$$\text{(C1)} \quad \widetilde{S}^p_m\big(\widetilde{Z}, \widetilde{S}^n_m(\widetilde{Y}_1, \widetilde{X}_1, \dots, \widetilde{X}_n), \dots, \widetilde{S}^n_m(\widetilde{Y}_p, \widetilde{X}_1, \dots, \widetilde{X}_n)\big) \approx \widetilde{S}^n_m\big(\widetilde{S}^p_n(\widetilde{Z}, \widetilde{Y}_1, \dots, \widetilde{Y}_p), \widetilde{X}_1, \dots, \widetilde{X}_n\big) \quad (m, n, p \in \mathbb{N}^+),$$

$$\text{(C2)} \quad \widetilde{S}^n_m(\lambda_j, \widetilde{X}_1, \dots, \widetilde{X}_n) \approx \widetilde{X}_j \quad (m = 1, 2, \dots,\ 1 \le j \le n,\ n \in \mathbb{N}^+),$$

$$\text{(C3)} \quad \widetilde{S}^n_n(\widetilde{Y}, \lambda_1, \dots, \lambda_n) \approx \widetilde{Y} \quad (n \in \mathbb{N}^+).$$

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where $\widetilde{Z}, \widetilde{Y}_1, \dots, \widetilde{Y}_p, \widetilde{X}_1, \dots, \widetilde{X}_n$ are variables for terms, $\widetilde{S}^n_m$ are operation symbols, and λ_i, $i = 1, \dots, m$, are nullary operation symbols.

The concept of a clone is one of the basic algebraic concepts. The models of the axioms (C1)–(C3) are called *abstract clones*. Each abstract clone is isomorphic to a concrete one, i.e. to a clone of operations defined on a set. A clone can be regarded as a category. The duals of those categories are the so-called *Lawvere theories* (see [1]). In this paper we consider partial clones; i.e. the partial many-sorted algebras of the type of clones which satisfy (C1)–(C3) as weak identities.

The many-sorted set $\left(W^{\text{lin}}_\tau(X_n)\right)_{n\in\mathbb{N}^+}$ is not closed under S^n_m. In [2] the second author proved the following:

Proposition 1.1. *If $f_i(t_1, \dots, t_{n_i}) \in W^{\text{lin}}_\tau(X_n)$, $s_1, \dots, s_n \in W^{\text{lin}}_\tau(X_m)$, and $\operatorname{var}(s_j) \cap \operatorname{var}(s_k) = \varnothing$ for $1 \le j < k \le n$, then $S^n_m(f_i(t_1, \dots, t_{n_i}), s_1, \dots, s_n) \in W^{\text{lin}}_\tau(X_m)$.*

This result leads to the partial many-sorted mapping

$$\overline{S}^n_m : W^{\text{lin}}_\tau(X_n) \times \left(W^{\text{lin}}_\tau(X_m)\right)^n \multimap\!\!\rightarrow W^{\text{lin}}_\tau(X_m)$$

defined by

$$\overline{S}^n_m(t, s_1, \dots, s_n) := \begin{cases} S^n_m(t, s_1, \dots, s_n) & \text{if } \operatorname{var}(s_j) \cap \operatorname{var}(s_k) = \varnothing \text{ for all } 1 \le j < k \le n, \\ \text{not defined} & \text{otherwise} \end{cases}$$

and to the many-sorted partial algebra

$$\text{clone}_{\text{lin}}\,\tau := \left(\left(W^{\text{lin}}_\tau(X_n)\right)_{n\in\mathbb{N}^+}; \left(\overline{S}^n_m\right)_{m,n\in\mathbb{N}^+}, (x_i)_{i\le n,\, n\in\mathbb{N}^+}\right)$$

which satisfies (C1)–(C3) as weak identities [2].

Remark 1.2. (i) If we consider an n-ary linear term $f_i(t_1, \dots, t_{n_i})$, then $n_i \le n$. Indeed, assume that $f_i(t_1, \dots, t_{n_i}) \in W^{\text{lin}}_\tau(X_n)$ and $n_i > n$. Hence, some variables in X_n must occur repeatedly in $f_i(t_1, \dots, t_{n_i})$, which is impossible.

(ii) According to the condition in Proposition 1.1, in S^n_m we have that $n \le m$. Otherwise, if $m < n$, then $s_1, \dots, s_n \in W^{\text{lin}}_\tau(X_m)$ means that there exist variables occurring more than once.

In [3] the nondeterministic superposition of terms of type τ was introduced as superposition of sets of terms of type τ (of tree languages) as follows: Let $\mathscr{P}(W_\tau(X_n))$ be the powerset (set of subsets) of $W_\tau(X_n)$. Then the operations

$$S^{nd,n}_m : \mathscr{P}(W_\tau(X_n)) \times \mathscr{P}((W_\tau(X_m)))^n \to \mathscr{P}(W_\tau(X_m))$$

for $m, n \in \mathbb{N}^+$ can be defined as follows:

(i) if $B = \{x_j\}$, $1 \le j \le n$, then $S^{nd,n}_m(B, B_1, \dots, B_n) := B_j$;

(ii) if $B = \{f_i(t_1, \dots, t_{n_i})\}$ and assumed that $S^{nd,n}_m(\{t_j\}, B_1, \dots, B_n)$ for $1 \le j \le n_i$ are already defined, then $S^{nd,n}_m(B, B_1, \dots, B_n) := \{f_i(r_1, \dots, r_n) : r_j \in S^{nd,n}_m(\{t_j\}, B_1, \dots, B_n),\ 1 \le j \le n_i\}$;

(iii) if B is an arbitrary nonempty subset of $W_\tau(X_n)$, then

$$S^{nd,n}_m(B, B_1, \dots, B_n) := \bigcup_{b\in B} S^{nd,n}_m(\{b\}, B_1, \dots, B_n).$$

If one of the sets $B, B_1, \dots, B_n$ is empty, then $S^{nd,n}_m(B, B_1, \dots, B_n) = \varnothing$.

Then we may consider the multi-based algebra

$$\mathscr{P}\text{-clone}\ \tau := \left((\mathscr{P}(W_\tau(X_n))_{n\in\mathbb{N}^+}; \left(S^{nd,n}_m\right)_{m,n\in\mathbb{N}^+}, (\{x_i\})_{i\le n,\, n\in\mathbb{N}^+}\right).$$

Remark 1.3. For the superposition operations $S^{nd,n}_m$ of sets of linear terms we have $n_i \le n \le m$ for each $i \in I$.

In [3] was proved that the many-sorted algebra $\mathscr{P}$-clone τ satisfies (C1) and (C3). The subalgebra $\mathscr{P}^+$-clone τ having arbitrary nonempty subsets of $W_\tau(X)$ as universes is an abstract clone; i.e., it satisfies (C1)–(C3).

2. Nondeterministic Superposition of Sets of Linear Terms

It is easy to see that the nondeterministic (for short nd-) superposition of sets of linear terms gives in general not sets of linear terms.

Example 2.1. Let $\tau = (2)$ with a binary operation symbol f and let $X_2 = \{x_1, x_2\}$. We consider the three subsets of $W^{\text{lin}}_{(2)}(X_2)$; namely, $B = \{f(x_2, x_1)\}$, $B_1 = \{x_1, x_2\}$, and $B_2 = \{x_1, f(x_1, x_2)\}$. Then

$$\begin{aligned}
S_2^{nd,2}(B, B_1, B_2) &= S_2^{nd,2}(\{f(x_2, x_1)\}, \{x_1, x_2\}, \{x_1, f(x_1, x_2)\}) \\
&= \big\{f(r_1, r_2) : r_1 \in S_2^{nd,2}(\{x_2\}, \{x_1, x_2\}, \{x_1, f(x_1, x_2)\}), \\
&\qquad r_2 \in S_2^{nd,2}(\{x_1\}, \{x_1, x_2\}, \{x_1, f(x_1, x_2)\})\big\} \\
&= \{f(r_1, r_2) : r_1 \in \{x_1, f(x_1, x_2)\}, r_2 \in \{x_1, x_2\}\} \\
&= \{f(x_1, x_1), f(x_1, x_2), f(f(x_1, x_2), x_1), f(f(x_1, x_2), x_2)\} \not\subseteq W^{\text{lin}}_{(2)}(X_2).
\end{aligned}$$

The following theorem gives a condition for the nondeterministic superposition of sets of linear terms to be a set of linear terms.

Theorem 2.2. *If $B \subseteq W^{\text{lin}}_\tau(X_n), B_1, \dots, B_n \subseteq W^{\text{lin}}_\tau(X_m)$ and if $\operatorname{var}(B_j) \cap \operatorname{var}(B_k) = \varnothing$ for $1 \le j < k \le n$ or if $B = \varnothing$, then $S_m^{nd,n}(B, B_1, \dots, B_n) \subseteq W^{\text{lin}}_\tau(X_m)$.*

The empty set can be regarded as a set of linear terms and if one of the sets $B, B_1, \dots, B_n$ is empty, then $S_m^{nd,n}(B, B_1, \dots, B_n) = \varnothing$ and therefore the theorem is satisfied in this case. So, for the proof we may assume that none of the sets $B, B_1, \dots, B_n$ is empty. We will use the following lemma.

Lemma 2.3. *Let $s, t \in W_\tau(X_n)$ with $\operatorname{var}(s) \cap \operatorname{var}(t) = \varnothing$. Let $B_1, \dots, B_n \subseteq W_\tau(X_m)$ with $\operatorname{var}(B_j) \cap \operatorname{var}(B_k) = \varnothing$ for $1 \le j < k \le n$. Then $\operatorname{var}(S_m^{nd,n}(\{s\}, B_1, \dots, B_n)) \cap \operatorname{var}(S_m^{nd,n}(\{t\}, B_1, \dots, B_n)) = \varnothing$.*

Proof. Proceed by induction on the complexity of s and t. Consider the cases:
1. $s = x_k$ and $t = x_l$, $x_k, x_l \in X_n$,
2. $s = x_k$ and $t = f_i(t_1, \dots, t_{n_i})$,
3. $s = f_i(s_1, \dots, s_{n_i})$ and $t = x_l$,
4. $s = f_j(s_1, \dots, s_{n_j})$ and $t = f_i(t_1, \dots, t_{n_i})$.

1. Since $\operatorname{var}(s) \cap \operatorname{var}(t) = \varnothing$, we have $x_k \neq x_l$ and by the nd-superposition, $S_m^{nd,n}(\{x_k\}, B_1, \dots, B_n) = B_k \neq B_l = S_m^{nd,n}(\{x_l\}, B_1, \dots, B_n)$ and then $\operatorname{var}(B_k) \cap \operatorname{var}(B_l) = \varnothing$ means $\operatorname{var}\left(S_m^{nd,n}(\{s\}, B_1, \dots, B_n)\right) \cap \operatorname{var}\left(S_m^{nd,n}(\{t\}, B_1, \dots, B_n)\right) = \varnothing$.

2. Inductively, we assume that

$$\operatorname{var}\left(S_m^{nd,n}(\{x_k\}, B_1, \dots, B_n)\right) = \operatorname{var}(B_k) \cap \operatorname{var}\left(S_m^{nd,n}(\{t_j\}, B_1, \dots, B_n)\right) = \varnothing$$

for $1 \le j \le n$. By the definition of nd-superposition we have

$$\begin{aligned}
&S_m^{nd,n}(\{f_i(t_1, \dots, t_{n_i})\}, B_1, \dots, B_n) \\
&= \big\{f_i(r_1, \dots, r_{n_i}) : r_j \in S_m^{nd,n}(\{t_j\}, B_1, \dots, B_n),\ 1 \le j \le n_i\big\}.
\end{aligned}$$

Since no r_j, $1 \le j \le n_i$, contains a variable from B_k, no term $f_i(r_1, \dots, r_{n_i})$ contains a variable from B_k and so

$$\operatorname{var}\left(S_m^{nd,n}(\{s\}, B_1, \dots, B_n)\right) \cap \operatorname{var}\left(S_m^{nd,n}(\{t\}, B_1, \dots, B_n)\right) = \varnothing.$$

3. Because of the commutativity of intersection the third case can be settled by analogy to the second case.

4. Inductively, we assume that

$$\operatorname{var}\left(S_m^{nd,n}(\{t_j\}, B_1, \dots, B_n)\right) \cap \operatorname{var}\left(S_m^{nd,n}(\{s\}, B_1, \dots, B_n)\right) = \varnothing$$

for $1 \leq j \leq n_i$. By the definition of nd-superposition, we have

$$S_m^{nd,n}(\{f_i(t_1,\ldots,t_{n_i})\},B_1,\ldots,B_n)$$
$$= \big\{f_i(r_1,\ldots,r_{n_i}) : r_l \in S_m^{nd,d}(\{t_l\},B_1,\ldots,B_n),\ 1 \leq j \leq n_i\big\}$$

and by hypothesis no r_l contains the variables from $S_m^{nd,n}(\{s\},B_1,\ldots,B_n)$. It follows that no term $f_i(r_1,\ldots,r_{n_i})$ contains the variables from $S_m^{nd,n}(\{s\},B_1,\ldots,B_n)$ and so $\operatorname{var}\big(S_m^{nd,n}(\{s\},B_1,\ldots,B_n)\big) \cap \operatorname{var}\big(S_m^{nd,n}(\{t\},B_1,\ldots,B_n)\big) = \varnothing$.

In the same way we proceed for a fixed t by induction on the complexity of s. □

Proof of Theorem 2.2. We give a proof on using the inductive definition of nd-superposition.

(i) If $B = \{x_j\}, x_j \in X_n$, then $S_m^{nd,n}(\{x_j\},B_1,\ldots,B_n) \subseteq W_\tau^{\mathrm{lin}}(X_m)$.

(ii) If $B = \{f_i(t_1,\ldots,t_{n_i})\}$ and we assume that $S_m^{nd,n}(\{t_j\},B_1,\ldots,B_n) \subseteq W_\tau^{\mathrm{lin}}(X_m)$ for $1 \leq j \leq n$, then $S_m^{nd,n}(\{f_i(t_1,\ldots,t_{n_i})\},B_1,\ldots,B_n) = \{f_i(r_1,\ldots,r_{n_i}) : r_j \in S_m^{nd,n}(\{t_j\},B_1,\ldots,B_n),\ 1 \leq j \leq n_i\}$. Since by the inductive hypothesis $S_m^{nd,n}(\{t_j\},B_1,\ldots,B_n) \subseteq W_\tau^{\mathrm{lin}}(X_m)$, all r_j are linear terms. We have to prove that $\operatorname{var}(r_j) \cap \operatorname{var}(r_k) = \varnothing$ for $1 \leq j < k \leq n_i$. Since $f_i(t_1,\ldots,t_{n_i}) \in W_\tau^{\mathrm{lin}}(X_m)$, $\operatorname{var}(t_j) \cap \operatorname{var}(t_k) = \varnothing$ for $1 \leq j < k \leq n_i$. Then all assumptions of Lemma 2.3 are satisfied and so $\operatorname{var}(S_m^{nd,n}(\{t_j\},B_1,\ldots,B_n)) \cap \operatorname{var}(S_m^{nd,n}(\{t_k\},B_1,\ldots,B_n)) = \varnothing$ for all $1 \leq j < k \leq n_i$. But then $\operatorname{var}(r_j) \cap \operatorname{var}(r_k) = \varnothing$ for $r_j \in S_m^{nd,n}(\{t_j\},B_1,\ldots,B_n), r_k \in S_m^{nd,n}(\{t_k\},B_1,\ldots,B_n)$ for $1 \leq j < k \leq n_i$ by Lemma 2.3. This shows that all terms $f_i(r_1,\ldots,r_{n_i})$ are linear and so $S_m^{nd,n}(\{f_i(t_1,\ldots,t_{n_i})\},B_1,\ldots,B_n) \subseteq W_\tau^{\mathrm{lin}}(X_m)$.

(iii) If $B \subseteq W_\tau^{\mathrm{lin}}(X_n)$ is an arbitrary set of linear terms, then by (iii) of the definition of nd-superposition we have

$$S_m^{nd,n}(B,B_1,\ldots,B_n) = \bigcup_{b \in B} S_m^{nd,n}(\{b\},B_1,\ldots,B_n) \subseteq W_\tau^{\mathrm{lin}}(X_m)$$

since by (ii) each set of the union consists of linear terms from $W_\tau^{\mathrm{lin}}(X_m)$.

Altogether, $S_m^{nd,n}(B,B_1,\ldots,B_n) \subseteq W_\tau^{\mathrm{lin}}(X_m)$. □

Applying Theorem 2.2 we define the many-sorted partial operations

$$\dot{S}_m^{nd,n} : \mathscr{P}(W_\tau(X_n)) \times (\mathscr{P}(W_\tau(X_m)))^n \multimap\!\!\rightarrow \mathscr{P}(W_\tau(X_m))$$

on the many-sorted set $(\mathscr{P}(W_\tau(X_n)))_{n \in \mathbb{N}^+}$ by

$$\dot{S}_m^{nd,n}(B,B_1,\ldots,B_n) := \begin{cases} S_m^{nd,n}(B,B_1,\ldots,B_n) & \text{if } \operatorname{var}(B_j) \cap \operatorname{var}(B_k) = \varnothing, \\ & \text{or } B = \varnothing \text{ for } 1 \leq j < k \leq n, \\ \text{not defined} & \text{otherwise} \end{cases}$$

for all $m,n \in \mathbb{N}^+$. We note that if one of the sets $B_1,\ldots,B_n$ is empty, then we have the first case and $\dot{S}_m^{nd,n}(B,B_1,\ldots,B_n) = S_m^{nd,n}(B,B_1,\ldots,B_n) = \varnothing$.

Then we may consider the partial multi-based algebra

$$\mathscr{P}\text{-}\operatorname{clone}^{\mathrm{lin}}\tau := \big(\big(\mathscr{P}\big(W_\tau^{\mathrm{lin}}(X_n)\big)_{n \in \mathbb{N}^+}; \big(\dot{S}_m^{nd,n}\big)_{m,n \in \mathbb{N}^+}, (\{x_i\})_{i \leq n,\, n \in \mathbb{N}^+}\big).$$

We call this algebra the *linear power clone of type τ*.

The following properties of $S_m^{nd,n}$ (see [3]) are also valid for $\dot{S}_m^{nd,n}$.

Theorem 2.4. *Assume that $A, A' \subseteq W_\tau^{\mathrm{lin}}(X_n)$, $B_1,\ldots,B_n,B_1',\ldots,B_n' \subseteq W_\tau^{\mathrm{lin}}(X_m)$ and $\operatorname{var}(B_i) \cap \operatorname{var}(B_j) = \varnothing$ for $1 \leq i < j \leq n$. Then*

(i) *If $A' \subseteq A$, then $\dot{S}_m^{nd,n}(A',B_1,\ldots,B_n) \subseteq \dot{S}_m^{nd,n}(A,B_1,\ldots,B_n)$.*

(ii) *If $B_i' \subseteq B_i$ for all $i = 1,\ldots,n$, then $\dot{S}_m^{nd,n}(A,B_1',\ldots,B_n') \subseteq \dot{S}_m^{nd,n}(A,B_1,\ldots,B_n)$.*

(iii) *If $b_i \in B_i$ for all $i = 1,\ldots,n$, then $\{S_m^n(b,b_1,\ldots,b_n)\} \subseteq \dot{S}_m^{nd,n}(A,B_1,\ldots,B_n)$.*

PROOF. By assumption $\dot{S}_m^{nd,n}(A, B_1, \ldots, B_n) = S_m^{nd,n}(A, B_1, \ldots, B_n) \subseteq W_\tau^{\mathrm{lin}}(X_m)$ and $\dot{S}_m^{nd,n}(A', B_1, \ldots, B_n) = S_m^{nd,n}(A', B_1, \ldots, B_n) \subseteq W_\tau^{\mathrm{lin}}(X_m)$. Applying the corresponding result from [3] we obtain (i). Now, $\dot{S}_m^{nd,n}(A', B_1', \ldots, B_n') = S_m^{nd,n}(A', B_1', \ldots, B_n')$ since $\mathrm{var}(B_i) \cap \mathrm{var}(B_j) = \varnothing$ and $B_i' \subseteq B_i, B_j' \subseteq B_j$ implies $\mathrm{var}(B_i') \cap \mathrm{var}(B_j') = \varnothing$ for $1 \leq i < j \leq n$. Using the corresponding properties of $S_m^{nd,n}$ (see [3]) we get (ii). From $\mathrm{var}(B_i) \cap \mathrm{var}(B_j) = \varnothing$ for $1 \leq i < j \leq n$ we obtain $\dot{S}_m^{nd,n}(\{a\}, \{b_1\}, \ldots, \{b_n\}) = S_m^{nd,n}(\{a\}, \{b_1\}, \ldots, \{b_n\}) = \{S(a, b_1, \ldots, b_n)\}$. By (i), we have $\dot{S}_m^{nd,n}(\{a\}, B_1, \ldots, B_n) \subseteq \dot{S}_m^{nd,n}(A, B_1, \ldots, B_n)$ and by (ii)

$$\dot{S}_m^{nd,n}(\{a\}, \{b_1\}, \ldots, \{b_n\}) \subseteq \dot{S}_m^{nd,n}(\{a\}, B_1, \ldots, B_n).$$

Altogether, this gives

$$\left\{S_m^n(b, b_1, \ldots, b_n)\right\} \subseteq \dot{S}_m^{nd,n}(A, B_1, \ldots, B_n). \quad \square$$

There are several ways of defining homomorphisms, subalgebras, and identities for partial algebras. Let $\mathscr{A}$ and $\mathscr{B}$ be partial algebras of the same type with indexed sets $\{f_i^{\mathscr{A}} : i \in I\}$ and $\{f_i^{\mathscr{B}} : i \in I\}$ of partial fundamental operations on A and B, respectively. A mapping $h : A \to B$ is said to be a *weak homomorphism* provided that for all fundamental operations we have: If $(a_1, \ldots, a_{n_i}) \in \mathrm{dom}\, f_i^{\mathscr{A}}$, then $(h(a_1), \ldots, h(a_{n_i})) \in \mathrm{dom}\, f_i^{\mathscr{B}}$ and for all $i \in I$,

$$h(f_i^{\mathscr{A}}(a_1, \ldots, a_{n_i})) = f_i^{\mathscr{B}}(h(a_1), \ldots, h(a_{n_i}))$$

where $\mathrm{dom}\, f_i^{\mathscr{A}}$ and $\mathrm{dom}\, f_i^{\mathscr{B}}$ are the domains of $f_i^{\mathscr{A}}$ and of $f_i^{\mathscr{B}}$, respectively.

Given $n \geq 1$, put $\overline{F}_\tau^n := \{f_i(x_1, \ldots, x_{n_i}) : i \in I,\ n = n_i\}$.

In [2], the author proved that $(\overline{F}_\tau^n)_{n \in \mathbb{N}^+}$ is a generating system of $\mathrm{clone}_{\mathrm{lin}}\, \tau$.

Lemma 2.5 [2]. *The many-sorted partial algebra* $\mathrm{clone}_{\mathrm{lin}}\, \tau$ *is free with respect to itself, freely generated by* $(\overline{F}_\tau^n)_{n \in \mathbb{N}^+}$.

To obtain a similar result for the many-sorted partial algebra $\mathscr{P}$-$\mathrm{clone}^{\mathrm{lin}}\, \tau$ we consider for each $i \in I$,

$$\widetilde{F}_\tau^n := \bigcup_{i \in I,\, 1 \leq n_i \leq n} \{\{f_i(x_1, \ldots, x_{n_i})\}\}.$$

Lemma 2.6. $(\widetilde{F}_\tau^n)_{n \in \mathbb{N}^+}$ *is a generating system of* $\mathscr{P}$-$\mathrm{clone}^{\mathrm{lin}}\, \tau$.

PROOF. Let $m, n \in \mathbb{N}^+$ and $\{x_i\} \subseteq X_n$. Since the singletons of variables belong to the type of $\mathscr{P}$-$\mathrm{clone}^{\mathrm{lin}}\, \tau$, these sets are generated. Now, we let $B = \{f_i(t_1, \ldots, t_{n_i})\} \subseteq W_\tau^{\mathrm{lin}}(X_m)$ and assume that $\{t_1\}, \ldots, \{t_{n_i}\} \subseteq W_\tau^{\mathrm{lin}}(X_m)$ are generated. Since $f_i(t_1, \ldots, t_{n_i}) \in W_\tau^{\mathrm{lin}}(X_m)$, $\mathrm{var}(t_j) \cap \mathrm{var}(t_k) = \varnothing$ for $1 \leq j < k \leq n_i$, this implies $\mathrm{var}(\{t_j\}) \cap \mathrm{var}(\{t_k\}) = \varnothing$ for $1 \leq j < k \leq n_i$. Thus,

$$\begin{aligned} B = \{f_i(t_1, \ldots, t_{n_i})\} &= \dot{S}_m^{nd,n_i}(\{f_i(x_1, \ldots, x_{n_i})\}, \{t_1\}, \ldots, \{t_{n_i}\}) \\ &= S_m^{nd,n_i}(\{f_i(x_1, \ldots, x_{n_i})\}, \{t_1\}, \ldots, \{t_{n_i}\}). \end{aligned}$$

Therefore, $B = \{f_i(t_1, \ldots, t_{n_i})\}$ is generated. Each set $B \subseteq W_\tau^{\mathrm{lin}}(X_m)$ can be written as $B = \bigcup_{b \in B} \{b\}$, moreover, $\{b\} \subseteq W_\tau^{\mathrm{lin}}(X_m)$. By the definition of $\dot{S}_m^{nd,n_i}$, B is generated. If B is empty, then B can be regarded as the empty union of elements of the generating set. $\square$

By Lemma 2.6 the many-sorted partial algebra $\mathscr{P}$-$\mathrm{clone}^{\mathrm{lin}}\, \tau$ is generated by $(\widetilde{F}_\tau^n)_{n \in \mathbb{N}^+}$.

Now, for any many-sorted mapping

$$(\varphi_n)_{n \in \mathbb{N}^+} : (\widetilde{F}_\tau^n)_{n \in \mathbb{N}^+} \to (\mathscr{P}(W_\tau^{\mathrm{lin}}(X_n))_{n \in \mathbb{N}^+}$$

we define the extension $(\overline{\varphi}_n)_{n \in \mathbb{N}^+}$ of $(\varphi_n)_{n \in \mathbb{N}^+}$ inductively as

$$(\overline{\varphi}_n)_{n \in \mathbb{N}^+} : (\widetilde{F}_\tau^n)_{n \in \mathbb{N}^+} \to (\mathscr{P}(W_\tau^{\mathrm{lin}}(X_n)))_{n \in \mathbb{N}^+}$$

with

(i) $\overline{\varphi}_n(\{x_j\}) := \{x_j\}$ for $x_j \in X_n$, $n \in \mathbb{N}^+$,

(ii) $\overline{\varphi}_m(\{f_i(t_1,\ldots,t_{n_i})\}) := \dot{S}_m^{nd,n_i}(\varphi_{n_i}(\{f_i(x_1,\ldots,x_{n_i})\}),\overline{\varphi}_m(\{t_1\}),\ldots,\overline{\varphi}_m(\{t_{n_i}\}))$ for $m, n_i \in \mathbb{N}^+$, $i \in I$, assumed that $\overline{\varphi}_m(\{t_j\})$ are already defined for $1 \le j \le n_i$,

(iii) $\overline{\varphi}_m(B) = \bigcup_{b\in B}\overline{\varphi}_m(\{b\})$ if $B \subseteq W_\tau^{\mathrm{lin}}(X_m)$ is an arbitrary set.

For $m, n_i \in \mathbb{N}^+, i \in I$, we have

$$\begin{aligned}\overline{\varphi}_m(\{f_i(x_1,\ldots,x_{n_i})\}) &= \dot{S}_m^{nd,n_i}(\varphi_{n_i}(\{f_i(x_1,\ldots,x_{n_i})\}),\overline{\varphi}_m(\{x_1\}),\ldots,\overline{\varphi}_m(\{x_{n_i}\}))\\ &= \dot{S}_m^{nd,n_i}(\varphi_{n_i}(\{f_i(x_1,\ldots,x_{n_i})\}),\{x_1\},\ldots,\{x_{n_i}\})\\ &= S_m^{nd,n_i}(\varphi_{n_i}(\{f_i(x_1,\ldots,x_{n_i})\}),\{x_1\},\ldots,\{x_{n_i}\})\\ &= \varphi_m(\{f_i(x_1,\ldots,x_{n_i})\}).\end{aligned}$$

This shows that $\overline{\varphi}_n$ extends φ_n. Moreover, we have

Lemma 2.7. *If $B \subseteq W_\tau^{\mathrm{lin}}(X_n)$, then* $\mathrm{var}(\overline{\varphi}_n(B)) \subseteq \mathrm{var}(B)$.

Proof. We may assume that $B \neq \varnothing$. If B is a singleton, then we give a proof by induction on the complexity of the linear term which forms the only element of the singleton B. If $B = \{x_j\}, x_j \in X_n$, $1 \le j \le n$, then

$$\mathrm{var}(\overline{\varphi}_n(B)) = \mathrm{var}(\overline{\varphi}_n(\{x_j\})) = \mathrm{var}(\{x_j\}) = \mathrm{var}(B).$$

If $B = \{f_i(t_1,\ldots,t_{n_i})\}$ and if $\mathrm{var}(\overline{\varphi}_n(\{t_j\})) \subseteq \mathrm{var}(\{t_j\})$ for all $1 \le j \le n_i \le n$, $n \in \mathbb{N}^+$, then

$$\begin{aligned}&\mathrm{var}(\overline{\varphi}_n(B)) = \mathrm{var}(\overline{\varphi}_n(\{f_i(t_1,\ldots,t_{n_i})\}))\\ &= \mathrm{var}\left(\dot{S}_m^{nd,n_i}(\varphi_{n_i}(\{f_i(x_1,\ldots,x_{n_i})\}),\overline{\varphi}_m(\{t_1\}),\ldots,\overline{\varphi}_m(\{t_{n_i}\}))\right)\\ &(\text{since } \mathrm{var}(\overline{\varphi}_m(\{t_j\})) \cap \mathrm{var}(\overline{\varphi}_m(\{t_k\})) = \varnothing \quad \text{for } 1 \le j < k \le n_i)\\ &\subseteq \bigcup_{j=1}^{n_i}\mathrm{var}(\overline{\varphi}_m(\{t_j\})) \subseteq \bigcup_{j=1}^{n_i}\mathrm{var}(\{t_j\}) = \mathrm{var}(\{f_i(t_1,\ldots,t_{n_i})\}) = \mathrm{var}(B).\end{aligned}$$

If B is an arbitrary nonempty subset of $W_\tau^{\mathrm{lin}}(X_n)$, then

$$\mathrm{var}(\overline{\varphi}_n(B)) = \mathrm{var}\Big(\bigcup_{b\in B}\overline{\varphi}_n(\{b\})\Big) = \bigcup_{b\in B}\mathrm{var}(\overline{\varphi}_n(\{b\})) \subseteq \bigcup_{b\in B}\mathrm{var}(\{b\}) = \mathrm{var}(B). \quad \square$$

From the previous lemma we observe that for any distinct sets of linear terms $\{t_j\}$ and $\{t_k\}$, $\mathrm{var}(\{t_j\}) \cap \mathrm{var}(\{t_k\}) = \varnothing$ implies $\mathrm{var}(\overline{\varphi}_n(\{t_j\})) \cap \mathrm{var}(\overline{\varphi}_n(\{t_k\})) = \varnothing$. Summarizing the above, we obtain the following:

Proposition 2.8. *The many-sorted partial algebra $\mathscr{P}$-clone$^{\mathrm{lin}}\,\tau$ is free with respect to itself, freely generated by $(\widetilde{F}_\tau^n)_{n\in\mathbb{N}^+}$.*

Proof. By Lemmas 2.6 and 2.7, $\mathscr{P}$-clone$^{\mathrm{lin}}\,\tau$ has a generating system and $\overline{\varphi}_n$ extends φ_n. For singletons $\{f_i(t_1,\ldots,t_{n_i})\} \subseteq W_\tau^{\mathrm{lin}}(X_m)$ $\mathrm{var}(\{t_j\}) \cap \mathrm{var}(\{t_k\}) = \varnothing$ implies $\mathrm{var}(\overline{\varphi}_n(\{t_j\})) \cap \mathrm{var}(\overline{\varphi}_n(\{t_k\})) = \varnothing$ for $1 \le j < k \le n_i$. That is, $(\varphi_{n_i}(\{f_i(x_1,\ldots,x_{n_i})\}),\overline{\varphi}_m(\{t_1\}),\ldots,\overline{\varphi}_m(\{t_{n_i}\}))$ belongs to the domain of $\dot{S}_m^{nd,n_i}$. Thus,

$$\begin{aligned}&\overline{\varphi}_m\big(\dot{S}_m^{nd,n_i}(\{f_i(x_1,\ldots,x_{n_i})\},\{t_1\},\ldots,\{t_{n_i}\})\big) = \overline{\varphi}_m(\{f_i(t_1,\ldots,t_{n_i})\})\\ &\quad= \dot{S}_m^{nd,n_i}(\varphi_{n_i}(\{f_i(x_1,\ldots,x_{n_i})\}),\overline{\varphi}_m(\{t_1\}),\ldots,\overline{\varphi}_m(\{t_{n_i}\}))\\ &\quad= \dot{S}_m^{nd,n_i}(\overline{\varphi}_{n_i}(\{f_i(x_1,\ldots,x_{n_i})\}),\overline{\varphi}_m(\{t_1\}),\ldots,\overline{\varphi}_m(\{t_{n_i}\})).\end{aligned}$$

In the case that B is an arbitrary subset of $W_\tau^{\mathrm{lin}}(X_n)$, we have $\overline{\varphi}_n(B) = \bigcup_{b\in B}\overline{\varphi}_n(\{b\})$, and therefore, $(\overline{\varphi}_n)_{n\in\mathbb{N}^+}$ is a weak endomorphism extending $(\varphi_n)_{n\in\mathbb{N}^+}$. $\square$

3. Properties of the Linear Power Clone

Our main question is whether the clone identities (C1)–(C3) are satisfied in the linear power clone $\mathscr{P}\text{-clone}^{\text{lin}}\,\tau$. But this is a partial many-sorted algebra; and in partial algebras there are various concepts of identities. An equation $s \approx t$ of terms over the many-sorted partial algebra $\mathscr{A}$ is said to be a *weak identity* in $\mathscr{A}$ if after evaluation, i.e. after all replacements of variables occurring in s and t by elements of A and operation symbols in s and t by the corresponding fundamental operations of $\mathscr{A}$, the one side and the other side are defined and equal.

Theorem 3.1. *The partial many-sorted algebra* $\mathscr{P}\text{-clone}^{\text{lin}}\,\tau$ *satisfies* (C1) *and* (C3) *as weak identities.*

Proof. We check the satisfaction of (C1)–(C3) as weak identities.

In (C1), i.e. in

$$\begin{aligned}&\widetilde{S}^p_m\big(\widetilde{Z},\widetilde{S}^n_m(\widetilde{Y}_1,\widetilde{X}_1,\dots,\widetilde{X}_n),\dots,\widetilde{S}^n_m(\widetilde{Y}_p,\widetilde{X}_1,\dots,\widetilde{X}_n)\big)\\ &\approx \widetilde{S}^n_m\big(\widetilde{S}^p_n(\widetilde{Z},\widetilde{Y}_1,\dots,\widetilde{Y}_p),\widetilde{X}_1,\dots,\widetilde{X}_n\big) \quad (m,n,p\in\mathbb{N}^+),\end{aligned}$$

we replace $\widetilde{S}^p_m$ by $\dot{S}^{nd,p}_m$, $\widetilde{S}^n_m$ by $\dot{S}^{nd,n}_m$, $\widetilde{S}^p_n$ by $\dot{S}^{nd,p}_n$, $\widetilde{Z}$ by $B\subseteq W^{\text{lin}}_\tau(X_m)$, $\widetilde{Y}_1,\dots,\widetilde{Y}_p$ by $B_1,\dots,B_p\in W^{\text{lin}}_\tau(X_n)$ and $\widetilde{X}_1,\dots,\widetilde{X}_n$ by $A_1,\dots,A_n\in W^{\text{lin}}_\tau(X_m)$ and obtain

$$\begin{aligned}&\dot{S}^{nd,p}_m\big(B,\dot{S}^{nd,n}_m(B_1,A_1,\dots,A_n),\dots,\dot{S}^{nd,n}_m(B_p,A_1,\dots,A_n)\big)\\ &\approx \dot{S}^{nd,n}_m\big(\dot{S}^{nd,p}_n(B,B_1,\dots,B_p),A_1,\dots,A_n\big).\end{aligned}$$

We assume that none of the sets $B_1,\dots,B_p,A_1,\dots,A_n$ is empty. If $\operatorname{var}(B_j)\cap\operatorname{var}(B_k)=\varnothing$ for $1\le j<k\le p$ and $\operatorname{var}(A_l)\cap\operatorname{var}(A_q)=\varnothing$ for $1\le l<q\le n$, then the right-hand side of this equation is defined. Then also $\dot{S}^{nd,n}_m(B_j,A_1,\dots,A_n)$, $j=1,\dots,p$, are defined and equal to $S^{nd,n}_m(B_j,A_1,\dots,A_n)$. It is left to show that $\operatorname{var}\big(S^{nd,n}_m(B_j,A_1,\dots,A_n)\big)\cap\operatorname{var}\big(S^{nd,n}_m(B_k,A_1,\dots,A_n)\big)=\varnothing$ for $1\le j<k\le p$. If both B_j and B_k consist of only one term, then by Lemma 2.3

$$\operatorname{var}\big(S^{nd,n}_m(B_j,A_1,\dots,A_n)\big)\cap\operatorname{var}\big(S^{nd,n}_m(B_k,A_1,\dots,A_n)\big)=\varnothing.$$

By (iii) of the definition of nd-superposition we have

$$S^{nd,n}_m(B_j,A_1,\dots,A_n)=\bigcup_{b\in B_j}S^{nd,n}_m(\{b\},A_1,\dots,A_n)$$

and $S^{nd,n}_m(\{b'\},B_k,A_1,\dots,A_n)=\bigcup_{b'\in B_k}(A_1,\dots,A_n)$ and so

$$\begin{aligned}&\operatorname{var}\big(S^{nd,n}_m(B_j,A_1,\dots,A_n)\big)\cap\operatorname{var}\big(S^{nd,n}_m(B_k,A_1,\dots,A_n)\big)\\ &=\operatorname{var}\Big(\bigcup_{b\in B_j}S^{nd,n}_m(\{b\},A_1,\dots,A_n)\Big)\cap\operatorname{var}\Big(\bigcup_{b'\in B_k}S^{nd,n}_m(\{b'\},A_1,\dots,A_n)\Big)\\ &=\bigcup_{b\in B_j}\operatorname{var}\big(S^{nd,n}_m(\{b\},A_1,\dots,A_n)\big)\cap\bigcup_{b'\in B_k}\operatorname{var}\big(S^{nd,n}_m(\{b'\},A_1,\dots,A_n)\big)\\ &=\bigcup_{b\in B_j,\,b'\in B_k}\Big(\operatorname{var}\big(S^{nd,n}_m(\{b\},A_1,\dots,A_n)\big)\cap\operatorname{var}\big(S^{nd,n}_m(\{b'\},A_1,\dots,A_n)\big)\Big)=\varnothing.\end{aligned}$$

Therefore, the left-hand side is defined and equal to

$$S^{nd,p}_m(B,S^{nd,n}_m(B_1,A_1,\dots,A_n),\dots,S^{nd,n}_m(B_p,A_1,\dots,A_n)).$$

(If one of the sets A_i or B_j is empty, then both sides are empty and (C1) is satisfied.) Since the total many-sorted algebra $\mathscr{P}\text{-clone}^{\text{lin}}\,\tau$ satisfies (C1) in any case both sides are equal and (C1) is a weak identity in $\mathscr{P}\text{-clone}^{\text{lin}}\,\tau$.

In (C2) $\widetilde{S}^n_m(\lambda_j, \widetilde{X}_1, \dots, \widetilde{X}_n) \approx \widetilde{X}_i$, $m, n \in \mathbb{N}^+$, $1 \le i \le n$, we replace λ_j by $\{x_j\}$, $\tilde{S}^n_m$ by $\dot{S}^{nd,n}_m$ and $\widetilde{X}_1, \dots, \widetilde{X}_n$ by $A_1, \dots, A_n \subseteq W^{\text{lin}}_\tau(X_m)$ and obtain $\dot{S}^{nd,n}_m(\{x_j\}, A_1, \dots, A_n) = S^{nd,n}_m(\{x_j\}, A_1, \dots, A_n) = \varnothing$ if $\text{var}(A_i) \cap \text{var}(A_k) = \varnothing$ for $1 \le i < k \le n$, i.e., if $\dot{S}^{nd,n}_m(\{x_j\}, A_1, \dots, A_n)$ is defined. But the right-hand side can be a nonempty set; i.e., (C2) is not satisfied. We remark that by the definition of linear terms the empty set is linear.

In (C3) $\widetilde{S}^n_n(\widetilde{Y}, \lambda_1, \dots, \lambda_n) \approx \widetilde{Y}$ we replace $\widetilde{Y}$ by $B \subseteq W^{\text{lin}}_\tau(X_n)$, $\widetilde{S}^n_n$ by $\dot{S}^{nd,n}_n$ and $\lambda_1, \dots, \lambda_n$ by $\{x_1\}, \dots, \{x_n\}$ and obtain on the left-hand side $\dot{S}^{nd,n}_n(B, \{x_1\}, \dots, \{x_n\})$. If $B = \{x_j\}$, then $\dot{S}^{nd,n}_n(\{x_j\}, \{x_1\}, \dots, \{x_n\}) = S^{nd,n}_n(\{x_j\}, \{x_1\}, \dots, \{x_n\}) = \{x_j\}$. Therefore, (C3) is satisfied. Now, put $B = \{f_i(t_1, \dots, t_{n_i})\}$. Then

$$\begin{aligned}\dot{S}^{nd,n}_n(\{f_i(t_1, \dots, t_{n_i})\}, \{x_1\}, \dots, \{x_n\}) &= S^{nd,n}_n(\{f_i(t_1, \dots, t_{n_i})\}, \{x_1\}, \dots, \{x_n\}) \\ &= \big\{f_i(r_1, \dots, r_{n_i}) : r_j \in S^{nd,n}_n(\{t_j\}, \{x_1\}, \dots, \{x_n\})\big\} = \{f_i(t_1, \dots, t_{n_i})\}\end{aligned}$$

and (C3) is satisfied. If B contains more than one element, then

$$\begin{aligned}\dot{S}^{nd,n}_n(B, \{x_1\}, \dots, \{x_n\}) &= S^{nd,n}_n(B, \{x_1\}, \dots, \{x_n\}) \\ &= \bigcup_{b \in B} S^{nd,n}_n(\{b\}, \{x_1\}, \dots, \{x_n\}) = \bigcup_{b \in B} \{b\} = B.\end{aligned}$$

If B is empty, then both sides are empty and (C3) is satisfied. $\square$

If we exclude the empty sequence $(\varnothing)_{n \in \mathbb{N}^+}$, then we get the following

Corollary 3.2. *The partial many-sorted algebra* $\mathscr{P}^+$*-clone*$^{\text{lin}}\,\tau := ((\mathscr{P}(W^{\text{lin}}_\tau(X_n)))_{n \in \mathbb{N}^+} \setminus \{(\varnothing)_{n \in \mathbb{N}^+}\};$ $\big(\dot{S}^{nd,n}_m\big)_{m,n \in \mathbb{N}^+}, (\{x_i\})_{i \le n,\, n \in \mathbb{N}^+}\big)$ *satisfies* (C1)–(C3) *as weak identities.*

4. Linear Nondeterministic Hypersubstitutions

A *hypersubstitution* of type τ is a mapping $\sigma : \{f_i : i \in I\} \to W_\tau(X)$ taking each n_i-ary operation symbol of type τ to an n_i-ary term of the same type. Hypersubstitutions may be considered as many-sorted mappings. Let $I_n \subseteq I$ with $n \in \mathbb{N}^+$ be the set of all indices such that f_j with $j \in I_n$ is n-ary and let $F^n_\tau := \{f_j : j \in I_n\}$. Then σ_n with $\sigma_n(F^n_\tau) \subseteq W_\tau(X_n)$ is defined as a mapping that sends operation symbols of arity n to n-ary terms. So, each hypersubstitution of type τ can be regarded as a many-sorted set of functions $\sigma := (\sigma_n)_{n \in \mathbb{N}^+}$. Nondeterministic hypersubstitutions were introduced in [3] as the mappings

$$\sigma_{nd} : \{f_i : i \in I\} \to \mathscr{P}(W_\tau(X))$$

that sends $\sigma(f_i) \subseteq W_\tau(X_{n_i})$ for $i \in I$. The linear nondeterministic hypersubstitutions were introduced in [4] as the mappings

$$\sigma_{nd} : \{f_i : i \in I\} \to \mathscr{P}(W^{\text{lin}}_\tau(X))$$

that send operation symbols to sets of linear terms and preserve arities.

In this section we consider nondeterministic hypersubstitutions as many-sorted mappings. In the linear case we define $\sigma^{\text{lin}}_{nd} := (\sigma^{\text{lin}}_{nd,n})_{n \in \mathbb{N}^+}$ with $\sigma^{\text{lin}}_{nd,n} : F^n_\tau \to \mathscr{P}(W^{\text{lin}}_\tau(X_n))$. For arbitrary nondeterministic hypersubstitutions σ_{nd} an extension $\hat{\sigma}_{nd}$ can be defined in steps as follows:

Definition 4.1. Let $n \in \mathbb{N}^+$.

(i) $\hat{\sigma}_{nd,n}[\varnothing] := \varnothing$.

(ii) $\hat{\sigma}_{nd,n}[\{x_i\}] := \{x_i\}$ for $i = 1, \dots, n$.

(iii) $\hat{\sigma}_{nd,n}[\{f_i(t_1, \dots, t_{n_i}\}] := S^{nd,n}_n(\sigma_{nd,n}(f_i), \hat{\sigma}_{nd,n}[\{t_1\}], \dots, \hat{\sigma}_{nd,n}[\{t_n\}])$ if we inductively assume that $\hat{\sigma}_{nd,n}[\{t_j\}]$, $1 \le j \le n$, are already known.

(iv) $\hat{\sigma}_{nd,n}[B] := \bigcup\{\hat{\sigma}_{nd,n}[\{t\}] \mid t \in B\}$ for any $\varnothing \ne B \subseteq W_\tau(X_n)$.

We put $\hat{\sigma}_{nd} := (\hat{\sigma}_{nd,n})_{n\in\mathbb{N}^+}$. Our question is whether the extension of a linear nondeterministic hypersubstitution sends sets of linear terms to sets of linear terms. This question was positively answered in [4]. This answer has the consequence that the product of every two linear nondeterministic hypersubstitutions is a linear nondeterministic hypersubstitution and therefore the set of all linear nondeterministic hypersubstitutions of type τ forms a monoid. We want now to derive this result from the following theorem which stems from our approach. We introduced already the concept of a weak homomorphism for partial algebras. For many-sorted partial algebras we may define a corresponding concept where weak homomorphisms are now sequences of mappings. If there is no risk of misunderstanding instead of σ_{nd}^{lin} we will write σ_{nd}.

Theorem 4.2. *For every linear nondeterministic hypersubstitution σ_{nd} its extension $\hat{\sigma}_{nd}$ is a weak endomorphism of the partial many-sorted linear power clone*

$$\mathscr{P}\text{-}\operatorname{clone}_{\text{lin}}\tau = \big((\mathscr{P}(W_\tau(X_n)))_{n\in\mathbb{N}^+};\big(\dot{S}_m^{nd,n}\big)_{m,n\in\mathbb{N}^+},(\{x_i\})_{i\le n,\,n\in\mathbb{N}^+}\big).$$

Proof. We have to prove that given arbitrary sets $A\in\mathscr{P}(W_\tau^{\text{lin}}(X_n))$ and $B_1,\dots,B_n\in\mathscr{P}(W_\tau^{\text{lin}}(X_m))$ we have: If $(A,B_1,\dots,B_n)\in\operatorname{dom}\dot{S}_m^{nd,n}(A,B_1,\dots,B_n)$, then $(\hat{\sigma}_{nd,n}[A],\hat{\sigma}_{nd,m}[B_1],\dots,\hat{\sigma}_{nd,m}[B_n])\in\operatorname{dom}\dot{S}_m^{nd,n}$ and

$$\hat{\sigma}_{nd,m}\big[\dot{S}_m^{nd,n}(A,B_1,\dots,B_n)\big]=\dot{S}_m^{nd,n}(\hat{\sigma}_{nd,n}[A],\hat{\sigma}_{nd,m}[B_1],\dots,\hat{\sigma}_{nd,m}[B_n]).$$

$(A,B_1,\dots,B_n)\in\operatorname{dom}\dot{S}_m^{nd,n}(A,B_1,\dots,B_n)$ means that $\operatorname{var}(B_i)\cap\operatorname{var}(B_j)=\varnothing$ for $1\le i<j\le n$ and that $\dot{S}_m^{nd,n}(A,B_1,\dots,B_n)=S_m^{nd,n}(A,B_1,\dots,B_n)$. Now we apply Lemma 3 of [4] which means, formulated in terms of our language, that for any nondeterministic linear hypersubstitution $\sigma_{nd}=(\sigma_{nd,n})_{n\in\mathbb{N}^+}$, for any set $B\subseteq W_\tau^{\text{lin}}(X_n)$ and for all $n\in\mathbb{N}^+$ we have

$$\operatorname{var}(B)\supseteq\operatorname{var}(\hat{\sigma}_{n,nd}[B]). \tag{$*$}$$

(We notice that this is even true for arbitrary nondeterministic hypersubstitutions and arbitrary sets of terms from $W_\tau(X_n)$.)

Then from $\operatorname{var}(B_i)\cap\operatorname{var}(B_j)=\varnothing$ and $\operatorname{var}(B_i)\supseteq\operatorname{var}(\hat{\sigma}_n[B_i])$ we infer that

$$\varnothing=\operatorname{var}(B_i)\cap\operatorname{var}(B_j)\supseteq\operatorname{var}(\hat{\sigma}_{nd,n}[B_i])\cap\operatorname{var}(\hat{\sigma}_{nd,n}[B_j]),\quad 1\le i<j\le n.$$

If $A\in\mathscr{P}(W_\tau^{\text{lin}}(X_n))$, then A consists only of linear terms t and

$$\hat{\sigma}_{nd,n}[A]=\bigcup\big\{\hat{\sigma}_{nd,n}[\{t\}]\mid t\in A\subseteq W_\tau^{\text{lin}}(X_n)\big\}.$$

If $\hat{\sigma}_{nd,n}[\{t\}]\subseteq W_\tau^{\text{lin}}(X_n)$ for all $t\in A$, then $\hat{\sigma}_{nd,n}[A]\subseteq W_\tau^{\text{lin}}(X_n)$. So, we have only to show that $t\in W_\tau^{\text{lin}}(X_n)$ implies $\hat{\sigma}_{nd,n}[\{t\}]\subseteq W_\tau^{\text{lin}}(X_n)$.

If $t=x_i\in X_n$, then $\hat{\sigma}_{nd,n}[\{x_i\}]=\{x_i\}\subseteq W_\tau^{\text{lin}}(X_n)$.

Assume that $t=f_i(t_1,\dots,t_{n_i})\in W_\tau^{\text{lin}}(X_n)$ and that $t_i\in W_\tau^{\text{lin}}(X_n)$ implies $\hat{\sigma}_{nd,n}[\{t_i\}]\subseteq W_\tau^{\text{lin}}(X_n)$.

By Definition 4.1, (iii) we have

$$\hat{\sigma}_{nd,n}[f_i(t_1,\dots,t_{n_i})]=S_n^{nd,n}(\sigma_{nd,n}(f_i),\hat{\sigma}_{nd,n}[\{t_1\}],\dots,\hat{\sigma}_{nd,n}[\{t_n\}]).$$

Here $\sigma_{nd,n}(f_i)\subseteq W_\tau^{\text{lin}}(X_n)$ since $\sigma_{nd,n}$ is a linear nondeterministic hypersubstitution. Since $\operatorname{var}(\{t_i\})=\varnothing$ implies $\operatorname{var}(\hat{\sigma}_{nd,n}[\{t_i\}])\cap\operatorname{var}(\hat{\sigma}_{nd,n}[\{t_j\}])=\varnothing$ for $1\le i<j\le n$ by (*), there follows $\hat{\sigma}_{nd,n}[\{t\}]\subseteq W_\tau^{\text{lin}}(X_n)$.

Altogether, $(A,B_1,\dots,B_n)\in\operatorname{dom}\dot{S}_m^{nd,n}(A,B_1,\dots,B_n)$, i.e. $A\subseteq W_\tau^{\text{lin}}(X_n),B_1,\dots,B_n\subseteq W_\tau^{\text{lin}}(X_n)$, $\operatorname{var}(B_i)\cap\operatorname{var}(B_j)=\varnothing$ for $1\le i<j\le n$ implies $(\hat{\sigma}_{nd,n}[A],\hat{\sigma}_{nd,m}[B_1],\dots,\hat{\sigma}_{nd,m}[B_n])\in\operatorname{dom}\dot{S}_m^{nd,n}$ and

$$\dot{S}_m^{nd,n}(\hat{\sigma}_{nd,n}[A],\hat{\sigma}_{nd,m}[B_1],\dots,\hat{\sigma}_{nd,m}[B_n])=S_m^{nd,n}(\hat{\sigma}_{nd,n}[A],\hat{\sigma}_{nd,m}[B_1],\dots,\hat{\sigma}_{nd,m}[B_n]).$$

From this and the endomorphism property of arbitrary nondeterministic hypersubstitutions (see Theorem 3.1.4 in [5]), i.e.

$$\hat{\sigma}_{nd,m}\big[S_m^{nd,n}(A,B_1,\dots,B_n)\big]=S_m^{nd,n}(\hat{\sigma}_{nd,n}[A],\hat{\sigma}_{nd,m}[B_1],\dots,\hat{\sigma}_{nd,m}[B_n]),$$

we obtain

$$\hat{\sigma}_{nd,m}\big[\dot{S}_m^{nd,n}(A,B_1,\dots,B_n)\big]=\dot{S}_m^{nd,n}(\hat{\sigma}_{nd,n}[A],\hat{\sigma}_{nd,m}[B_1],\dots,\hat{\sigma}_{nd,m}[B_n]).\quad\square$$

As a corollary of Theorem 4.2 we get Proposition 2.5 from [4]:

Corollary 4.3. *The extension of a nondeterministic linear hypersubstitution sends the sets of linear terms to sets of linear terms.*

In [4] the authors defined a product $\circ_{nd}$ on the set $\mathrm{Hyp}^{nd}_{\mathrm{lin}}(\tau)$ of all nondeterministic linear hypersubstitutions by $\sigma_{nd,1} \circ_{nd} \sigma_{nd,2} := \hat{\sigma}_{nd,1} \circ \sigma_{nd,2} := \hat{\sigma}_{nd,1} \circ \sigma_{nd,2}$. Since the nondeterministic identity hypersubstitution $\sigma_{nd,id}$, sending each operation symbol f_i, $i \in I$, to the singleton $\{f_i(x_1, \ldots, x_{n_i})\}$ is linear, the further consequence is the following theorem which was also proved in [4]:

Theorem 4.4. *The set of all nondeterministic linear hypersubstitutions forms a many-sorted submonoid of the many-sorted monoid of all nondeterministic hypersubstitutions.*

5. Deterministic and Nondeterministic Linear Hypersubstitutions

An important result in automata theory says that each finite nondeterministic automaton is equivalent to a deterministic one, i.e. it recognizes the same formal language [6]. In this section we will study the interconnection between deterministic and nondeterministic linear hypersubstitutions. Here let $(\mathrm{Hyp}^{nd}_{\mathrm{lin}}(\tau); \circ_{nd}, \sigma_{id,nd})$ be the one-sorted monoid of linear hypersubstitutions. By definition, for each $i \in I$, $\sigma_{nd}(f_i)$ is a set of n_i-ary linear terms. Therefore, for every $\sigma_{nd} \in \mathrm{Hyp}^{nd}_{\mathrm{lin}}$ we may consider the following set:

$$\sigma^*_{nd} := \{\rho : \rho \in \mathrm{Hyp}^{\mathrm{lin}}(\tau) \text{ and } \rho(f_i) \in \sigma_{nd}(f_i) \text{ for all } i \in I\}.$$

Here $\mathrm{Hyp}^{\mathrm{lin}}(\tau)$ denotes the set of all linear hypersubstitutions of type τ, i.e. the set of all mappings sending the operation symbols f_i to single linear terms preserving the arity. Let $(\mathrm{Hyp}^{nd}_{\mathrm{lin}}(\tau))^*$ be the set of all σ^*_{nd} with $\sigma_{nd} \in \mathrm{Hyp}^{nd}_{\mathrm{lin}}(\tau)$.

Then we have

Proposition 5.1. $\left(\mathrm{Hyp}^{nd}_{\mathrm{lin}}(\tau)\right)^* = \mathrm{Hyp}^{\mathrm{lin}}(\tau)$.

Proof. $(\mathrm{Hyp}^{nd}_{\mathrm{lin}}(\tau))^* \subseteq \mathrm{Hyp}^{\mathrm{lin}}(\tau)$ is clear. Let $\rho \in \mathrm{Hyp}^{\mathrm{lin}}(\tau)$ be a linear hypersubstitution of type τ. To show that $\rho \in (\mathrm{Hyp}^{nd}_{\mathrm{lin}}(\tau))^*$ we need a linear nd-hypersubstitution σ_{nd} such that $\rho(f_i) \in \sigma_{nd}(f_i)$ for all $i \in I$. We choose σ_{nd} with $\sigma_{nd}(f_i) = \{\rho(f_i)\}$ for all $i \in I$. Then $\rho(f_i) \in \sigma_{nd}(f_i)$ is satisfied. $\square$

For the extension of a single linear hypersubstitution ρ we use the same notation $\hat{\rho}$ as we used for the extension of a nondeterministic linear hypersubstitution. It is inductively in steps:

(i) $\hat{\rho}[x_j] = x_j$, $j \in \{1, \ldots, n, \ldots\}$,

(ii) $\hat{\rho}[f_i(t_1, \ldots, t_{n_i})] = S^n_m(\rho(f_i), \hat{\rho}[t_1], \ldots, \hat{\rho}[t_{n_i}])$, $i \in I$.

This extension sends the linear terms to linear terms. The connection between arbitrary deterministic and nondeterministic hypersubstitutions was considered in [7]. We will apply these results to nondeterministic linear hypersubstitutions. At first we recall the following definition from [5].

Definition 5.2. Each nondeterministic hypersubstitution σ_{nd} induces an extension mapping $\bar{\sigma}_{nd}$ by

(i) $\bar{\sigma}_{nd}[\{t\}] = \{\hat{\rho}[t] \mid \rho \in \sigma^*_{nd}\}$ (for $t = x_j$ the right-hand side consists only of x_j).

(ii) $\bar{\sigma}_{nd}[B] = \bigcup_{b \in B} \bar{\sigma}_{nd}[\{b\}], \bar{\sigma}[\varnothing] = \varnothing$.

As a consequence, if B is a set of linear terms, then $\bar{\sigma}[B]$ is a set of linear terms.

Using the new extension we define the binary operation

$$\diamond_{nd} : \mathrm{Hyp}^{nd}_{\mathrm{lin}}(\tau) \times \mathrm{Hyp}^{nd}_{\mathrm{lin}}(\tau) \to \mathrm{Hyp}^{nd}_{\mathrm{lin}}(\tau)$$

by $\sigma_{nd,1} \diamond_{nd} \sigma_{nd,2} := \bar{\sigma}_{nd,1} \circ \sigma_{nd,2}$.

The operation $\diamond_{nd}$ has the following properties (see [7]):

Proposition 5.3. *Let $\sigma_{1,nd}, \sigma_{2,nd} \in \mathrm{Hyp}^{nd}(\tau)$ and $B \subseteq W_\tau(X)$. Then*

(i) $(\sigma_{1,nd} \diamond_{nd} \sigma_{2,nd})(f_i) = \{(\rho_1 \circ_h \rho_2)(f_i) \mid \rho_1 \in \sigma^*_{1,nd},\ \rho_2 \in \sigma^*_{2,nd},\ i \in I\}$.

(ii) $\overline{(\sigma_{1,nd} \diamond_{nd} \sigma_{2,nd})}[B] = (\bar{\sigma}_{1,nd} \circ \bar{\sigma}_{2,nd})[B]$.

Proposition 5.1 holds also for submonoids.

Proposition 5.4. *Let $\mathscr{M}$ be a submonoid of $\big(\mathrm{Hyp}^{\mathrm{lin}}(\tau); \diamond_{nd}, \sigma_{id,nd}\big)$ or of $\big(\mathrm{Hyp}^{\mathrm{lin}}(\tau); \circ_{nd}, \sigma_{id,nd}\big)$. Then M^* forms a submonoid of $\big(\mathrm{Hyp}_{\mathrm{lin}}(\tau); \circ_h, \sigma_{id}\big)$.*

Proof. Since $\sigma_{id}(f_i) = f_i(x_1, \dots, x_{n_i}) \in \{f_i(x_1, \dots, x_{n_i})\} = \sigma_{id,nd}(f_i)$, we get $\sigma_{id} \in M^*$.

Let $\rho \in M^*$. Then there exists $\sigma_{nd} \in M \subseteq \mathrm{Hyp}^{nd}_{\mathrm{lin}}(\tau)$ such that $\rho(f_i) \in \sigma(f_i)$ for all $i \in I$, i.e. $\rho \in \mathrm{Hyp}_{\mathrm{lin}}(\tau)$ and thus $M^* \subseteq \mathrm{Hyp}_{\mathrm{lin}}(\tau)$.

Let now $\rho_1, \rho_2 \in M^*$. Then there exist $\sigma_{1,nd}, \sigma_{2,nd} \in M$ such that $\rho_1(f_i) \in \sigma_{1,nd}(f_i)$, $\rho_2(f_i) \in \sigma_{2,nd}(f_i)$ for all $i \in I$. By Proposition 5.3(i), $(\rho_1 \circ_h \rho_2)(f_i) \in (\sigma_{1,nd} \diamond_{nd} \sigma_{2,nd})$ for all $i \in I$. Since M is a submonoid of $\big(\mathrm{Hyp}^{nd}_{\mathrm{lin}}(\tau); \diamond_{nd}, \sigma_{id,nd}\big)$, we get $\sigma_{1,nd} \diamond_{nd} \sigma_{2,nd} \in M$ and so $\rho_1 \circ_h \rho_2 \in M^*$.

In the second case we use that for every term $t \in W_\tau(X)$ from $\rho(f_i) \in \sigma_{nd}(f_i)$ for all $i \in I$ and $\sigma \in \mathrm{Hyp}^{nd}_{\mathrm{lin}}(\tau)$, $\rho \in \mathrm{Hyp}_{\mathrm{lin}}(\tau)$ there follows $\hat{\rho}[t] \in \hat{\sigma}[\{t\}]$. Indeed, by Definition 5.2(i) $\hat{\rho}[t] \in \bar{\sigma}[\{t\}]$ and then by (iv) of Definition 4.1, $\hat{\rho}[t] \in \bar{\sigma}[\{t\}] \subseteq \hat{\sigma}[\{t\}]$.

Let $\rho_1, \rho_2 \in M^*$. Then there exist $\sigma_{1,nd}, \sigma_{2,nd} \in M$ with $\rho_1(f_i) \in \sigma_{1,nd}(f_i)$, $\rho_2(f_i) \in \sigma_{2,nd}(f_i)$ for all $i \in I$. Then

$$(\rho_1 \circ \rho_2)(f_i) = \hat{\rho}_1[\rho_2(f_i)] \in \hat{\sigma}_{1,nd}[\{\rho_2(f_i)\}] \subseteq \hat{\sigma}_{1,nd}[\sigma_{2,nd}(f_i)] = (\sigma_{1,nd} \circ_{nd} \sigma_{2,nd})(f_i)$$

since $\{\rho_2(f_i)\} \subseteq \sigma_{2,nd}(f_i)$ for all $i \in I$. Since M is a submonoid of $\big(\mathrm{Hyp}^{nd}_{\mathrm{lin}}(\tau); \circ_{nd}, \sigma_{id}\big)$, the product $\sigma_{1,nd} \circ_{nd} \sigma_{2,nd}$ belongs to M and thus $\rho_1 \circ \rho_2 \in M^*$. $\square$

We remark that this proof differs only slightly from the proof of Proposition 5.2 in [7].

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