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DIFFERENTIABILITY OF MAPPINGS OF THE SOBOLEV SPACE W^1_{n-1} WITH CONDITIONS ON THE DISTORTION FUNCTION

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Abstract: We define two scales of the mappings that depend on two real parameters p and q, with $n-1 \le q \le p < \infty$, as well as a weight function θ. The case $q = p = n$ and $\theta \equiv 1$ yields the well-known mappings with bounded distortion. The mappings of a two-index scale are applied to solve a series of problems of global analysis and applications. The main result of the article is the a.e. differentiability of mappings of two-index scales.

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Introduction

This article studies the mappings that depend on two real parameters p and q with $n-1 \le q \le p < \infty$, as well as a weight function θ. The foundations for this theory are laid in [1]. The case $q = p = n$ and $\theta \equiv 1$ yields the well-known mappings with bounded distortion, whose study Reshetnyak had initiated in the 1960s; see [2]. The theory of mappings with bounded distortion can be regarded as a natural generalization of the theory of analytic functions on the Euclidean space of arbitrary dimension $n \ge 2$. Consider a domain $\Omega \subset \mathbb{R}^n$, meaning a connected open set, in the Euclidean space $\mathbb{R}^n$, with $n \ge 2$. An element $f : \Omega \to \mathbb{R}^n$ of the Sobolev space $W^1_{n,\mathrm{loc}}(\Omega)$ is called a *mapping with bounded distortion* whenever

$$|Df(x)|^n \le KJ(x, f) \quad \text{for almost all } x \in \Omega, \tag{1}$$

where $K \in [1, \infty)$ is a constant, $Df(x) = \bigl(\frac{\partial f_i}{\partial x_j}\bigr)_{i,j=1,\dots,n}$ is the Jacobi matrix, and $J(x, f) = \det Df(x)$. Reshetnyak established the main topological properties of these mappings by proving that every nonconstant mapping with bounded distortion is continuous, open, and discrete [2, Chapter II, § 6]. It is known that every homeomorphism with bounded distortion is quasiconformal.

In accordance with Corollary 30 of this article, every continuous, open, and discrete mapping $f : \Omega \to \mathbb{R}^n$ of the Sobolev space $W^1_{n-1,\mathrm{loc}}(\Omega)$ satisfying the condition

$$|\operatorname{adj} Df(x)|^{\frac{n}{n-1}} \le K'J(x, f) \quad \text{for almost all } x \in \Omega$$

is a mapping with bounded distortion; i.e., f also satisfies (1); here $\operatorname{adj} A$ is the adjoint matrix, while $K' \in [1, \infty)$ is a constant. We include the mappings of $W^1_{n-1,\mathrm{loc}}(\Omega)$ into the two two-index scales of mappings; see Definitions 1 and 3 below, depending on the two real parameters $n-1 \le q \le p < \infty$. The goal of this article is to show that the mappings of two-index scales are a.e. differentiable and inherit many properties of mappings with bounded distortion.

The membership in the Sobolev space $W^1_{n-1,\mathrm{loc}}(\Omega)$ immediately deprives the mapping of quite a few properties familiar in quasiconformal analysis: differentiability, Luzin's $\mathscr{N}$-property, Luzin's $\mathscr{N}^{-1}$-property, and many others. Thus, the proofs of the main statements of this article are new.

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The objects we propose to study are motivated in [1]: This class of mappings arises naturally in the pullback problem for differential forms of codegree 1 and has applications in nonlinear elasticity.

Section 1 defines the two-index scales of mappings and presents some of their properties. In Subsection 1.1 we establish the properties of the generalized Poletsky function. The construction of its new proof is then applied in the proof of the properties of pulled-back functions and capacity inequalities; see Subsections 1.2 and 1.3. Section 2 establishes the a.e. differentiability of mappings of the classes under study subject to the restriction $n-1<q\le p<n+\frac{1}{n-2}$ imposed on the parameters. Section 3 justifies Liouville-type Theorems, while in Section 4 we apply the mappings under study to classify Riemannian manifolds. The main tool is the capacity inequality of Subsection 1.2.

1. Definition of Two-Index Scales of Mappings and the Properties of the Mappings

Recall some concepts of use in this article. Given $k\ge 0$, $\delta\in(0,\infty]$, and $A\subset\mathbb{R}^n$, put

$$\mathbb{H}^k_\delta(A)=\frac{\omega_k}{2^k}\inf\Big\{\sum_{i\in\mathbb{N}}(\operatorname{diam}S_i)^k:\operatorname{diam}S_i<\delta, A\subset\bigcup_{i\in\mathbb{N}}S_i\Big\},$$

where $\omega_k=\frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)}$ and the infimum is taken over all countable coverings $\{S_i\}$ of A. If A cannot be covered by a countable collection of sets of this form then put $\mathbb{H}^k_\delta(A)=\infty$. The limit $\mathbb{H}^k(A)=\lim_{\delta\to0}\mathbb{H}^k_\delta(A)$ exists and is called the *k-dimensional Hausdorff measure* of A. In $\mathbb{R}^n$ the n-dimensional Hausdorff measure of $A\subset\mathbb{R}^n$ coincides with the n-dimensional Lebesgue measure.

A mapping $f:\Omega\to\mathbb{R}^n$ is called *approximately differentiable* at $x\in\Omega$ whenever there exists a linear mapping $L:\mathbb{R}^n\to\mathbb{R}^n$ such that

$$\lim_{r\to0}\frac{\mathbb{H}^n(\{y\in B(x,r):|f(y)-f(x)-L(y-x)|>\varepsilon\})}{r^n}=0$$

for every $\varepsilon>0$.

It is known [3; 4, Theorem 12.2; 5; 6, Theorem 3.1.4] that a mapping $f:\Omega\to\mathbb{R}^n$ is approximately differentiable a.e. in Ω if and only if f has an approximate partial derivative a.e. in Ω.

Assume that the Sobolev function space $W^1_p(\Omega)$ is defined on a domain $\Omega\subset\mathbb{R}^n$, with $p\in[1,\infty]$. Say that $f=(f_1,f_2,\dots,f_n):\Omega\to\mathbb{R}^n$ is in the *Sobolev space* $W^1_p(\Omega)$, with $p\in[1,\infty]$, whenever all coordinates f_i lie in $W^1_p(\Omega)$.

Every $f:\Omega\to\mathbb{R}^n$ in $W^1_1(\Omega)$ has partial derivatives a.e. in Ω; consequently, f is approximately differentiable a.e. in Ω.

Given an $n\times n$-matrix A, let $\operatorname{adj}A$ stand for the adjoint matrix defined by the condition $A\operatorname{adj}A=I\det A$ whenever the determinant of A is nonzero, and extended by continuity in the topology of $\mathbb{R}^{n\times n}$ otherwise.

Nonnegative measurable functions $\theta,\omega:\mathbb{R}^n\to(0,\infty)$ are called *weights.*

The following scale of mappings is defined in [1, 7].

Definition 1. A mapping $f:\Omega\to\mathbb{R}^n$ belongs to $\mathscr{ID}(\Omega;q,p;\theta,1)$, where $n-1\le q\le p<\infty$, and is called a *mapping with bounded θ-weighted interior (q,p)-distortion*, whenever

(1) f is continuous, open, and discrete;

(2) f is in $W^1_{n-1,\mathrm{loc}}(\Omega)$;

(3) the Jacobian $J(x,f)$ is nonnegative for almost all $x\in\Omega$;

(4) f has finite codistortion: $\operatorname{adj}Df(x)=0$ a.e. on $Z=\{x\in\Omega:\det Df(x)=0\}$;

(5) the function of the local θ-weighted (q,p)-distortion

$$\Omega\ni x\mapsto\mathscr{K}^{\theta,1}_{q,p}(x,f)=\begin{cases}\dfrac{\theta^{\frac{n-1}{q}}(x)|\operatorname{adj}Df(x)|}{J(x,f)^{\frac{n-1}{p}}} & \text{if } J(x,f)\ne0,\\ 0 & \text{otherwise}\end{cases}$$

is in $L_\varrho(\Omega)$, where ϱ is found from the condition $\frac{1}{\varrho}=\frac{n-1}{q}-\frac{n-1}{p}$, and $\varrho=\infty$ for $q=p$.

Put $\mathscr{K}_{q,p}^{\theta,1}(f;\Omega) = \left\|\mathscr{K}_{q,p}^{\theta,1}(\cdot,f) \mid L_{\varrho}(\Omega)\right\|$.

REMARK 2. In this article we prove that $\mathscr{I}\mathscr{D}(\Omega;n,n;1,1)$ coincides with the space of mappings with bounded distortion (1). Therefore, it is hardly surprising that the mappings in $\mathscr{I}\mathscr{D}(\Omega;q,p;\theta,1)$ inherit many properties of the mappings with bounded distortion [2, 8, 9].

The definition of two-index scale, given below, differs from that above and rests on the behavior of exterior distortion coefficient.

The following space of mappings is defined and studied in [1, 10][1)]

DEFINITION 3. A mapping $f:\Omega\to\mathbb{R}^n$ is in $\mathscr{O}\mathscr{D}(\Omega;q,p;\theta,1)$, where $n-1\le q\le p<\infty$, or is called a *mapping with bounded θ-weighted exterior (q, p)-distortion*, whenever

(1) f is continuous, open, and discrete;

(2) f is in $W^1_{n-1,\mathrm{loc}}(\Omega)$;

(3) the Jacobian $J(x,f)$ is nonnegative for almost all $x\in\Omega$;

(4) f has finite distortion: $Df(x)=0$ a.e. on the zero set $Z=\{x\in\Omega:\det Df(x)=0\}$ of the Jacobian;

(5) the function of the local $(\theta,1)$-weighted (q,p)-distortion

$$\Omega\ni x\mapsto K_{q,p}^{\theta,1}(x,f)=\begin{cases}\dfrac{\theta^{\frac{1}{q}}(x)|Df(x)|}{J(x,f)^{\frac{1}{p}}} & \text{if } J(x,f)\ne 0,\\ 0 & \text{otherwise}\end{cases}$$

is in $L_{\varkappa}(\Omega)$, where $\varkappa$ can be found from the condition $\frac{1}{\varkappa}=\frac{1}{q}-\frac{1}{p}$, and $\varkappa=\infty$ for $q=p$. Put $K_{q,p}^{\theta,1}(f;\Omega)=\left\|K_{q,p}^{\theta,1}(\cdot,f)\mid L_{\varkappa}(\Omega)\right\|$.

The connection of mappings in $\mathscr{I}\mathscr{D}(\Omega;q,p;\theta,1)$ for $n-1\le q\le p<\infty$ to the space of degree $n-1$ forms was studied in [1, 11], and the connection of mappings in $\mathscr{O}\mathscr{D}(\Omega;q,p;\theta,1)$ to the Sobolev spaces, in [12–14].

In Theorem 8 below we will prove that

$$\mathscr{O}\mathscr{D}(\Omega;q,p;\theta,1)\subset\mathscr{I}\mathscr{D}(\Omega;q,p;\theta,1) \tag{2}$$

provided that $n-1<q\le p<\infty$.

If $x=\sum_{l=1}^n x_l e_l$, where $\{e_l\}$ is the standard basis for $\mathbb{R}^n$; then Pr_i is the projection along the ith axis: $\mathrm{Pr}_i(x)=x-x_ie_i$, while pr_j is a projection to the jth coordinate axis: $\mathrm{pr}_j(x)=x_j$ for $x\in\mathbb{R}^n$.

A continuous mapping $f:\Omega\to\Omega'$ enjoys *Luzin's property on hypersurfaces*, briefly expressed as $f\in\mathrm{LPHP}$, whenever for $\mathscr{H}^1$-almost all $x_j\in\mathrm{pr}_j(\Omega)$ the mapping $\mathrm{Pr}_i\circ f:\mathrm{pr}_j^{-1}(x_j)\cap\Omega\to\mathbb{R}^{n-1}$ enjoys Luzin's $\mathscr{N}$-property with respect to the $(n-1)$-dimensional Hausdorff measure; i.e., $\mathscr{H}^{n-1}(\mathrm{Pr}_i\circ f(A))=0$ for every $A\subset\mathrm{pr}_j^{-1}(x_j)\cap\Omega$ with $\mathscr{H}^{n-1}(A)=0$ for $i,j=1,\dots,n$.

Not every continuous mapping in $W^1_{n-1,\mathrm{loc}}(\Omega)$ enjoys Luzin's property on hypersurfaces, but this holds whenever $f\in W^1_{n-1,\mathrm{loc}}(\Omega)$ is a homeomorphism [15] or a continuous, open, and discrete mapping [16, Proposition 3.3].

REMARK 4. Condition 2 of the above definition entails the properties of the mapping f, see [1, Proposition 2]:

(1) f is approximately differentiable a.e.;

(2) $\operatorname{adj} Df\in L_{1,\mathrm{loc}}$;

(3) $f\in\mathrm{LPHP}$.

These properties appear in the hypotheses of Theorem 5.

Denote by Z the zero set $\{x\in\Omega:\det Df(x)=0\}$ of the Jacobian of a mapping in a Sobolev space.

[1)] The definition of scale of mappings in [1] differs from that of [10] in Subsection 2: here we require only that $f\in W^1_{n-1,\mathrm{loc}}(\Omega)$ instead of $f\in W^1_{q,\mathrm{loc}}(\Omega)$ for $n-1<q$ in [10]. Nevertheless, the methods of [1] enable us to reprove all results of [10] even under the so-weakened assumption.

Theorem 5 [17, Theorem 2]. *Suppose that a homeomorphism $f:\Omega\to\Omega'$ enjoys the properties:*

(1) *f is approximately differentiable a.e.;*

(2) $\operatorname{adj} Df\in L_{1,\mathrm{loc}}$*;*

(3) $f\in \mathrm{LPHP}$*;*

(4) *f has finite codistortion:* $\operatorname{adj} Df(x)=0$ *a.e. on* $Z=\{x\in\Omega:\det Df(x)=0\}$.

Then

(5) *the inverse homeomorphism $f^{-1}:\Omega'\to\Omega$ is in $W^1_{1,\mathrm{loc}}(\Omega')$;*

(6) *f^{-1} has finite distortion: $Df^{-1}(y)=0$ a.e. on the zero set $Z'=\{y\in\Omega':\det Df^{-1}(y)=0\}$ of the Jacobian of f^{-1}.*

Many estimates below rely on Theorem 6. To state the latter, we introduce the distortion function: Given two numbers $1\le s\le r<\infty$ and a continuous mapping $\psi:\Omega'\to\mathbb{R}^n$ in $W^1_{1,\mathrm{loc}}(\Omega')$ with $\Omega'\subset\mathbb{R}^n$, define the (exterior) operator distortion function:

$$\Omega'\ni y\mapsto K^{1,\omega}_{s,r}(y,\psi)=\begin{cases}\frac{|D\psi(y)|}{\omega^{\frac1r}(\psi(y))|J(y,\psi)|^{\frac1r}} & \text{for } y\in\Omega'\setminus Z',\\ 0 & \text{otherwise.}\end{cases}$$

Theorem 6. *Suppose that*

(1) *a homeomorphism $f:\Omega\to\Omega'$ is in $\mathscr{OD}(\Omega;q,p;\theta,1)$, where $n-1<q\le p<\infty$,*

(2) *the weight function $\omega(x)=\theta^{-\frac{n-1}{q-(n-1)}}(x)$ is locally summable.*

Then

(3) *The inverse homeomorphism f^{-1} belongs to $W^1_{s,\mathrm{loc}}(\Omega')$, where $s=\frac{p}{p-n+1}$;*

(4) *f^{-1} has finite distortion: $Df^{-1}(y)=0$ a.e. on $Z'=\{y\in\Omega:\det Df^{-1}(y)=0\}$;*

(5) *$K^{1,\omega}_{s,r}(\cdot,f^{-1})\in L_\varrho(\Omega')$, where $r=\frac{q}{q-n+1}$, $s=\frac{p}{p-n+1}$, and $\frac1\varrho=\frac1s-\frac1r=\frac{n-1}{p}-\frac{n-1}{q}=\frac{n-1}{\varkappa}$;*

(6) *the homeomorphism f^{-1} induces via the change of variables the bounded operator*

$$f^{-1*}:L^1_r(\Omega;\omega)\cap W^1_{\infty,\mathrm{loc}}\to L^1_s(\Omega').$$

Moreover,

$$\beta_{q,p}\big\|K^{1,\omega}_{s,r}(\cdot,f^{-1})\mid L_\varrho(\Omega')\big\|\le\|f^{-1*}\|\le\big\|K^{1,\omega}_{s,r}(\cdot,f^{-1})\mid L_\varrho(\Omega')\big\|,\tag{3}$$

where $\beta_{q,p}$ is some constant, and

$$\big\|K^{1,\omega}_{s,r}(\cdot,f^{-1})\mid L_\varrho(\Omega')\big\|\le\big\|K^{\theta,1}_{q,p}(\cdot,f)\mid L_\varkappa(\Omega)\big\|^{n-1}.\tag{4}$$

The right-hand side of (3) is established using the change-of-variable formula [6, 18]: *For every nonnegative measurable function $u:A\to\mathbb{R}$, where A is a measurable set in $\mathbb{R}^n$, we have*

$$\int\limits_A u(x)|J(x,\varphi)|\,dx=\int\limits_{\mathbb{R}^n}\Big(\sum_{x\in\varphi^{-1}(y)\cap(A\setminus\Sigma)}u(x)\Big)\,dy,\tag{5}$$

where $\varphi:A\to\mathbb{R}^n$ is an approximately differentiable mapping, $\Sigma\subset A$ is the singular set of φ; i.e., the set of measure zero outside which φ enjoys Luzin's $\mathscr N$-property, while $J(x,\varphi)=\det D\varphi(x)$ is the Jacobian of φ. If φ enjoys Luzin's $\mathscr N$-property then $\Sigma=\varnothing$. We can always assume that $\Sigma\cap Z=\varnothing$.

Proof of Theorem 6 rests on the methods of [17, Theorem 4; 10, 19]. If hypotheses 1 and 2 are met then by Remark 4 and Theorem 5 the inverse homeomorphism f^{-1} belongs to $W^1_{1,\mathrm{loc}}(\Omega')$, and claim 4 holds: f^{-1} has finite distortion; i.e., $Df^{-1}(y)=0$ a.e. on Z'. Let us now establish the inequality in (4).

Observe first of all that $\varkappa = \varrho(n-1)$. Condition 2 of the theorem implies that $\operatorname{adj} Df(x) = 0$ a.e. on Z. Moreover, the inequality $|\operatorname{adj} Df(x)| \leq |Df(x)|^{n-1}$ yields

$$\big\|K^{1,\omega}_{s,r}(\cdot, f^{-1}) \mid L_\varrho(\Omega')\big\|^\varrho = \int\limits_{\Omega'\setminus Z'} \omega^{-\frac{\varrho}{r}}(f^{-1}(y))\bigg(\frac{|Df^{-1}|(y)}{|J(y,f^{-1})|^{\frac{1}{r}}}\bigg)^\varrho dy$$

$$= \int\limits_{\Omega\setminus Z} \omega^{-\frac{\varrho}{r}}(x)\bigg(\frac{|J(x,f)|^{\frac{1}{r}}|\operatorname{adj} Df|(x)}{|J(x,f)|}\bigg)^\varrho |J(x,f)|\,dx$$

$$\leq \int\limits_{\Omega\setminus Z} \bigg(\frac{\theta^{\frac{n-1}{q}}(x)|Df|^{n-1}(x)}{|J(x,f)|^{\frac{n-1}{p}}}\bigg)^\varrho dx = \big\|K^{\theta,1}_{q,p}(\cdot, f) \mid L_\varkappa(\Omega)\big\|^{(n-1)\varrho}. \tag{6}$$

Thus, (4) is justified.

Let us show that every $u \in L^1_r(\Omega;\omega) \cap W^1_{\infty,\mathrm{loc}}$ satisfies

$$\big\|f^{-1*}u \mid L^1_s(\Omega')\big\| \leq \big\|K^{1,\omega}_{s,r}(\cdot, f^{-1}) \mid L_\varrho(\Omega')\big\| \cdot \big\|f \mid L^1_r(\Omega;\omega)\big\|.$$

Since $u \circ f^{-1}$ is in $\mathrm{ACL}(\Omega')$, we see that

$$\big\|f^{-1*}u \mid L^1_s(\Omega')\big\| \leq \bigg(\int\limits_{\Omega'\setminus Z'} (|\nabla u|(f^{-1}(y))|Df^{-1}|(y))^s\,dy\bigg)^{\frac{1}{s}}$$

$$\leq \bigg(\int\limits_{\Omega'\setminus Z'} |\nabla u|^s(f^{-1}(y))\omega(f^{-1}(y))^{\frac{s}{r}}|J(y,f^{-1})|^{\frac{s}{r}}\omega(f^{-1}(y))^{-\frac{s}{r}}\frac{|Df^{-1}|^s(y)}{|J(y,f^{-1})|^{\frac{s}{r}}}\,dy\bigg)^{\frac{1}{s}}.$$

Applying Hölder's inequality for $s < r$ and then changing the variable in the first factor, we deduce the estimate

$$\big\|f^{-1*}u \mid L^1_s(\Omega')\big\| \leq \bigg(\int\limits_{\Omega'} \big(K^{1,\omega}_{s,r}(\cdot, f^{-1})\big)^\rho\,dy\bigg)^{1/\rho}\bigg(\int\limits_{\Omega} |\nabla u|^r(x)\omega(y)\,dy\bigg)^{1/r},$$

while for $q = p$ the left factor equals $\|K^{1,\omega}_{s,r}(\cdot, f^{-1}) \mid L_\infty(\Omega')\|$. Consequently, the right-hand side of (3) is justified.

The left estimate in (3) is proved in [10, Theorem 1], see also [19, Theorem 1], and is covered by our next statement. $\square$

Proposition 7. *Suppose that the weight function $\omega : \Omega \to (0,\infty)$ is locally summable. Consider a homeomorphism $\varphi : \Omega \to \Omega'$ with the property:*

(1) the inverse homeomorphism φ^{-1} induces via the change of variables the bounded operator $\varphi^{-1*} : L^1_r(\Omega;\omega) \cap W^1_{\infty,\mathrm{loc}} \to L^1_s(\Omega')$, where $n-1 < s \leq r < \infty$.

Then

(2) $\varphi^{-1} \in W^1_{s,\mathrm{loc}}(\Omega')$;

(3) φ^{-1} has finite distortion: $D\varphi^{-1}(y) = 0$ a.e. on Z';

(4) $K^{1,\omega}_{s,r}(\cdot, \varphi^{-1}) \in L_\varrho(\Omega')$, where $\frac{1}{\varrho} = \frac{1}{s} - \frac{1}{r}$ and $\varrho = \infty$ if $s = r$, and the norm of φ^{-1} satisfies (3).

Theorem 8. *If $f : \Omega \to \Omega'$ is in $\mathscr{OD}(\Omega; q, p; \theta, 1)$, where $n-1 < q \leq p < \infty$, then f also belongs to $\mathscr{ID}(\Omega; q, p; \theta, 1)$. Moreover,*

$$\big\|\mathscr{K}^{\theta,1}_{q,p}(\cdot, f) \mid L_\varrho(\Omega)\big\| \leq \big\|K^{\theta,1}_{q,p}(\cdot, f) \mid L_\varkappa(\Omega)\big\|^{n-1}. \tag{7}$$

PROOF. The first three conditions in Definitions 1 and 3 coincide. If a mapping has finite distortion then it also has finite codistortion.

It remains to prove that

$$\mathscr{K}_{q,p}^{\theta,1}(f;\Omega) \leq K_{q,p}^{\theta,1}(f;\Omega)^{n-1}.$$

On the one hand, we have (4),

$$\left\|K_{s,r}^{1,\omega}(\cdot,f^{-1}) \mid L_{\varrho}(\Omega')\right\| \leq \left\|K_{q,p}^{\theta,1}(\cdot,f) \mid L_{\varkappa}(\Omega)\right\|^{n-1},$$

and on the other,

$$\left\|\mathscr{K}_{q,p}^{\theta,1}(\cdot,f) \mid L_{\varrho}(\Omega)\right\| = \left\|K_{s,r}^{1,\omega}(\cdot,f^{-1}) \mid L_{\varrho}(\Omega')\right\|.$$

The last relation was proved (see (13) in [1]). Note that the proof of (4) as well as the proof of (13) of [1] does not use the local summability of $\omega(x)$. These two relations yield (7). $\square$

REMARK 9. Theorem 8 justifies (2). The conclusion of Theorem 8 is that every statement proved for the mappings in $\mathscr{I}\mathscr{D}(\Omega;q,p;\theta,1)$ also holds for mappings in $\mathscr{O}\mathscr{D}(\Omega;q,p;\theta,1)$ with the adjustment: The term $\left\|\mathscr{K}_{q,p}^{\theta,1}(\cdot,f) \mid L_{\varrho}(\Omega)\right\|$ in an (in)equality should be replaced with $\left\|K_{q,p}^{\theta,1}(\cdot,f) \mid L_{\varkappa}(\Omega)\right\|^{n-1}$, while the equality should be replaced with inequality in accordance with (7). This implies in particular that all statements of [1] hold for the mappings in $\mathscr{O}\mathscr{D}(\Omega;q,p;\theta,1)$ with the indicated change in the (in)equalities; see the claims of Subsections 1.1 and 1.2 for instance.

1.1. Properties of the generalized Poletsky function. Take a continuous mapping $f:\Omega\to\mathbb{R}^n$ and a domain D compactly embedded into Ω, meaning that D is bounded and $\overline{D}\subset\Omega$, written briefly as $D\Subset\Omega$, and take $y\notin f(\partial D)$. Denote by $\mu(y,f,D)$ the *degree* of f at y with respect to D. Say that f is *sense-preserving* whenever $\mu(y,f,D)>0$ for all domains $D\Subset\Omega$ and all points $y\in f(D)\setminus f(\partial D)$. For $A\subset\Omega$ refer as the *multiplicity function* to $\mathbb{R}^n\ni y\mapsto N(y,f,A)=\#\{f^{-1}(y)\cap A\}$. Moreover, put $N(f,A)=\sup_{y\in\mathbb{R}^n}N(y,f,A)$.

Suppose that $f:\Omega\to\mathbb{R}^n$ is a continuous, open, and discrete mapping. A domain $D\Subset\Omega$ is called *normal* whenever $f(\partial D)=\partial f(D)$. A *normal neighborhood* of $x\in\Omega$ is a normal domain $U\subset\Omega$ such that $\overline{U}\cap f^{-1}(f(x))=\{x\}$. The quantity $i(x,f)=\mu(f(x),f,U)$ is independent of the choice of a normal neighborhood U of x (see [2, Chapter II, § 2; 8, part 1] for instance) and is called the *local index* of f at x. A point $x\in\Omega$ is called a *branch point* of f whenever f is not a homeomorphism of any neighborhood of x. Denote the collection of all branch points of f by B_f.

In the following two lemmas we state propositions of interest in their own right. The first of them is applied both in the proof of the main result of this subsection, which is Theorem 13, and in the proof of Lemma 11, so that we can use the sign of the approximative Jacobian to draw the conclusion on sense-preservation.

Lemma 10. *Assume that $f:\Omega\to\mathbb{R}^n$ is a continuous, open, and discrete mapping in $\mathscr{O}\mathscr{D}(\Omega;q,p;\theta,1)$, where $n-1\leq q\leq p<\infty$. Then for every open connected set $U\subset\Omega$ the set $\{x\in U\setminus B_f : J(x,f)\neq 0\}$ has positive measure.*

PROOF. If, on the contrary, $J(x,f)=0$ a.e. on a connected set $U\subset\Omega\setminus B_f$ on which f is a homeomorphism then $Df(x)=0$ a.e. on U because f has finite distortion. Then f is constant on U, and consequently, f cannot be open. $\square$

Lemma 11. *If $f:\Omega\to\mathbb{R}^n$ is a continuous, open, and discrete mapping in $\mathscr{O}\mathscr{D}(\Omega;q,p;\theta,1)$, where $n-1<q\leq p<\infty$, then f is differentiable a.e. on $\Omega\setminus(B_f\cup\{x : J(x,f)=0\})$ and sense-preserving.*

PROOF OF LEMMA 11 repeats that of Lemma 16 of [1] almost verbatim.

DEFINITION 12. For a sense-preserving, continuous, open, and discrete mapping $f:\Omega\to\mathbb{R}^n$ and a normal domain $D\Subset\Omega$, define the Poletsky function $g_D:V\to\mathbb{R}^n$ on $V=f(D)$ by putting

$$V\ni y\mapsto g_D(y)=\Lambda\sum_{x\in f^{-1}(y)\cap D} i(x,f)x, \tag{8}$$

where $\Lambda\in(0,\infty)$ will be fixed below.

Poletsky introduced in [20] a function of the form (8) for mappings with bounded distortion. The next statement presents the properties of the Poletsky function for the classes of mappings under consideration.

Theorem 13 [1, 21]. *Suppose that $f : \Omega \to \mathbb{R}^n$ belongs to $\mathscr{ID}(\Omega; q, p; \theta, 1)$ or $\mathscr{OD}(\Omega; q, p; \theta, 1)$. Then*

(1) *the function g_D defined in* (8) *is continuous and belongs to* $\mathrm{ACL}(V)$*;*

(2) *$Dg_D(y) = 0$ a.e. on $Z' \cup \Sigma'$;*

(3) *if the weight function $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$ is locally summable then g_D defined in* (8) *is in $W^1_s(V)$, where $s = \frac{p}{p-(n-1)}$; furthermore,*

$$\|Dg_D \mid L_s(V)\| \le \Lambda N(f, D)^{\frac{s-1}{s}} \left\|\mathscr{K}^{\theta,1}_{q,p}(\cdot, f) \mid L_\varrho(D)\right\| \Bigg(\int\limits_D \omega(x)\,dx\Bigg)^{\frac{1}{r}} \tag{9}$$

for $f \in \mathscr{ID}(\Omega; q, p; \theta, 1)$ and

$$\|Dg_D \mid L_s(V)\| \le \Lambda N(f, D)^{\frac{s-1}{s}} \left\|K^{\theta,1}_{q,p}(\cdot; f) \mid L_\varkappa(D)\right\|^{n-1} \Bigg(\int\limits_D \omega(x)\,dx\Bigg)^{\frac{1}{r}} \tag{10}$$

for $f \in \mathscr{OD}(\Omega; q, p; \theta, 1)$.

Proof. As Remark 19 of [1] points out, *if a continuous, open, and discrete mapping $f : \Omega \to \mathbb{R}^n$ is sense-preserving and satisfies only the conditions:*

(4) *f is approximately differentiable a.e. on Ω;*

(5) *f has finite codistortion, see Definition* (1)*;*

(6) $\operatorname{adj} Df \in L_1(D)$ *for the normal domain $D \Subset \Omega$ in the definition of Poletsky function;*

(7) *f enjoys Luzin's property on hypersurfaces* ($f \in \mathrm{LPHP}$).

Then claims 1 *and* 2 *of Theorem* 13 *hold for the Poletsky function g_D.*

Theorem 14 of [1] establishes (9). It remains to prove (10), and for that we use the notation of the theorem.

Given a continuous, open, and discrete mapping $f : \Omega \to \mathbb{R}^n$ and $x \in \Omega$, denote by $U(x, f, r)$ the connected component of $f^{-1}(B(f(x), r))$ that contains x.

Fix an arbitrary normal domain $D \Subset \Omega$ and $y \in V = f(D)$. Since f is continuous and discrete, while D is a normal domain; it follows that $f^{-1}(y) \cap D = \{q_1, \dots, q_s\}$ is finite. According to Lemmas 21 and 22 of [1], there exists a positive number r_0 with $0 < r_0 < \operatorname{loc}(y, V)$ such that the normal neighborhoods $U(q_i, f, r_0)$, for $i = 1, \dots, s$, are disjoint. Choose $r_1 \le r_0$ so that $B(y, r_1) \cap f\big(D \setminus \bigcup_{i=1}^s U(q_i, f, r_0)\big) = \varnothing$. Then the connected components of $f^{-1}(B(y, r_1))$ intersecting D are $U(q_i, f, r_1)$ for $i = 1, \dots, s$. Put

$$U = \bigcup_{i=1}^s U(q_i, f, r_1) = \bigcup_{i=1}^s U_i \quad \text{where } U_i = U(q_i, f, r_1).$$

By analogy with Theorem 3.1.8 of [6], take a Borel set $\Sigma \subset U$ of zero $\mathbb{H}^n$-measure outside which f enjoys Luzin's $\mathscr{N}$-property. Put $Z = \{x \in U \setminus \Sigma : J(x, f) = 0\}$, which may also be assumed a Borel set.

Subdivide the complement $U \setminus \Sigma$ into a countable collection of disjoint measurable sets F_k, for $k \in \mathbb{N}$, such that $\bigcup_{k\in\mathbb{N}} F_k = U \setminus \Sigma$ and $f : F_k \to \mathbb{R}^n$ is Lipschitz for all $k \in \mathbb{N}$ [7, Theorem 3.1.8]. Each set $F_k \setminus Z$ becomes the union of a countable collection of disjoint measurable sets F_{km}, for $m \in \mathbb{N}$, such that $f|_{F_{km}} : F_{km} \to \mathbb{R}^n$ is bi-Lipschitz [6, Lemma 3.2.2]. By Lebesgue's Theorem, we may assume that F_{km} consists only of points of density 1, while by Rademacher's Theorem, all points of F_{km} are differentiability points of $f|_{F_{km}} : F_{km} \to \mathbb{R}^n$. Denote $\{F_{km}\}_{k,m\in\mathbb{N}}$ by $\{E_l\}_{l\in\mathbb{N}}$. This yields the subdivision $U = \Sigma \cup Z \cup \bigcup_{l\in\mathbb{N}} E_l$ into disjoint subsets. It corresponds to the decomposition of the image $f(U) = Z' \cup \Sigma' \cup \bigcup_{l\in\mathbb{N}} E'_l$, where $Z' = f(\Sigma)$, $\Sigma' = f(Z)$, and $E'_l = f(E_l)$ are not necessarily disjoint. Denote by $\psi_l : E'_l \to E_l$ the inverse mapping to $f : E_l \to E'_l$.

Fix an arbitrary cube $Q = Q(y,r) \subset B(y,r_1) \subset f(D)$ with $r \in (0, M(y))$, where $M(y) > 0$ is an arbitrary number satisfying $Q(y, M(y)) \subset B(y,r_1)$. The points $y \in Q\backslash(Z'\cup\Sigma')$ have the representation

$$g_D(y) = \Lambda \sum_{x\in f^{-1}(y)\cap D} i(x,f)x = \Lambda \sum_{i=1}^{s}\sum_{p=1}^{k(i)} \alpha^z_{i,p} \circ \beta^{-1}(y) = \Lambda \sum_{l\in\mathbb{N}} \psi'_l(y). \tag{11}$$

Observe that at each $y \in Q \setminus (Z' \cup \Sigma')$ at most $N(f,D)$ terms are defined in the rightmost sum. Thus, for $y \in Q \setminus (Z' \cup \Sigma')$ we obtain

$$|Dg_D(y)|^s \le \Lambda^s \Big|\sum_{l\in\mathbb{N}} D\psi'_l(y)\Big|^s \le \Lambda^s N(f,D)^{s-1} \sum_{l\in\mathbb{N}} |D\psi'_l(y)|^s.$$

Basing on that, by analogy with (6) and appreciating (4), for $p = q$ we have

$$\begin{aligned}
\big\||Dg_D| \mid L^1_s(Q)\big\|^s &= \int\limits_{Q\backslash(Z'\cup\Sigma')} |Dg_D(y)|^s\,dy \\
&\le \Lambda^s N(f,D)^{s-1} \int\limits_{Q\backslash(Z'\cup\Sigma')} \sum_{l\in\mathbb{N}} |D\psi'_l|^s(y)\,dy \\
&\le \Lambda^s N(f,D)^{s-1} \sum_{l\in\mathbb{N}} \int\limits_{E'_l\cap Q} |D\psi'_l|^s(y)\,dy \\
&\le \Lambda^s N(f,D)^{s-1} \sum_{l\in\mathbb{N}} \big\|K^{\theta,1}_{p,p}(\cdot,f) \mid L_\infty(E_l \cap f^{-1}(Q))\big\|^{(n-1)s} \int\limits_{E_l\cap f^{-1}(Q)} \omega(x)\,dx \\
&\le \Lambda^s N(f,D)^{s-1} \big\|K^{\theta,1}_{p,p}(\cdot,f) \mid L_\infty(D)\big\|^{(n-1)s} \int\limits_{f^{-1}(Q)} \omega(x)\,dx.
\end{aligned}$$

Since the family of cubes $Q(y,r) \subset f(D)$, for $r \in (0, M(y))$, constitutes a Vitali covering of $f(D)$, there exists a countable family of disjoint cubes $Q_k \subset f(D)$ covering $f(D)$ up to a set of measure zero. Applying the above estimates for $\big\||Dg_D| \mid L^1_s(Q)\big\|^s$ to each cube Q_k and adding up the resulting inequalities over $k \in \mathbb{N}$, we obtain (10).

If $q < p$ then, applying Hölder's inequality twice and using the argument of Theorem 6, we successively deduce

$$\begin{aligned}
\Big(\int\limits_Q |Dg_D|^s(y)\,dy\Big)^{1/s} &\le \Lambda N(f,D)^{\frac{s-1}{s}} \Big(\sum_{l\in\mathbb{N}} \int\limits_{E'_l\cap Q} |D\psi'_l|^s(y)\,dy\Big)^{1/s} \\
&\le \Lambda N(f,D)^{\frac{s-1}{s}} \Bigg[\sum_{l\in\mathbb{N}} \Big(\int\limits_{E_l\cap f^{-1}(Q)} K^{\theta,1}_{q,p}(x;f)^{\varkappa}\,dx\Big)^{\frac{s}{\varrho}} \Big(\int\limits_{E_l\cap f^{-1}(Q)} \omega(x)\,dx\Big)^{s/r}\Bigg]^{1/s} \\
&\le \Lambda N(f,D)^{\frac{s-1}{s}} \Bigg[\Big(\sum_{l\in\mathbb{N}} \int\limits_{E_l\cap f^{-1}(Q)} K^{\theta,1}_{q,p}(x;f)^{\varkappa}\,dx\Big)^{\frac{s}{\varrho}} \Big(\sum_{l\in\mathbb{N}} \int\limits_{E_l\cap f^{-1}(Q)} \omega(x)\,dx\Big)^{s/r}\Bigg]^{1/s} \\
&\le \Lambda N(f,D)^{\frac{s-1}{s}} \big\|K^{\theta,1}_{q,p}(\cdot;f) \mid L_{\varkappa}(f^{-1}(Q))\big\|^{n-1} \Big(\int\limits_{f^{-1}(Q)} \omega(x)\,dx\Big)^{1/r},
\end{aligned} \tag{12}$$

where $\varrho = \frac{rs}{r-s} = \frac{pq}{(p-q)(n-1)}$.

By analogy with the case $q = p$, consider a countable family of disjoint cubes $Q_k \subset f(D)$ covering $f(D)$ up to a set of measure zero. Applying (12) to each cube Q_k and using Hölder's inequality, we deduce

$$\begin{aligned}
\Bigl(\int\limits_{f(D)} |Dg_D|^s(y)\,dy\Bigr)^{1/s} &\le \Bigl(\sum_{k\in\mathbb{N}}\int\limits_{Q_k} |Dg_D|^s(y)\,dy\Bigr)^{1/s} \\
\le \Lambda N(f,D)^{\frac{s-1}{s}}&\Bigl[\sum_{k\in\mathbb{N}}\Bigl(\int\limits_{f^{-1}(Q_k)} K^{\theta,1}_{q,p}(x;f)^{\varkappa}\,dx\Bigr)^{\frac{s}{\varrho}}\Bigl(\int\limits_{f^{-1}(Q_k)} \omega(x)\,dx\Bigr)^{s/r}\Bigr]^{1/s} \\
\le \Lambda N(f,D)^{\frac{s-1}{s}}&\Bigl[\Bigl(\sum_{k\in\mathbb{N}}\int\limits_{f^{-1}(Q_k)} K^{\theta,1}_{q,p}(x;f)^{\varkappa}\,dx\Bigr)^{\frac{s}{\varrho}}\Bigl(\sum_{k\in\mathbb{N}}\int\limits_{f^{-1}(Q_k)} \omega(x)\,dx\Bigr)^{s/r}\Bigr]^{1/s} \\
\le \Lambda N(f,D)^{\frac{s-1}{s}}&\bigl\|K^{\theta,1}_{q,p}(\cdot;f)\mid L_{\varkappa}(D)\bigr\|^{n-1}\Bigl(\int\limits_D \omega(x)\,dx\Bigr)^{1/r}. \qquad (13)
\end{aligned}$$

Consequently, (10) is established, and so the proof of Theorem 13 is complete. $\square$

1.2. The first pulled-back function and the first capacity inequality. Consider a continuous mapping $f : \Omega \to \mathbb{R}^n$ and a domain D compactly embedded into Ω, meaning that D is bounded and $\overline{D} \subset \Omega$, briefly written $D \Subset \Omega$, and take $y \notin f(\partial D)$. Denote by $\mu(y, f, D)$ the *degree* of f at y with respect to D. Say that f *is sense-preserving* whenever $\mu(y, f, D) > 0$ for all $D \Subset \Omega$ and all $y \in f(D) \setminus f(\partial D)$. If $A \subset \Omega$ then the *multiplicity function* is the mapping $\mathbb{R}^n \ni y \mapsto N(y, f, A) = \#\{f^{-1}(y) \cap A\}$. Moreover, put $N(f, A) = \sup_{y\in\mathbb{R}^n} N(y, f, A)$.

In this subsection we consider the pulled-back function (14) (see below) and establish for the latter some new properties.

Definition 14. Suppose that $f : \Omega \to \mathbb{R}^n$ is a sense-preserving, continuous, open, and discrete mapping. For each compactly supported smooth function $u \in C^1_0(\Omega)$ whose support $\operatorname{supp} u$ lies in a compactly embedded domain $D \Subset \Omega$, define the *pulled-back function* $w : f(\Omega) \to \mathbb{R}$ as

$$f(\Omega) \ni y \mapsto w(y) = \begin{cases} \max\limits_{x\in f^{-1}(y)} u(x), & y \in f(\operatorname{supp} u), \\ 0, & y \notin f(\operatorname{supp} u). \end{cases} \qquad (14)$$

In the next lemma we state the topological properties of the pulled-back function w whose proofs are independent of the differential properties of f.

Lemma 15 [22]. *The function w enjoys the properties:*
(1) $\operatorname{supp} w \subset f(\operatorname{supp} u)$ *is compact;*
(2) w *is continuous.*

The next lemma states the main differential properties of w.

Lemma 16. *Suppose that a continuous, open, and discrete mapping $f : \Omega \to \mathbb{R}^n$ is in $\mathscr{I}\mathscr{D}(\Omega; q, p; \theta, 1)$ or $\mathscr{O}\mathscr{D}(\Omega; q, p; \theta, 1)$. Then the pulled-back function w enjoys Property* ACL *on $f(\Omega)$. Moreover, $\nabla w = 0$ a.e. on $Z' \cup \Sigma'$; here Z' and Σ' are the same as in the statement of Theorem* 13.

Proof. For $f \in \mathscr{I}\mathscr{D}(\Omega; q, p; \theta, 1)$ this lemma is proved in [1]. The same proof works for $f \in \mathscr{O}\mathscr{D}(\Omega; q, p; \theta, 1)$. $\square$

Theorem 17. *Consider a domain Ω in $\mathbb{R}^n$, a mapping $f : \Omega \to \mathbb{R}^n$ in $\mathscr{I}\mathscr{D}(\Omega; q, p; \theta, 1)$, where $n - 1 \le q \le p < \infty$, and a locally summable weight function $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$. Then*

(1) *the pulled-back function w enjoys Property* ACL *on the domain $f(\Omega)$; and, moreover, $\nabla w = 0$ a.e. on $Z' \cup \Sigma'$, where Z' and Σ' are the same as in the statement of Theorem* 13;

(2) *for a compactly embedded domain $D \Subset \Omega$ with $\operatorname{supp} u \subset D$, the pulled-back function $w : f(\Omega) \to \mathbb{R}$, specified in* (14), *in the case $n-1 < q \leq p < \infty$ we have the estimate*

$$\Bigl(\int\limits_{f(D)} |\nabla w|^s(y)\,dy\Bigr)^{\frac{1}{s}} \leq \bigl\|\mathscr{K}^{\theta,1}_{q,p}(\cdot, f) \mid L_\varrho(D)\bigr\| \Bigl(\int\limits_{D} |\nabla u|^r(x)\omega(x)\,dx\Bigr)^{\frac{1}{r}} \tag{15}$$

for $f \in \mathscr{I}\mathscr{D}(\Omega; q, p; \theta, 1)$ and

$$\Bigl(\int\limits_{f(D)} |\nabla w|^s(y)\,dy\Bigr)^{\frac{1}{s}} \leq \bigl\|K^{\theta,1}_{q,p}(\cdot; f) \mid L_\varkappa(D)\bigr\|^{n-1} \Bigl(\int\limits_{D} |\nabla u|^r(x)\omega(x)\,dx\Bigr)^{\frac{1}{r}} \tag{16}$$

for $f \in \mathscr{O}\mathscr{D}(\Omega; q, p; \theta, 1)$;

In the case $n-1 = q \leq p < \infty$ we have the estimate

$$\Bigl(\int\limits_{f(D)} |\nabla w|^s(y)\,dy\Bigr)^{\frac{1}{s}} \leq \bigl\|\mathscr{K}^{\theta,1}_{n-1,p}(\cdot, f) \mid L_s(D)\bigr\| \cdot \|\nabla u \mid L_\infty(D)\|$$

for $f \in \mathscr{I}\mathscr{D}(\Omega; n-1, p; \theta, 1)$ and

$$\Bigl(\int\limits_{f(D)} |\nabla w|^s(y)\,dy\Bigr)^{\frac{1}{s}} \leq \bigl\|K^{\theta,1}_{n-1,p}(\cdot; f) \mid L_s(D)\bigr\|^{n-1} \cdot \|\nabla u \mid L_\infty(D)\| \tag{17}$$

for $f \in \mathscr{O}\mathscr{D}(\Omega; n-1, p; \theta, 1)$.

Here $s = \frac{p}{p-(n-1)}$, $r = \frac{q}{q-(n-1)}$, and $\frac{1}{\varrho} = \frac{1}{s} - \frac{1}{r}$, while $\omega = \theta^{-\frac{n-1}{q-(n-1)}} \in L_{1,\mathrm{loc}}(\Omega)$.

Proof of Theorem 17 rests on Lemma 16 and the construction of the proof of the latter. The estimate in (15) is proved in [1]. We obtain the estimate in (16) in accordance with Remark 9.

Consider the case $q = n-1 < p < \infty$ in detail. Observe that the collection of cubes $Q(y,r) \subset f(D)$ for $y \in f(D)$ and $r \in (0, r_y)$ constitutes a Vitali covering of $f(D)$. Basing on (11), in each cube $Q = Q(y,r) \subset f(D)$ we can express the value of w at $y \in \beta_z \setminus (Z' \cup \Sigma')$ as

$$w(y) = \max_{x \in f^{-1}(y)} u(x) = \max_{\substack{i=1,\dots,s\\ p=1,\dots,k(i)}} u \circ \alpha^z_{i,p} \circ \beta^{-1}(y) = \max_{l \in \mathbb{N}} u(\psi'_l(y))$$

because $\#\{l : \psi'_l(y) \cap f^{-1}(y) \neq \varnothing\} \leq N(f, D)$ for almost all $y \in Q$. Moreover, for almost all $y \in Q \setminus (Z' \cup \Sigma')$ we have

$$\frac{\partial}{\partial y_j} w(y) = \frac{\partial}{\partial y_j} \max_{\substack{i=1,\dots,s\\ p=1,\dots,k(i)}} u \circ \alpha^z_{i,p} \circ \beta^{-1}(y) = \frac{\partial}{\partial y_j}(u \circ \psi'_m)(y)$$

for some $m \in \mathbb{N}$; the choice of m depends on $y \in f(D)$. Consequently,

$$|\nabla w(y)|^s \leq \sum_{l \in \mathbb{N}} |\nabla(u \circ \psi'_l)(y)|^s$$

for almost all $y \in Q \setminus (Z' \cup \Sigma')$; observe that at most $N(f, D)$ terms in this sum are nonzero.

Put

$$T_m = \Bigl\{x \in D \setminus (Z' \cup \Sigma') : \theta(x) \geq \frac{1}{m},\ \mathscr{K}^{\theta,1}_{n-1,p}(x, f) \leq m\Bigr\}, \quad m \in N.$$

For the parameters $n-1 = q < q' < p < \infty$, $s = \frac{p}{p-(n-1)}$, $r' = \frac{q'}{q'-(n-1)}$, and $\frac{1}{\varrho'} = \frac{1}{s} - \frac{1}{r'}$ we have $\omega = \theta^{-\frac{n-1}{q'-(n-1)}} \in L_1(D \setminus T_m)$ and $\mathscr{K}^{\theta,1}_{n-1,p}(\cdot, f) \in L_{\rho'}(D \setminus T_m)$, as well as the estimates for the norm in L_s of the derivative of the pulled-back function:

$$
\begin{aligned}
\int\limits_{Q\setminus f(T_m)} |\nabla w(y)|^s \, dy &= \int\limits_{Q\setminus(Z'\cup\Sigma'\cup f(T_m))} |\nabla w(y)|^s \, dy \\
&\leq \sum_{l\in\mathbb{N}} \int\limits_{(E'_l\cap Q)\setminus f(T_m)} |Du|^s(\psi'_l(y))|D\psi'_l(y)|^s \, dy \\
\leq \sum_{l\in\mathbb{N}} \int\limits_{(E'_l\cap Q)\setminus f(T_m)} & |Du|^s(\psi'_l(y))\omega^{\frac{s}{r'}}(\psi'_l(y))J(y,\psi'_l(y))^{\frac{s}{r'}} \frac{|D\psi'_l(y)|^s}{\omega^{\frac{s}{r'}}(\psi'_l(y))J(y,\psi'_l(y))^{\frac{s}{r'}}} \, dy \\
&\leq \sum_{l\in\mathbb{N}} \Bigg(\int\limits_{(E'_l\cap Q)\setminus f(T_m)} |Du|^{r'}(\psi'_l(y))\omega(\psi'_l(y))J(y,\psi'_l(y)) \, dy \Bigg)^{\frac{s}{r'}} \\
\times \Bigg(\int\limits_{(E'_l\cap Q)\setminus f(T_m)} \frac{|D\psi'_l(y)|^{\rho'}}{\omega^{\frac{\rho}{r'}}(\psi'_l(y))J(y,\psi'_l)^{\frac{\rho}{r'}}} \, dy \Bigg)^{\frac{s}{\rho'}} &\leq \Bigg(\sum_{l\in\mathbb{N}} \int\limits_{(E_l\cap f^{-1}(Q))\setminus T_m} |Du|^{r'}(x)\omega(x) \, dx \Bigg)^{\frac{s}{r'}} \\
&\times \Bigg(\sum_{l\in\mathbb{N}} \int\limits_{(E_l\cap f^{-1}(Q))\setminus T_m} \bigg(\frac{J(x,f)^{\frac{1}{r'}}|\operatorname{adj} Df(x)|}{\omega^{\frac{1}{r'}}(x)J(x,f)} \bigg)^{\rho'} J(x,f) \, dx \Bigg)^{\frac{s}{\rho'}} \\
= \Bigg(\int\limits_{f^{-1}(Q)\setminus T_m} |Du|^{r'}(x)\omega(x) \, dx \Bigg)^{\frac{s}{r'}} & \Bigg(\int\limits_{f^{-1}(Q)\setminus(T_m\cup Z)} \bigg(\frac{\theta^{\frac{n-1}{q'}}(x)|\operatorname{adj} Df(x)|}{J(x,f)^{\frac{n-1}{p}}} \bigg)^{\rho'} \, dx \Bigg)^{\frac{s}{\rho'}}.
\end{aligned}
$$

Consequently, for the chosen cube Q we see that

$$
\begin{aligned}
\Bigg(\int\limits_{Q\setminus f(T_m)} |\nabla w(y)|^s \, dy \Bigg)^{\frac{1}{s}} &\leq \Bigg(\int\limits_{f^{-1}(Q)\setminus T_m} \theta^{(\frac{n-1}{q'}-1)\rho'}(x) \big(\mathscr{K}^{\theta,1}_{n-1,p}(x,f)\big)^{\rho'} \, dx \Bigg)^{\frac{1}{\rho'}} \\
&\times \|\nabla u \mid L_\infty(D)\| \Bigg(\int\limits_{f^{-1}(Q)\setminus T_m} \omega(x) \, dx \Bigg)^{\frac{1}{r'}}.
\end{aligned} \tag{18}
$$

In (18) we can pass to the limit as $q' \to n-1$ by the Lebesgue Dominated Convergence Theorem. This yields

$$
\Bigg(\int\limits_{Q\setminus f(T_m)} |\nabla w(y)|^s \, dy \Bigg)^{\frac{1}{s}} \leq \big\|\mathscr{K}^{\theta,1}_{n-1,p}(\cdot,f) \mid L_\varrho(f^{-1}(Q)\setminus T_m)\big\| \|\nabla u \mid L_\infty(D)\|, \tag{19}
$$

because as $q' \to n-1$ we have $\varrho' \to \varrho$ and $r' \to \infty$, and the collection $\theta^{(\frac{n-1}{q'}-1)\rho'}(x)$ $(\mathscr{K}^{\theta,1}_{n-1,p}(x,f)^{\rho'})$ is bounded on $f^{-1}(Q) \setminus T_m$ and tends to 1 $(\mathscr{K}^{\theta,1}_{n-1,p}(x,f)^{\rho})$ for almost all $x \in f^{-1}(Q) \setminus T_m$. Passing in (19) to the limit as $m \to \infty$ yields (17):

$$
\Bigg(\int\limits_{Q} |\nabla w(y)|^s \, dy \Bigg)^{\frac{1}{s}} \leq \big\|\mathscr{K}^{\theta,1}_{n-1,p}(\cdot,f) \mid L_s(f^{-1}(Q))\big\| \cdot \|\nabla u \mid L_\infty(D)\| \tag{20}
$$

because $\varrho = s$.

Continuing the argument as in the proof of (13), we obtain (15). Indeed, by Vitali's Theorem, we can refines from the covering $Q(y,r) \subset f(D)$ for $y \in f(D)$ and $r \le r_y$ a countable family of disjoint cubes $Q_k \subset f(D)$ covering $f(D)$ up to a set of measure zero. Applying (20) to each cube Q_k, we find that

$$\begin{aligned}\Bigl(\int\limits_{f(D)} |\nabla w|^s(y)\,dy\Bigr)^{1/s} &\le \Bigl(\sum_{k\in\mathbb{N}}\int\limits_{Q_k} |\nabla w|^s(y)\,dy\Bigr)^{1/s}\\ &\le \Bigl(\sum_{k\in\mathbb{N}}\int\limits_{f^{-1}(Q_k)} \mathscr{K}^{\theta,1}_{n-1,p}(x;f)^s\,dx\Bigr)^{1/s}\|\nabla u \mid L_\infty(D)\|\\ &\le \bigl\|\mathscr{K}^{\theta,1}_{n-1;p}(\cdot;f) \mid L_s(D)\bigr\| \cdot \|\nabla u \mid L_\infty(D)\|. \end{aligned} \tag{21}$$

Consequently, (15) is justified, and so Theorem 17 is established for $q < p$.

For $q = p$ the proof simplifies clearly. $\square$

In a weighted Sobolev space we can define the capacity of a condenser as follows:

Definition 18. An ordered triple $E = (F_0, F_1; D)$ of nonempty sets, where D is an open set in $\mathbb{R}^n$, while F_1 and F_0 are closed subsets of $\overline{D}$, is called a *condenser* in $D \subset \mathbb{R}^n$. The quantity

$$\operatorname{Cap}^\omega_p(E) = \operatorname{Cap}^\omega_p(F_0, F_1; D) = \inf \int\limits_D |\nabla g|^p(z)\omega(z)\,dz,$$

where the infimum is taken over all functions g continuous and absolutely continuous on almost all lines parallel to the coordinate axes ($g \in \mathrm{ACL}(D) \cap C(D)$) such that $g \ge 1$ ($g \le 0$) in some neighborhood of F_1 (F_0), is called the *ω-weighted p-capacity* of the condenser $E = (F_0, F_1; D)$. If $\omega \equiv 1$ then we simply write $\operatorname{Cap}_p(E)$ instead of $\operatorname{Cap}^1_p(E)$. In the case $p = \infty$ the capacity $\operatorname{Cap}_\infty E$ equals $\inf \|\nabla g \mid L_\infty(D)\|$, where the infimum is taken over all functions admissible for E.

If U is an open set and C is a compact subset of U then we denote the condenser $E = (\partial U, C; U)$ by $E = (U, C)$.

Observe that, considering instead of g the cutoff $\max(0, \min(1, g))$, we may assume that the functions, tested in Definition 18, are bounded below by 0 and above by 1.

Using (15) of Theorem 17, we can obtain capacity estimates that generalize those of [10, Section 1.3]. Theorem 17 implies the following:

Theorem 19. *Suppose that $f \in \mathscr{I}\mathscr{D}(\Omega; q, p; \theta, 1)$ or $f \in \mathscr{O}\mathscr{D}(\Omega; q, p; \theta, 1)$, where $n-1 < q \le p < \infty$, and the weight function $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$ is locally summable. If $E = (A, C)$ is a condenser in Ω with a normally embedded domain $A \Subset \Omega$ and a compact set $C \subset A$ then*

$$(\operatorname{Cap}_s f(E))^{1/s} \le \bigl\|\mathscr{K}^{\theta,1}_{q,p}(\cdot, f) \mid L_\varrho(A \setminus C)\bigr\| \bigl(\operatorname{Cap}^\omega_r E\bigr)^{1/r} \tag{22}$$

for $f \in \mathscr{I}\mathscr{D}(\Omega; q, p; \theta, 1)$ and

$$(\operatorname{Cap}_s f(E))^{1/s} \le \bigl\|K^{\theta,1}_{q,p}(\cdot, f) \mid L_\varkappa(A \setminus C)\bigr\|^{n-1} \bigl(\operatorname{Cap}^\omega_r E\bigr)^{1/r} \tag{23}$$

for $f \in \mathscr{O}\mathscr{D}(\Omega; q, p; \theta, 1)$. Here $s = \frac{p}{p-(n-1)}$ and $r = \frac{q}{q-(n-1)}$, while $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$.

Inequality (22) is proved in [1]. Inequality (23) is established in accordance with Remark 9.

Proposition 20. *Suppose that $f \in \mathscr{I}\mathscr{D}(\Omega; q, p; \theta, 1)$ or $f \in \mathscr{O}\mathscr{D}(\Omega; q, p; \theta, 1)$, where $n-1 = q \le p < \infty$, and the weight function $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$ is locally summable. If $E = (A, C)$ is a condenser in Ω with a normally embedded $A \Subset \Omega$ domain and a compact set $C \subset A$ then*

$$(\operatorname{Cap}_s f(E))^{1/s} \le \bigl\|\mathscr{K}^{\theta,1}_{n-1,p}(\cdot, f) \mid L_s(A \setminus C)\bigr\| \operatorname{Cap}_\infty E \tag{24}$$

for $f \in \mathscr{I}\mathscr{D}(\Omega; q, p; \theta, 1)$ and

$$(\operatorname{Cap}_s f(E))^{1/s} \le \bigl\|K^{\theta,1}_{n-1,p}(\cdot, f) \mid L_{\frac{s}{n-1}}(A \setminus C)\bigr\|^{n-1} \operatorname{Cap}_\infty E$$

for $f \in \mathscr{O}\mathscr{D}(\Omega; q, p; \theta, 1)$. Here $s = \frac{p}{p-(n-1)}$ and $r = \frac{q}{q-(n-1)}$.

To prove Proposition 20, apply (21).

1.3. The second pulled-back function and the second capacity inequality. The results stated in this and previous subsections underlie our estimates for the norms of pulled-back functions which generalize the results of [10, Section 1.2] to the mappings in $\mathscr{OD}(\Omega; q, p; \theta, 1)$. Generalizations for $\mathscr{ID}(\Omega; q, p; \theta, 1)$ are obtained in [1].

Definition 21. Suppose that $f : \Omega \to \mathbb{R}^n$ is a sense-preserving, continuous, open, and discrete mapping, and take a positive number Λ. Given a smooth compactly supported function $u \in C_0^1(\Omega)$, define the *pulled-back function* $v = f_* u : f(\Omega) \to \mathbb{R}$ as

$$f(\Omega) \ni y \mapsto v(y) = \begin{cases} \Lambda \sum\limits_{x \in f^{-1}(y)} i(x, f) u(x), & y \in f(\operatorname{supp} u), \\ 0, & y \notin f(\operatorname{supp} u). \end{cases} \tag{25}$$

The positive constant Λ is fixed later. Since $\operatorname{supp} u$ is a compact set in Ω and f is continuous and discrete, the sum in (25) contains only finitely many terms, which guarantees that the function v is well defined.

The next lemma states the topological properties of the pulled-back function v whose proofs are independent of the differential properties of f.

Lemma 22 [8, 9]. *The function v enjoys the properties:*
(1) $\operatorname{supp} v \subset f(\operatorname{supp} u)$ *is a compact set;*
(2) v *is continuous.*

Let us establish the main differential properties of v.

Lemma 23. *Suppose that a continuous, open, discrete mapping $f : \Omega \to \mathbb{R}^n$ in $\mathscr{OD}(\Omega; q, p; \theta, 1)$ enjoys* (1)–(3) *of Theorem* 13. *Then the pulled-back function v enjoys Property* ACL *on $f(\Omega)$. Moreover, $\nabla v = 0$ a.e. on the union $Z' \cup \Sigma'$.*

Here Z' and Σ' are the same as in the statement of Theorem 13.

Proof. By Remark 9, this claim follows from the embedding $\mathscr{OD}(\Omega; q, p; \theta, 1) \subset \mathscr{ID}(\Omega; q, p; \theta, 1)$ and Lemma 31 of [1]. □

The following result strengthens Theorem 4 of [10].

Theorem 24. *Consider a domain Ω in $\mathbb{R}^n$ and a mapping $f : \Omega \to \mathbb{R}^n$ in $\mathscr{OD}(\Omega; q, p; \theta, 1)$, where $n - 1 < q \le p < \infty$. Suppose that the weight function $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$ is locally summable. The pulled-back function $v = f_* u : f(\Omega) \to \mathbb{R}$, where $u \in C_0^1(\Omega)$, for every domain $D \Subset \Omega$ with $\operatorname{supp} u \subset D$ satisfies*

$$\bigl\| v \mid L_s^1\bigl(f(D)\bigr)\bigr\| \le \Lambda N(f, D)^{\frac{s-1}{s}} \bigl\| K_{q,p}^{\theta,1}(\cdot, f) \mid L_\varkappa(D)\bigr\|^{n-1} \bigl\| u \mid L_r^1(D, \omega)\bigr\|, \tag{26}$$

where $r = \frac{q}{q-(n-1)}$ and $s = \frac{p}{p-(n-1)}$, while $\frac{1}{\varrho} = \frac{1}{s} - \frac{1}{r}$.

Proof. By Remark 9, the claim follows from the embedding $\mathscr{OD}(\Omega; q, p; \theta, 1) \subset \mathscr{ID}(\Omega; q, p; \theta, 1)$ and Theorem 32 of [1]. □

The concept of capacity of a condenser in a weighted Sobolev space is given above in Definition 18.

Using (26), we can obtain the capacity estimates that generalize those of [10, Section 1.3]. Theorem 24 yields a Väisälä-type capacity inequality.

Corollary 25. *Suppose that $f : \Omega \to \mathbb{R}^n$ is a mapping in $\mathscr{OD}(\Omega; q, p; \theta, 1)$ and the weight function $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$ is locally summable.*

(1) *For a condenser $E = (A, C)$ in Ω with $A \Subset \Omega$ and a compact subset C of A,*

$$(\operatorname{Cap}_s f(E))^{1/s} \le \frac{\bigl\| K_{q,p}^{\theta,1}(\cdot, f) \mid L_\varkappa(A \setminus C)\bigr\|^{n-1} \cdot (N(f, A))^{(s-1)/s}}{M(f, C)} \bigl(\operatorname{Cap}_r^\omega E\bigr)^{1/r},$$

where $s = \frac{p}{p-(n-1)}$ *and* $r = \frac{q}{q-(n-1)}$, *while* $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$ *and*

$$M(f,C) = \inf_{x\in f(C)} \sum_{z\in f^{-1}(x)\cap C} i(z,f).$$

(2) *For a condenser* $E=(A,C)$ *in* Ω *with a normal domain* $A \Subset \Omega$ *and a compact subset* C *of* A,

$$(\operatorname{Cap}_s f(E))^{1/s} \le \frac{\|K^{\theta,1}_{q,p}(\cdot,f) \mid L_\varkappa(A\setminus C)\|^{n-1}}{(N(f,A))^{1/s}} \big(\operatorname{Cap}^\omega_r E\big)^{1/r},$$

where $s = \frac{p}{p-(n-1)}$ *and* $r = \frac{q}{q-(n-1)}$, *while* $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$.

Proof. By Remark 9, the claim follows from the embedding $\mathscr{OD}(\Omega;q,p;\theta,1) \subset \mathscr{ID}(\Omega;q,p;\theta,1)$ and Corollary 37 of [1]. $\square$

2. Almost Everywhere Differentiability of the Mappings in $\mathscr{ID}(\Omega;q,p;\theta,1)$ and $\mathscr{OD}(\Omega;q,p;\theta,1)$, with $n-1 \le q \le p < n+\frac{1}{n-2}$

Lemma 11 shows that the mappings in $\mathscr{ID}(\Omega;q,p;\theta,1)$ are differentiable a.e. on $\Omega\setminus(B_f\cup Z)$. In this section we prove that the mappings in this space are differentiable a.e. in Ω under the condition $n-1 \le q \le p < n+\frac{1}{n-2}$ on the summability exponents. As a corollary, we find that $|B_f\setminus Z| = 0$; i.e., B_f lies in Z with the exception of a subset of measure zero.

A proof of differentiability rests on estimates (22) and (24) of the type of the Poletsky inequality [20] for the capacity of the pulled-back condenser in terms of the weighted capacity of the original condenser.

Using Theorem 19, we can prove the following

Theorem 26. *Every mapping in* $\mathscr{ID}(\Omega;q,p;\theta,1)$ *or* $\mathscr{OD}(\Omega;q,p;\theta,1)$, *where* $n-1 \le q \le p < n+\frac{1}{n-2}$, *is differentiable a.e. in* Ω *under the condition that the weight function* $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$ *is locally summable in* Ω.

Proof. Fix an arbitrary mapping $f \in \mathscr{ID}(\Omega;q,p;\theta,1)$ and a normal domain $D \Subset \Omega$. As the condenser $E=(A,C)$, consider the spherical layer in which $A = B(x,2t) \Subset \Omega$ and $C = \overline{B(x,t)}$ with $t>0$. Put the function $u(z)$ in (15) and (16) equal to $\min\{t^{-1}\operatorname{dist}(z,S(x,2t)),1\}$ if $z\in B(x,2t)$. The function u is obviously admissible for the capacity of the spherical condenser E. Moreover,

$$\Big(\int_A |\nabla u|^r(z)\omega(z)\,dz\Big)^{\frac1r} \le \frac1t\Big(\int_A \omega(z)\,dz\Big)^{\frac1r} \tag{27}$$

because $|\nabla_z(\operatorname{dist}(z,S(x,2t)))| = 1$.

Case I: $n-1 < q \le p$ and $n < p < n+\frac{1}{n-2}$. In this case $s\in(n-1,n)$. From [23] (see also [24]) we infer the left-hand side of the next inequality, while from the relations[2)] between capacities, the right-hand side:

$$\operatorname{diam}(f(\overline{B(x,t)}))^{n-s} \le C(n,s)\operatorname{cap}\big(f(\overline{B(x,t)}); w^1_s(\mathbb{R}^n)\big) \le C(n,s)\operatorname{Cap}_s(f(B(x,2t), f(\overline{B(x,t)})).$$

[2)]For $s\in(1,n)$ the space $w^1_s(\mathbb{R}^n)$ is defined as the completion of the space $C^\infty_0(\mathbb{R}^n)$ of compactly supported functions in the norm $\|u \mid w^1_s(\mathbb{R}^n)\| = \|\nabla u \mid L_s(\mathbb{R}^n)\|$. This space is known to be isomorphic to the space of Riesz potentials [23, 24]: $u\in w^1_s(\mathbb{R}^n)$ if and only if $u = |x|^{1-n} * f$, where $f\in L_s(\mathbb{R}^n)$ with $s\in(1,n)$. Refer as the *s-capacity* of a compact set $K\subset D$ in $w^1_s(\mathbb{R}^n)$, with $s\in(1,n)$, to

$$\operatorname{cap}\big(K; w^1_s(\mathbb{R}^n)\big) = \inf\|\nabla u \mid L_s(\mathbb{R}^n)\|^s,$$

where the infimum is taken over all functions $u\in \mathrm{ACL}(D)\cap C(D)$ with $u\ge1$ on K. It is obvious that $\operatorname{cap}\big(C; w^1_s(\mathbb{R}^n)\big) \le \operatorname{Cap}_s(E)$ for the condenser $E=(A,C)$.

This estimate and (27) show that

$$\frac{\operatorname{diam}(f(\overline{B(x,t)}))}{t} \le C'\biggl(\frac{1}{|B(x,2t)|}\int\limits_{B(x,2t)} \bigl(\mathscr{K}^{\theta,1}_{q,p}(z,f)\bigr)^{\rho}\,dz\biggr)^{\frac{s}{\rho(n-s)}}\biggl(\frac{1}{|B(x,2t)|}\int\limits_{B(x,2t)} \omega(z)\,dz\biggr)^{\frac{s}{r(n-s)}}.$$

Passage to the limit as $t\to 0$ yields

$$\varlimsup_{z\to x}\frac{|f(z)-f(x)|}{|z-x|} \le C'\bigl(\mathscr{K}^{\theta,1}_{q,p}(x,f)\bigr)^{\frac{s}{n-s}}\omega(x)^{\frac{s}{r(n-s)}} < \infty \tag{28}$$

a.e. in D. By Stepanov's Theorem, the mapping f is differentiable a.e. in D.

Case II: $n-1<q\le p\le n$. In this case $s\ge n$, but we can use Hölder's inequality for any $\alpha\in(n-1,n)$ to infer that

$$\operatorname{Cap}_\alpha(f(B(x,2t)),f(\overline{B(x,t)}))^{\frac{1}{\alpha}} \le |f(B(x,2t))|^{\frac{s-\alpha}{s\alpha}}\operatorname{Cap}_s(f(B(x,2t),f(\overline{B(x,t)}))^{\frac{1}{s}}$$

and reduce the second case to the previous one with the help of estimates (22) and (27):

$$\operatorname{Cap}_s(f(B(x,2t)),f(\overline{B(x,t)}))^{\frac{1}{s}} \le \bigl\|\mathscr{K}^{\theta,1}_{q,p}(\cdot,f)\mid L_\varrho(A)\bigr\|\cdot\frac{1}{t}\biggl(\int\limits_A \omega(z)\,dz\biggr)^{\frac{1}{r}}.$$

Indeed, the last two inequalities imply that

$$\operatorname{Cap}_\alpha(f(B(x,2t)),f(\overline{B(x,t)}))^{\frac{1}{\alpha}} \le |f(B(x,2t))|^{\frac{s-\alpha}{s\alpha}}\bigl\|\mathscr{K}^{\theta,1}_{q,p}(\cdot,f)\mid L_\varrho(A)\bigr\|\cdot\frac{1}{t}\biggl(\int\limits_A \omega(z)\,dz\biggr)^{\frac{1}{r}}.$$

By analogy with case I, we obtain

$$(\operatorname{diam}(f(\overline{B(x,t)}))^{n-\alpha} \le C(n,\alpha)\operatorname{Cap}_\alpha(f(B(x,2t),f(\overline{B(x,t)}))).$$

Comparing the last two relations, we deduce the inequality

$$\biggl(\frac{\operatorname{diam}(f(\overline{B(x,t)}))}{t}\biggr)^{\frac{n-\alpha}{\alpha}} \le C'(n,\alpha)\biggl(\frac{|f(B(x,2t))|}{|B(x,2t)|}\biggr)^{\frac{s-\alpha}{s\alpha}}$$
$$\times\biggl(\frac{1}{|B(x,2t)|}\int\limits_{B(x,2t)} \bigl(\mathscr{K}^{\theta,1}_{q,p}(z,f)\bigr)^{\rho}\,dz\biggr)^{\frac{1}{\rho}}\biggl(\frac{1}{|B(x,2t)|}\int\limits_{B(x,2t)} \omega(z)\,dz\biggr)^{\frac{1}{r}}.$$

Passing to the limit as $t\to 0$ yields

$$\varlimsup_{z\to x}\frac{|f(z)-f(x)|}{|z-x|} \le C'\overline{\Phi}'(x)^{\frac{s-\alpha}{s(n-\alpha)}}\bigl(\mathscr{K}^{\theta,1}_{q,p}(x,f)\bigr)^{\frac{\alpha}{n-\alpha}}\omega(x)^{\frac{\alpha}{r(n-\alpha)}} < \infty \tag{29}$$

for almost all $x\in D$. Here $\overline{\Phi}'(x)$ is the upper derivative of the $N(f,D)$-quasiadditive set function[3)] Φ (see [25, 26]) defined on the open set $U\Subset D$ as $\Phi(U)=|f(U)|$; it is known, see [26, Corollary 2] for instance, that

$$\overline{\Phi}'(x) = \varlimsup_{\operatorname{rad}B\to 0,\,x\in B}\frac{\Phi(B)}{|B|} < \infty \tag{30}$$

a.e. in D. By Stepanov's Theorem, f is differentiable a.e. in D.

3) Recall (see [25, 26]) that a set function Φ defined on bounded open sets $U'\subset\Omega'$ and taking finite nonnegative values is called *monotone* whenever $\Phi(V')\le\Phi(U')$ for all bounded open sets $V'\subset U'\subset\Omega'$, and $N(f,D)$-*quasiadditive* whenever $\sum_{i=1}^{\infty}\Phi(U_i')\le N(f,D)\Phi\bigl(\bigcup_{i=1}^{\infty}U_i'\bigr)$ for all disjoint bounded open sets $U_i'\subset\Omega'$, $i\in\mathbb{N}$.

CASE III: $n-1=q\le p<n+\frac{1}{n+2}$. If $p\in\big(n,n+\frac{1}{n+2}\big)$ then, using (24) instead of (28), we obtain

$$\varlimsup_{z\to x}\frac{|f(z)-f(x)|}{|z-x|}\le C'\big(\mathscr{K}^{\theta,1}_{n-1,p}(x,f)\big)^{\frac{s}{n-s}}<\infty, \tag{31}$$

while if $n-1=q\le p\le n$ then we arrive, instead of (29), at the inequality

$$\varlimsup_{z\to x}\frac{|f(z)-f(x)|}{|z-x|}\le C'\overline{\Phi}'(x)^{\frac{s-\alpha}{s(n-\alpha)}}\big(\mathscr{K}^{\theta,1}_{n-1,p}(x,f)\big)^{\frac{\alpha}{n-\alpha}}<\infty \tag{32}$$

for almost all $x\in D$, where $\alpha\in(n-1,n)$ is an arbitrary number). As above, by Stepanov's Theorem f is differentiable a.e. in D.

Since D is an arbitrary normal domain in Ω, it follows that f is differentiable a.e. in Ω. $\square$

Proposition 27. *Suppose that $f:\Omega\to\mathbb{R}^n$ is a mapping in $\mathscr{I}\mathscr{D}(\Omega;q,p;\theta,1)$ or $\mathscr{O}\mathscr{D}(\Omega;q,p;\theta,1)$, where $n-1\le q\le p<n+\frac{1}{n-2}$, and the weight function $\omega(x)=\theta^{-\frac{n-1}{q-(n-1)}}(x)$ is locally summable in the case $n-1<q$. Then*

(1) $\overline{\Phi}'(x)=J(x,f)$ a.e. in Ω, where $\overline{\Phi}'(x)$ is defined in (30);

(2) $B_f\subset Z=\{x\in\Omega : J(x,f)=0\}$ up to a set of measure zero, i.e., $|B_f\setminus Z|=0$;

(3) $f\in\mathscr{I}\mathscr{D}(\Omega;q,p;\theta,1)$ has finite distortion.

PROOF. (1) and (2) At the points of the local homeomorphism the required equality is proved in [17, Section 2.3]. If $x\in B_f$ is a point of differentiability then it is known [2] that $J(x,f)=0$. It is easy to show that $\overline{\Phi}'(x)=0$ in this case; see [2, Lemma 2.9] for instance.

(3) We can now express (29) as

$$|Df(x)|\le C'J(x,f)^{\frac{s-\alpha}{s\alpha}}\big(\mathscr{K}^{\theta,1}_{q,p}(x,f)\big)^{\frac{\alpha}{n-\alpha}}\omega(x)^{\frac{\alpha}{r(n-\alpha)}}<\infty$$

and (32) as

$$|Df(x)|\le C'J(x,f)^{\frac{s-\alpha}{s\alpha}}\big(\mathscr{K}^{\theta,1}_{n-1,p}(x,f)\big)^{\frac{\alpha}{n-\alpha}}<\infty$$

for almost all $x\in\Omega$.

By analogy with the above, rearrange (28) as

$$|Df(x)|\le C'\big(\mathscr{K}^{\theta,1}_{q,p}(x,f)\big)^{\frac{s}{n-s}}\omega(x)^{\frac{s}{r(n-s)}}<\infty$$

and (31) as

$$|Df(x)|\le C'\big(\mathscr{K}^{\theta,1}_{n-1,p}(x,f)\big)^{\frac{s}{n-s}}<\infty$$

for almost all $x\in\Omega$.

Hence, we infer that the partial derivatives vanish a.e. on the zero set of the Jacobian. Consequently, f has finite distortion: $Df(x)=0$ a.e. on the zero set Z of the Jacobian; recall that in all formulas $\mathscr{K}^{\theta,1}_{q,p}(x,f)$ vanishes a.e. on Z. $\square$

Proposition 28. *Suppose that $f:\Omega\to\mathbb{R}^n$ is a mapping in $\mathscr{I}\mathscr{D}(\Omega;q,p;\theta,1)$, where $n-1\le q\le p<n+\frac{1}{n-2}$, and that $\frac{1}{\rho}=\frac{n-1}{q}-\frac{n-1}{p}$. Take, moreover, $p'=\frac{s}{s-(n-1)}$, the number q' found from the condition $\frac{n-1}{\rho}=\frac{1}{q'}-\frac{1}{p'}$, and the weight function $\bar{\theta}(x)$ equal to $\theta^{\frac{(n-1)^2q'}{q}}(x)$. Then $1\le q'\le p'<\infty$ and the mapping $f:D\to f(D)$, where $D\Subset\Omega$ is a normal domain, induces the bounded operator*

$$f^*:L^1_{p'}(f(D))\cap W^1_{\infty,\mathrm{loc}}(f(D))\to L^1_{q',\bar{\theta}}(D) \tag{33}$$

as $f^*(g)=g\circ f$ for $g\in L^1_{p'}(f(D))\cap W^1_{\infty,\mathrm{loc}}(f(D))$. Its norm satisfies

$$\|f^*\|\le N(f,D)^{\frac{1}{p'}}\big\|K^{\bar{\theta},1}_{q',p'}(\cdot,f)\mid L_{\frac{\rho}{n-1}}(D)\big\|; \tag{34}$$

moreover,

$$\big\|K^{\bar{\theta},1}_{q',p'}(\cdot,f)\mid L_{\frac{\rho}{n-1}}(D)\big\|\le\big\|\mathscr{K}^{\theta,1}_{q,p}(\cdot,f)\mid L_{\varrho}(D)\big\|^{n-1}.$$

PROOF. STEP 1. In this part of the proof the arguments assume that the weight function $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$ is locally summable.

Fix an arbitrary $f \in \mathscr{I}\mathscr{D}(\Omega; q, p; \theta, 1)$ and a normal domain $D \Subset \Omega$. As the condenser $E = (A, C)$ consider the spherical layer with $A = B(x, 2t) \Subset \Omega$ and $C = \overline{B(x,t)}$, where $t > 0$ is some small number. Put the function $u(z)$ in (15) and (16) equal to $\min\{t^{-1}\operatorname{dist}(z, S(x, 2t)), 1\}$ for $z \in B(x, 2t)$. The function u is obviously admissible for the capacity of the spherical condenser E. Moreover,

$$\biggl(\int\limits_A |\nabla u|^r(z)\omega(z)\,dz\biggr)^{\frac{1}{r}} \leq \frac{1}{t}\biggl(\int\limits_{A\setminus C} \omega(z)\,dz\biggr)^{\frac{1}{r}} \leq \frac{1}{t}\biggl(\int\limits_A \omega(z)\,dz\biggr)^{\frac{1}{r}}$$

because $|\nabla_z(\operatorname{dist}(z, S(x, 2t)))| = 1$ a.e. in $B(x, 2t) \setminus \overline{B(x,t)}$.

CASE I: $n - 1 < q \leq n \leq p < n + \frac{1}{n-2}$. In this case $s \in (n-1, n]$. From [27, Proposition 6], see also [22] for $s = n$, we infer the inequality

$$\operatorname{Cap}_s^{n-1}(f(B(x, 2t)), f(\overline{B(x,t)})) \geq C(n, s)\frac{\operatorname{diam}(f(\overline{B(x,t)}))^s}{|f(B(x, 2t))|^{1-n+s}}. \tag{35}$$

This estimate and (15) show that

$$\begin{gathered}\frac{\operatorname{diam}(f(\overline{B(x,t)}))}{t} \leq C'\biggl(\frac{|f(B(x,2t))|}{|B(x,2t)|}\biggr)^{\frac{1-n+s}{s}} \\ \times\biggl(\frac{1}{|B(x,2t)|}\int\limits_{B(x,2t)} \bigl(\mathscr{K}_{q,p}^{\theta,1}(z,f)\bigr)^{\rho}\,dz\biggr)^{\frac{n-1}{\rho}}\biggl(\frac{1}{|B(x,2t)|}\int\limits_{B(x,2t)} \omega(z)\,dz\biggr)^{\frac{n-1}{r}}.\end{gathered} \tag{36}$$

Passage to the limit as $t \to 0$ yields

$$\varlimsup_{z\to x}\frac{|f(z) - f(x)|}{|z - x|} \leq C'\overline{\Phi}'(x)^{\frac{1-n+s}{s}}\bigl(\mathscr{K}_{q,p}^{\theta,1}(x,f)\bigr)^{n-1}\omega(x)^{\frac{n-1}{r}} < \infty \tag{37}$$

for almost all $x \in D$. Here $\overline{\Phi}'(x)$ is the upper derivative of the $N(f, D)$-quasiadditive set function Φ, see [26], defined on the open set $U \Subset D$ as $\Phi(U) = |f(U)|$; it is known, see [26, Corollary 2] for instance, that

$$\overline{\Phi}'(x) = \varlimsup_{\operatorname{rad} B\to 0,\, x\in B}\frac{\Phi(B)}{|B|} < \infty$$

a.e. in D.

CASE II: $n - 1 < q \leq p < n$. In this case $s > n$ and (35) does not work. Nevertheless, Hölder's inequality implies that

$$\operatorname{Cap}_n(f(B(x,2t)), f(\overline{B(x,t)}))^{\frac{1}{n}} \leq |f(B(x,2t))|^{\frac{s-n}{sn}}\operatorname{Cap}_s(f(B(x,2t)), f(\overline{B(x,t)}))^{\frac{1}{s}},$$

and we reduce the second case to the previous one using the capacity estimate (22):

$$\operatorname{Cap}_s(f(B(x,2t)), f(\overline{B(x,t)}))^{\frac{1}{s}} \leq \bigl\|\mathscr{K}_{q,p}^{\theta,1}(\cdot, f) \mid L_\varrho(A)\bigr\| \cdot \frac{1}{t}\biggl(\int\limits_A \omega(z)\,dz\biggr)^{\frac{1}{r}}.$$

Indeed, from the last two inequalities we see that

$$\operatorname{Cap}_n(f(B(x,2t)), f(\overline{B(x,t)}))^{\frac{1}{n}} \leq |f(B(x,2t)|^{\frac{s-n}{sn}}\bigl\|\mathscr{K}_{q,p}^{\theta,1}(\cdot, f) \mid L_\varrho(A)\bigr\| \cdot \frac{1}{t}\biggl(\int\limits_A \omega(z)\,dz\biggr)^{\frac{1}{r}},$$

while (35) yields

$$\operatorname{Cap}_n^{n-1}(f(B(x,2t), f(\overline{B(x,t)}))) \geq C(n)\frac{\operatorname{diam}(f(\overline{B(x,t)}))^n}{|f(B(x,2t))|}.$$

Comparing the last two relations, we deduce (36) and then arrive at (37). By Theorem 26, f is differentiable a.e. in Ω. Thus, (37) shows that

$$|Df(x)| \le C' J(x,f)^{\frac{1-n+s}{s}} \left(\mathscr{K}^{\theta,1}_{q,p}(x,f)\right)^{n-1} \omega(x)^{\frac{n-1}{r}} < \infty \tag{38}$$

for almost all $x \in D$.

Step 2. Rearrange (38) as

$$\begin{cases} \left(\omega(x)^{-\frac{n-1}{r}} \frac{|Df(x)|}{J(x,f)^{\frac{s-(n-1)}{s}}}\right)^{\frac{\rho}{n-1}} \le C_1^{\rho} (\mathscr{K}^{\theta,1}_{q,p}(x,f))^{\rho} & \text{if } \rho < \infty, \\ \left(\omega(x)^{-\frac{n-1}{s}} \frac{|Df(x)|}{J(x,f)^{\frac{s-(n-1)}{s}}}\right)^{\frac{1}{n-1}} \le C_1 \mathscr{K}^{\theta,1}_{p,p}(x,f) & \text{if } \rho = \infty, \end{cases} \tag{39}$$

where $s = \frac{p}{p-(n-1)}$, $r = \frac{q}{q-(n-1)}$, and $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$, while C_1 is some constant depending only on n, p, and q. We obtained (39) for the exponents $n-1 < q \le p < n + \frac{1}{n-2}$ on assuming that the weight function $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$ is locally summable. Using the well-known inequality

$$\frac{|Df(x)|}{J(x,f)^{\frac{1}{n}}} \le \left(\frac{|Df(x)|}{J(x,f)^{\frac{n-1}{n}}}\right)^{n-1} \quad \text{for a.e. } x \in D,$$

we can check directly that the inequalities of (39) are valid with $C_1 = 1$. This yields the last inequality of Proposition 28.

Since the right-hand side of (39) is summable for $\rho < \infty$, and bounded for $\rho = \infty$, so is the left-hand side. Consequently, the function

$$\Omega \setminus Z \ni x \mapsto \omega(x)^{-\frac{n-1}{r}} \frac{|Df(x)|}{J(x,f)^{\frac{s-(n-1)}{s}}} = \frac{(\theta(x)^{\frac{(n-1)^2 q'}{q}})^{\frac{1}{q'}} |Df(x)|}{J(x,f)^{\frac{s-(n-1)}{s}}} = \frac{\bar\theta(x)^{\frac{1}{q'}} |Df(x)|}{J(x,f)^{\frac{1}{p'}}} \tag{40}$$

belongs to $L_{\frac{\rho}{n-1}}(\Omega \setminus Z)$, where $s = \frac{p}{p-(n-1)}$, $r = \frac{q}{q-(n-1)}$, $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$, and $\bar\theta(x) = \theta(x)^{\frac{(n-1)^2 q'}{q}}$, while q' is found from the condition $\frac{1}{q'} = \frac{n-1}{\rho} + \frac{1}{p'}$. It is easy to verify that $1 \le q' \le p' < \infty$.

We can see from (39) and (40) that the exterior function of the local $(\bar\theta, 1)$-weighted (q', p')-distortion

$$D \ni x \mapsto K^{\bar\theta,1}_{q',p'}(x,f) = \begin{cases} \frac{\bar\theta(x)^{\frac{1}{q'}} |Df(x)|}{|J(x,f)|^{\frac{1}{p'}}} & \text{if } J(x,f) \ne 0, \\ 0 & \text{otherwise} \end{cases}$$

is in $L_{\frac{\rho}{n-1}}(\Omega)$. Moreover, $\frac{n-1}{\rho} = \frac{1}{q'} - \frac{1}{p'}$, and $\rho = \infty$ for $q = p$.

It is clear that if $g \in L^1_{p'}(f(D)) \cap W^1_{\infty,\mathrm{loc}}(f(D))$ then $g \circ f$ is in $\mathrm{ACL}(D)$. It remains to estimate the norm $\|\nabla(g \circ f) \mid L^1_{q',\bar\theta}(D)\|$. Observe that $\nabla(g \circ f)(x) = 0$ a.e. on $Z \cap D$ because f has finite distortion. Applying (6), we infer that

$$\|\nabla(g \circ f) \mid L^1_{q',\bar\theta}(D)\|^{q'} = \int\limits_D |\nabla(g \circ f)|^{q'}(x) \bar\theta(x)\, dx$$

$$\le \int\limits_{D\setminus Z} |\nabla g|^{q'}(f(x)) |Df(x)|^{q'} \bar\theta(x)\, dx = \int\limits_{D\setminus Z} |\nabla g|^{q'}(f(x)) J(x,f)^{\frac{q'}{p'}} \frac{\bar\theta(x) |Df(x)|^{q'}}{J(x,f)^{\frac{q'}{p'}}}\, dx$$

$$\le \left(\int\limits_{D\setminus Z} |\nabla g|^{p'}(f(x)) J(x,f)\, dx\right)^{\frac{q'}{p'}} \left(\int\limits_{D\setminus Z} \frac{\bar\theta(x)^{\frac{p'}{p'-q'}} |Df(x)|^{\frac{p'q'}{p'-q'}}}{J(x,f)^{\frac{q'}{p'-q'}}}\, dx\right)^{\frac{p'-q'}{p'}}$$

$$\le \left(N(f,D) \int\limits_{f(D)} |\nabla g|^{p'}(y)\, dy\right)^{\frac{q'}{p'}} \left(\int\limits_{D\setminus Z} \left(\frac{\bar\theta(x)^{\frac{1}{q'}} |Df(x)|}{J(x,f)^{\frac{1}{p'}}}\right)^{\frac{\rho}{n-1}} dx\right)^{\frac{p'-q'}{p'}}.$$

Consequently,

$$\big\|f^*(g) \mid L^1_{q',\bar\theta}(D)\big\| \le N(f,D)^{\frac{1}{p'}}\big\|K^{\bar\theta,1}_{q',p'}(\cdot,f) \mid L_{\frac{\rho}{n-1}}(D)\big\|\cdot\big\|g \mid L^1_{p'}(f(D))\big\|,$$

and (34) is justified.

Case III: $n-1=q\le p<n+\frac{1}{n+2}$. In this case we obtain instead of (39) the inequalities

$$\begin{cases} \Big(\theta(x)^{n-1}\frac{|Df(x)|}{J(x,f)^{\frac{s-(n-1)}{s}}}\Big)^{\frac{s}{n-1}} \le C_1^s\big(\mathscr{K}^{\theta,1}_{n-1,p}(x,f)\big)^s & \text{if } s<\infty,\\ \Big(\theta(x)^{n-1}\frac{|Df(x)|}{J(x,f)}\Big)^{\frac{1}{n-1}} \le C_1\mathscr{K}^{\theta,1}_{n-1,n-1}(x,f) & \text{if } n-1=q=p, \end{cases}$$

where $s=\frac{p}{p-(n-1)}$, while C_1 is a constant depending only on n, p, and q. Since the right-hand side of the last inequality is summable for $s<\infty$, and bounded for $s=\infty$, so is the left-hand side. Therefore, for $s<\infty$ the function

$$\Omega\setminus Z\ni x\mapsto \theta(x)^{n-1}\frac{|Df(x)|}{J(x,f)^{\frac{1}{p'}}}$$

belongs to $L_{\frac{s}{n-1}}(\Omega\setminus Z)$, where $s=\frac{p}{p-(n-1)}$ and $p'=\frac{s}{s-(n-1)}$, while the number q' is found from the condition $\frac{n-1}{s}=\frac{1}{q'}-\frac{1}{p'}$ and equals 1. It is easy to verify that $1=q'\le p'<\infty$ and $\bar\theta(x)=\theta(x)^{n-1}$.

Consequently, for $n-1=q=p$ we have $s=\infty$ and $1=q'=p'$, while the function

$$\Omega\setminus Z\ni x\mapsto \theta(x)^{n-1}\frac{|Df(x)|}{J(x,f)}$$

belongs to $L_\infty(\Omega)$ and $\bar\theta(x)=\theta(x)^{n-1}$.

Furthermore, in the same fashion as at step 2, for $g\in L^1_{p'}(f(D))\cap W^1_{\infty,\mathrm{loc}}(f(D))$ we show that

$$\big\|f^*(g) \mid L^1_{1,\bar\theta}(D)\big\| \le N(f,D)^{\frac{1}{p'}}\big\|K^{\bar\theta,1}_{1,p'}(\cdot,f) \mid L_{\frac{s}{n-1}}(D)\big\|\cdot\big\|g \mid L^1_{p'}(f(D))\big\|, \tag{41}$$

justifying (34). □

The following statements concern the case of the weight function $\theta\equiv 1$.

Corollary 29. *Suppose that $f:\Omega\to\mathbb{R}^n$ is a mapping in $\mathscr{I}\mathscr{D}(\Omega;q,p;1,1)$, where $n-1\le q\le p\le n$. Then*

(1) *f enjoys Luzin's $\mathscr{N}^{-1}$-property: $|f^{-1}(E)|=0$ whenever $E\subset\Omega'$ with $|E|=0$;*
(2) *$J(x,f)>0$ a.e. in Ω;*
(3) *the set B_f of branch points has measure zero.*

Proof. (1) In this case $1\le q'\le p'\le n$ and $\bar\theta\equiv 1$, while the operator

$$f^*: L^1_{p'}(f(D))\cap W^1_{\infty,\mathrm{loc}}(f(D))\to L^1_{q'}(D) \tag{42}$$

of (33), where D is a normal domain, cannot be bounded in the absence of Luzin's $\mathscr{N}^{-1}$-property; i.e., if $|f(E)|=0$ then $|E|=0$; this property is the claim of Theorem 4 of [28].

(2) Luzin's $\mathscr{N}^{-1}$-property implies that $J(x,f)>0$ a.e. in Ω, for otherwise the Change-of-Variables Theorem yields $|f(Z\setminus\Sigma)|=0$, whence we infer that $|Z|=0$. Here the singular set Σ is of measure zero.

(3) Proposition 27 yields $|B_f\setminus Z|=0$. Since $|Z|=0$, it follows that $|B_f|=0$. □

Corollary 30. *Each mapping $f : \Omega \to \mathbb{R}^n$ in $\mathscr{I}\mathscr{D}(\Omega; n, n; 1, 1)$ has bounded distortion.*

Proof. By Corollary 29, every mapping $f : \Omega \to \mathbb{R}^n$ in $\mathscr{I}\mathscr{D}(\Omega; n, n; 1, 1)$ enjoys Luzin's $\mathscr{N}^{-1}$-property: $|f^{-1}(E)| = 0$ whenever $E \subset \Omega'$ with $|E| = 0$; its Jacobian $J(x, f)$ is positive a.e. in Ω and the set B_f of branch points is of measure zero. From (39) with $\rho = \infty$ and $\theta \equiv 1$ we infer that

$$\left(\frac{|Df(x)|}{J(x,f)^{\frac{1}{n}}}\right)^{\frac{1}{n-1}} \leq C_1 \bigl\|\mathscr{K}^{1,1}_{n,n}(x, f) \mid L_\infty(\Omega)\bigr\|.$$

Hence, we obtain (1) with the constant

$$K = C_1^{n(n-1)} \bigl\|\mathscr{K}^{1,1}_{n,n}(x, f) \mid L_\infty(\Omega)\bigr\|^{n(n-1)}.$$

Now (42) yields $f \in W^1_{n,\mathrm{loc}}(\Omega)$. $\square$

Remark 31. The differentiability of mappings in $\mathscr{I}\mathscr{D}(\Omega; n-1, n; 1, 1)$ is proved in [16], which also established the finiteness of distortion for the mappings of this class, as well as the properties stated in Corollary 29. Note that the proofs in this article differ from the arguments of [16]. For instance, in the proof of Luzin's $\mathscr{N}^{-1}$-property the author of [16] cites [28], while here we cite the earlier article [29]. Moreover, the method for proving Poletsky-type capacity inequalities used in [16] is based on [30], while here the method relies on [17, 21], and so on.

3. Liouville-Type Theorems

The next result enables us to generalize Liouville's Theorem.

Theorem 32. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping with bounded $(\theta, 1)$-weighted (p, q)-distortion, $n - 1 < q \leq p < \infty$, and the weight function $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$ is summable. If there exist a continuum $C \subset \mathbb{R}^n$ with nonempty interior and a sequence $A_k \supset C$, for $k \in N$, of bounded open sets exhausting[4)] $\mathbb{R}^n$ such that $\operatorname{cap}^\omega_r(A_k, C) \to 0$ as $k \to \infty$ then*

$$\operatorname{cap}_s(\mathbb{R}^n \setminus f(\mathbb{R}^n), f(C); \mathbb{R}^n) = 0,$$

where $s = \frac{p}{p-(n-1)}$ and $r = \frac{q}{q-(n-1)}$, while $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$.

Proof. The condenser $E_k = (A_k, C)$ satisfies (22):

$$(\operatorname{cap}_s(f(E_k)))^{\frac{1}{s}} \leq \mathscr{K}^{\theta,1}_{p,q}(f; \mathbb{R}^n)\bigl(\operatorname{cap}^\omega_r(E_k)\bigr)^{\frac{1}{r}}.$$

By assumption, $\operatorname{cap}^\omega_r(E_k) \to 0$ as $k \to \infty$. Then we deduce the capacity estimate:

$$\operatorname{cap}_s(f(\mathbb{R}^n), f(C); \mathbb{R}^n) \leq \lim_{k\to\infty} \operatorname{cap}_s(f(E_k)) = 0.$$

Hence, $\operatorname{cap}_s(f(\mathbb{R}^n), f(C); \mathbb{R}^n) = 0$ as required. $\square$

Corollary 33. *If the hypotheses of Theorem 32 hold and $q, p \in (n-1, n)$ then $f(\mathbb{R}^n) = \mathbb{R}^n$.*

Proof. Observe that $s = \frac{p}{p-(n-1)} > n$. Suppose that $f(\mathbb{R}^n) \neq \mathbb{R}^n$. The nonempty complement $\mathbb{R}^n \setminus f(\mathbb{R}^n)$ contains a point y and $f(C)$ contains a point x, and the points are at a finite distance. The s-capacity of the pair $F_0 = \{x\}$ and $F_1 = \{y\}$ of singletons in $L^1_s(\mathbb{R}^n)$ is comparable to the positive quantity $|x-y|^{n-s}$, and it is known [23] that a lower bound for the s-capacity follows from the boundedness of the operator embedding $L^1_s(\mathbb{R}^n)$ into the space of Hölder functions of exponent $1 - \frac{n}{s}$. Thus, $\operatorname{cap}_s(\mathbb{R}^n \setminus f(\mathbb{R}^n), f(C); \mathbb{R}^n) > 0$. The latter contradicts Theorem 32. $\square$

Corollary 34. *Suppose that the hypotheses of Theorem 32 hold and $n - 1 < q \leq p = n$. If $\operatorname{cap}_n(\mathbb{R}^n \setminus f(\mathbb{R}^n), f(C); \mathbb{R}^n) > 0$ then the set of mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ with bounded $(\theta, 1)$-weighted (n, q)-distortion is empty.*

Proof. Indeed, assuming the contrary, we get $f : \mathbb{R}^n \to \mathbb{R}^n$ with bounded $(\theta, 1)$-weighted (n, q)-distortion. The conclusion of Theorem 32 for f contradicts the hypotheses of the corollary. $\square$

Remark 35. Therefore, for $\theta \equiv 1$ and $q = n$ we arrive at Corollary 2.12 of [8].

[4)] In other words, $\bigcup_{k=1}^\infty A_k = \mathbb{R}^n$ and $A_k \subset A_{k+1}$ for $k \in N$.

4. Classification of Riemannian Manifolds

Following [31] and given two Riemannian manifolds $\mathbb{M}$ and $\mathbb{N}$, define the classes of mappings $W^1_{p,\mathrm{loc}}(\mathbb{M},\mathbb{N})$. Below we integrate with respect to the Hausdorff measures on the Riemannian manifolds determined by the corresponding Riemannian metrics, denoting by $d\mu$ the Hausdorff measure on $\mathbb{M}$ and by $d\nu$, that on $\mathbb{N}$.

Definition 36. A mapping $f:\mathbb{M}\to\mathbb{N}$ between two Riemannian spaces $\mathbb{M}$ and $\mathbb{N}$ is in $W^1_{p,\mathrm{loc}}(\mathbb{M},\mathbb{N})$ whenever the function $[f]_y:\mathbb{M}\to\mathbb{R}$ defined as $[f]_y(x)=\mathrm{loc}(f(x),y)$ belongs to $L_{p,\mathrm{loc}}(\mathbb{M},\mathbb{R})$ for all $y\in\mathbb{N}$ and there exists $g\in L_{p,\mathrm{loc}}(M,\mathbb{R})$ such that $|\nabla[f]_y|(x)\le g(x)$ a.e. on $\mathbb{M}$ for all $y\in\mathbb{N}$.

The formal differential $Df(x):T_xM\to T_{f(x)}N$ of a given mapping in $W^1_{p,\mathrm{loc}}(\mathbb{M},\mathbb{N})$ is defined almost everywhere.

Assume henceforth that the manifolds are oriented and have the same dimension n. The condition that the manifolds are oriented enables us to define the class of mappings with bounded θ-weighted interior (q,p)-distortion by analogy with Definition 1.

Definition 37. A mapping $f:\mathbb{M}\to\mathbb{N}$ is in $\mathscr{I}\mathscr{D}(\mathbb{M},\mathbb{N};q,p;\theta,1)$, where $n-1\le q\le p<\infty$, and is called a *mapping with bounded θ-weighted interior (q,p)-distortion*, whenever

(1) f is continuous, open, and discrete;
(2) f lies in the Sobolev space $W^1_{n-1,\mathrm{loc}}(\mathbb{M})$;
(3) the Jacobian $J(x,f)$ is nonnegative for almost all $x\in\mathbb{M}$;
(4) f has finite codistortion: $\operatorname{adj}Df(x)=0$ a.e. on $Z=\{x\in\Omega:\det Df(x)=0\}$;
(5) the function

$$\mathbb{M}\ni x\mapsto \mathscr{K}^{\theta,1}_{q,p}(x,f)=\begin{cases}\dfrac{\theta^{\frac{n-1}{q}}(x)|\operatorname{adj}Df(x)|}{J(x,f)^{\frac{n-1}{p}}} & \text{if } J(x,f)\ne0,\\ 0 & \text{otherwise}\end{cases}$$

of the local θ-weighted (q,p)-distortion is in $L_\varrho(\Omega)$, where ϱ is found from the condition $\frac{1}{\varrho}=\frac{n-1}{q}-\frac{n-1}{p}$, and $\varrho=\infty$ for $q=p$.

Put $\mathscr{K}^{\theta,1}_{q,p}(f;\mathbb{M})=\big\|\mathscr{K}^{\theta,1}_{q,p}(\cdot,f)\mid L_\varrho(\mathbb{M})\big\|$.

For $p=q$ and $\omega\equiv1$ this class of mappings was studied in [32] under some additional assumptions.

By analogy with the above, for an arbitrary smooth function $u:\mathbb{M}\to\mathbb{R}$, $u\in C^1_0(\mathbb{M})$, we define the pulled-back function $w:f(\mathbb{M})\to\mathbb{R}$; see Definition 14. As in the proof of Lemma 23, we infer that $w\in\mathrm{ACL}(f(\mathbb{M}))$: since Property $\mathrm{ACL}(f(\mathbb{M}))$ is local, for every point $y\in f(M)$, by choosing a sufficiently small neighborhood covered by one chart we reduce the problem to the case of the Euclidean space. Furthermore, as in Theorem 17, in the case $n-1<q\le p<\infty$ we can generalize the estimate (15) for the pulled-back function to Riemannian manifolds:

$$\Bigg(\int\limits_{f(D)}|\nabla w|^s(y)\,d\nu\Bigg)^{\frac1s}\le\big\|\mathscr{K}^{\theta,1}_{q,p}(\cdot,f)\mid L_\varrho(D)\big\|\Bigg(\int\limits_{D}|\nabla u|^r(x)\omega(x)\,d\mu\Bigg)^{\frac1r}\tag{43}$$

for every $u\in C^1_0(D)$, where $D\Subset\mathbb{M}$. Here $r=\frac{q}{q-(n-1)}$ and $s=\frac{p}{p-(n-1)}$, while $\omega=\theta^{-\frac{n-1}{q-(n-1)}}\in L_{1,\mathrm{loc}}(\mathbb{M})$.

We can apply these results to classification of manifolds. Riemannian spaces can be classified in accordance with the behavior of the s-capacity of their compact sets.

The main method of this classification relies on nonlinear potential theory. Recall that a *condenser* in $\mathbb{M}$ is a pair (A,C), where $A\subset\mathbb{M}$ is an open connected set, while $C\subset A$ is compact. The ω-weighted *p-capacity* of $E=(A,C)$ is defined as

$$\operatorname{cap}^\omega_p(E)=\inf\Bigg\{\int\limits_A|\nabla u|^p\omega(x)\,d\mu : u\in C_0(A)\cap\mathrm{ACL}(A)\text{ and }u\ge1\text{ on }C\Bigg\}.\tag{44}$$

Here $C_0(A)$ is the collection of compactly supported continuous functions on A. If $\omega \equiv 1$ then we use the notation $\operatorname{cap}_p(E)$ instead of $\operatorname{cap}^1_p(E)$.

Definition 38. A manifold $\mathbb{M}$ is called *r-parabolic* (or *ω-r-parabolic*) whenever $\operatorname{cap}_r(\mathbb{M}, C) = 0$ (respectively $\operatorname{cap}^{\omega}_r(\mathbb{M}, C) = 0$) for each compact set $C \subset \mathbb{M}$ with nonempty interior. Otherwise, $\mathbb{M}$ is called *r-hyperbolic* (or *ω-r-hyperbolic*).

Let us prove a theorem on the behavior of parabolicity under the action of mappings with bounded $(\theta, 1)$-weighted (p, q)-distortion.

Theorem 39. *Consider a mapping $f : \mathbb{M} \to \mathbb{N}$ of Riemannian manifolds in $\mathscr{I}\mathscr{D}(\mathbb{M}, \mathbb{N}; q, p; \theta, 1)$, where $n-1 < q \leq p < \infty$. If $r = \frac{q}{q-(n-1)}$ and $s = \frac{p}{p-(n-1)}$, the weight function $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$ is summable on $\mathbb{M}$, while the Riemannian manifold $\mathbb{M}$ is ω-r-parabolic, then $\mathbb{N}$ is s-parabolic.*

Proof. Take a compact set $C \subset \mathbb{N}$ and verify that

$$\operatorname{cap}_s(\mathbb{N}, C) = 0. \tag{45}$$

Observe that $f^{-1}(C)$ is a closed but not necessarily compact set. However, we can exhaust $f^{-1}(C)$ with compact sets:

$$f^{-1}(C) = \bigcup_{l=1}^{\infty} K_l,$$

where each set $K_l \subset \mathbb{M}$ is compact and $K_l \subset K_{l+1}$ for $l \in \mathbb{N}$.

Fix $l \in N$. Since the Riemannian manifold $\mathbb{M}$ is ω-r-parabolic, in accordance with Definition (44) there is a sequence of functions $u_m \in C_0(\mathbb{M}) \cap \mathrm{ACL}(\mathbb{M})$ such that $u_m \geq 1$ on K_l and

$$\biggl(\int\limits_{\mathbb{M}} |\nabla u_m|^r(x)\omega(x)\,d\mu\biggr)^{\frac{1}{r}} \leq \frac{1}{m}.$$

Associate to u_m the pulled-back function w_m and insert both functions into (43). This yields

$$\begin{aligned}(\operatorname{cap}_s(\mathbb{N}, f(K_l)))^{1/s} &\leq \biggl(\int\limits_{\mathbb{N}} |\nabla w_m|^s(y)\,d\nu\biggr)^{\frac{1}{s}} \\ &\leq \bigl\|\mathscr{K}^{\theta,1}_{q,p}(\cdot, f) \mid L_{\varrho}(\mathbb{M})\bigr\| \biggl(\int\limits_{\mathbb{M}} |\nabla u_m|^r(x)\omega(x)\,d\mu\biggr)^{\frac{1}{r}} \leq \frac{\mathscr{K}^{\omega,1}_{q,p}(f; \mathbb{M})}{m}.\end{aligned} \tag{46}$$

Since $m \in \mathbb{N}$ is arbitrary, it follows that $\operatorname{cap}_s(\mathbb{N}, f(K_l)) = 0$.

Since $\bigcup_{l=1}^{\infty} f(K_l) = C$, the semiadditivity of capacity implies (45). Thus, if $\mathbb{M}$ is ω-r-parabolic then $\mathbb{N}$ is s-parabolic too. $\square$

Corollary 40. *Consider a mapping $f : \mathbb{M} \to \mathbb{N}$ of Riemannian manifolds in $\mathscr{I}\mathscr{D}(\mathbb{M}, \mathbb{N}; q, p; \theta, 1)$, where $n-1 < q \leq p < \infty$. If $r = \frac{q}{q-(n-1)}$ and $s = \frac{p}{p-(n-1)}$, the weight function $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$ is summable on $\mathbb{M}$, while the Riemannian manifold $\mathbb{N}$ is r-hyperbolic, then $\mathbb{M}$ is ω-s-hyperbolic.*

Remark 41. Theorem 39 generalizes Theorem A of [32] proved under the condition that $f : \mathbb{M} \to \mathbb{N}$ enjoys Luzin's $\mathscr{N}$-property, $q = p$, and $\omega \equiv 1$.

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