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QUASIEQUATIONAL BASES OF CANTOR ALGEBRAS

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Abstract: There are continuum many quasivarieties of Cantor algebras having an ω-independent quasiequational basis but no independent quasiequational basis whose intersection does have an independent quasiequational basis.

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§ 1. Introduction

The question about existence of independent quasiequational bases for algebraic structures having no finite quasiequational bases was raised in the papers of Maltsev [1] and Tarski [2]. Kartashov proved in [3] that there are continuum many quasivarieties of unars having no independent quasiequational basis. Gorbunov constructed in [4] an example of a quasivariety with two unary operations in the signature which has no independent quasiequational basis, but which has an ω-independent quasiequational basis. He also found a finite unary algebra that has neither an independent quasiequational basis nor an ω-independent quasiequational basis [5, Corollary 6.4.2]. The results analogous to those mentioned above were obtained also for the variety of pointed abelian groups and for the quasivariety of unoriented graphs containing nonbipartite graphs in the paper by Kravchenko and Yakovlev [6], as well as in the paper of Yakovlev and the first author [7], for the variety of differential groupoids and for a certain quasivariety of unary algebras with two operations in the signature—in the paper of Kravchenko, Nurakunov, and Schwidefsky [8]; cf. also [9]. The results on independent axiomatizability were obtained also for some quasivarieties of groups in the papers of Budkin [10–12] and for some quasivarieties of graphs in the paper of Sizyi [13]. In the paper of Kravchenko, Nurakunov, and Schwidefsky [14], a general sufficient condition was found for existence of continuum many quasivarieties possessing the above properties.

In this paper, we continue the study of independent axiomatizability of quasivarieties. The main result is the construction of continuum many quasivarieties of Cantor algebras which have no independent quasiequational basis, have an ω-independent quasiequational basis, and whose intersection does have an independent quasiequational basis; cf. Theorem 1 and Corollary 1. We note that the statement concerning existence of continuum many quasivarieties of Cantor algebras with no independent quasiequational basis and with an ω-independent quasiequational basis is a corollary of the general result of Kravchenko, Nurakunov, and Schwidefsky [14] mentioned above. Here, we give a direct proof of this fact which nonetheless employs some ideas from [8, 14]. Moreover, we present quasiequational bases of the considered quasivarieties in explicit form. We note also that Q-universality of the variety $\mathbf{C}_{mn}$ of Cantor algebras was established in the paper of Sheremet [15].

§ 2. Definitions and Auxiliaries

In this section, we present necessary information from the theory of quasivarieties. All definitions that do not appear here can be found in the monograph of Gorbunov [5].

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Let $P = \{p_0, p_1, \dots, p_n, \dots\}$ denote the set of primes ordered in the natural way and let $\mathscr{P}_{\mathrm{fin}}(X)$ be the set of finite subsets of a set X. The set of nonnegative integers we denote by ω. Given a nonempty finite set $F \subseteq \omega$, we put $[F] = \prod_{i \in F} p_i$. Also, $[\varnothing] = 1$.

Definition 1. A sentence φ of type σ is a *quasi-identity*, if φ is of the form

$$\forall\, x_1 \dots x_n \underset{i<k}{\&}\, p_i(x_1, \dots, x_n) \;\longrightarrow\; p(x_1, \dots, x_n)$$

where $p_i(x_1, \dots, x_n)$, ... , $p_{k_1}(x_1, \dots, x_n)$, and $p(x_1, \dots, x_n)$ are atomic formulas of type σ.

In other words, φ is a quasi-identity if φ is a universal Horn sentence with a nonempty positive matrix.

Definition 2. Let $\mathbf{K}_0$ be a class of algebraic structures of an arbitrary fixed type σ. A class $\mathbf{K} \subseteq \mathbf{K}_0$ is a $\mathbf{K}_0$-*quasivariety*, if there is a set Σ of quasi-identities of type σ such that $\mathbf{K}$ coincides with the class of all structures from $\mathbf{K}_0$ in which all quasi-identities from Σ hold; i.e.,

$$\mathbf{K} = \mathrm{Mod}(\Sigma) \cap \mathbf{K}_0 = \{\mathscr{A} \in \mathbf{K}_0 \mid \mathscr{A} \models \Sigma\}.$$

In this case, the set of quasi-identities Σ is a *quasiequational basis* of $\mathbf{K}$ *relative to* $\mathbf{K}_0$.

For an arbitrary class $\mathbf{K}$, the set of all $\mathbf{K}$-quasivarieties forms a complete lattice under the set-theoretic inclusion which is called the *lattice of* $\mathbf{K}$-*quasivarieties* and is denoted by $\mathrm{Lq}(\mathbf{K})$.

Definition 3. A quasiequational basis Σ of a class $\mathbf{K}$ relative to a class $\mathbf{K}_0$ is *independent*, if $\mathbf{K} \neq \mathrm{Mod}\big(\Sigma \backslash \{\varphi\}\big) \cap \mathbf{K}_0$ for every $\varphi \in \Sigma$.

In other words, a quasiequational basis Σ of $\mathbf{K}$ relative to $\mathbf{K}_0$ is independent if, for any $\varphi \in \Sigma$, there is an algebraic structure $\mathscr{A} \in \mathbf{K}_0$ such that $\mathscr{A} \models \Sigma \backslash \{\varphi\}$ and $\mathscr{A} \models \neg\varphi$.

Definition 4. A quasiequational basis Σ of $\mathbf{K}$ relative to $\mathbf{K}_0$ is ω-*independent* if $\Sigma = \bigcup_{n<\omega} \Sigma_n$, where $\Sigma_n \cap \Sigma_m = \varnothing$ for any different m, $n < \omega$ and $\mathbf{K} \neq \mathrm{Mod}\big(\Sigma \backslash \Sigma_n\big) \cap \mathbf{K}_0$ for any $n < \omega$.

In other words, a quasiequational basis Σ of $\mathbf{K}$ relative to $\mathbf{K}_0$ is ω-independent, if for any $n < \omega$ there is an algebraic structure $\mathscr{A}_n \in \mathbf{K}_0$ such that $\mathscr{A}_n \models \Sigma_m$ for all $m \neq n$ and $\mathscr{A}_n \models \neg\varphi$ for some $\varphi \in \Sigma_n$.

A sufficient condition for a quasivariety to have no independent quasiequational basis can be obtained from the statement by Gorbunov in [4]:

Proposition 1 [5, Proposition 6.3.1]. *Let* $\mathbf{K}$ *be an arbitrary quasivariety and let* $\mathbf{K}_0$ *be a proper subquasivariety of* $\mathbf{K}$. *If* $\mathbf{K}_0$ *has an infinite independent quasiequational basis relative to* $\mathbf{K}$, *then for each quasivariety* $\mathbf{K}_1 \in \mathrm{Lq}(\mathbf{K})$ *which includes* $\mathbf{K}_0$ *and which is finitely axiomatizable relative to* $\mathbf{K}$, *the number of upper covers of* $\mathbf{K}_0$ *in* $\mathrm{Lq}(\mathbf{K}_1)$ *is infinite.*

§ 3. Quasivarieties of Cantor Algebras

We recall here some basic definitions that concern Cantor algebras and present the auxiliary statements that we will use in the proof of the main result; cf. Theorem 1.

Definition 5. Let $0 < m < n < \omega$. The variety $\mathbf{C}_{mn}$ of algebraic structures of type $\sigma_{mn} = \{\lambda_1, \dots, \lambda_m, \alpha_1, \dots, \alpha_n\}$, consisting of m n-ary function symbols $\lambda_1, \dots, \lambda_m$ and n m-ary function symbols $\alpha_1, \dots, \alpha_n$ is defined by the identities

$$\forall\, x_1 \dots x_m \quad \lambda_i\big(\alpha_1(x_1, \dots x_m), \dots, \alpha_n(x_1, \dots x_m)\big) = x_i, \quad 1 \leqslant i \leqslant m;$$

$$\forall\, x_1 \dots x_n \quad \alpha_j\big(\lambda_1(x_1, \dots x_n), \dots, \lambda_m(x_1, \dots x_n)\big) = x_j, \quad 1 \leqslant j \leqslant n.$$

A *Cantor algebra* is an arbitrary structure from $\mathbf{C}_{mn}$. (We assume usually that m and n are fixed and can be easily recovered from the context.)

We now present a construction from the papers of Smirnov [16, 17] which we will need later. We assume here that $m = 1$ and $\lambda = \lambda_1$.

We put $\sigma_{1n}^* = \{\alpha_1, \dots, \alpha_n\}$ and let $\mathbf{R}$ denote the quasivariety of algebraic structures of type σ_{1n}^* which is defined by the quasi-identity

$$\forall\, xy \quad \alpha_1(x) = \alpha_1(y) \;\&\; \cdots \;\&\; \alpha_n(x) = \alpha_n(y) \;\longrightarrow\; x = y.$$

It is not hard to see that, for every structure $\mathscr{C} \in \mathbf{C}_{1n}$, its σ_{1n}^*-reduct $\mathscr{C}^*$ belongs to $\mathbf{R}$. Given a structure $\mathscr{A} \in \mathbf{R}$ and $a,\, a_1,\, \dots,\, a_n \in A$, we put

$$A^{(0)} = A, \quad \alpha_1^{(0)} = \alpha_1, \quad \dots, \quad \alpha_n^{(0)} = \alpha_n;$$
$$\lambda^{(0)}(a_1, \dots, a_n) = a \text{ if and only if } \alpha_1(a) = a_1,\ \dots,\ \alpha_n(a) = a_n.$$

Let $k < \omega$. Suppose that we have already defined the set $A^{(k)}$ and the operations $\alpha_1^{(k)}$, ... , $\alpha_n^{(k)}$, as well as the partial operation $\lambda^{(k)}$ on $A^{(k)}$. We put

$$A^{(k+1)} = A^{(k)} \cup \bigl\{\lambda^{(k+1)}(b_1, \dots, b_n) \mid b_1, \dots, b_n \in A^{(k)},\ \lambda^{(k)}(b_1, \dots, b_n) \text{ is not defined}\bigr\};$$

$$\alpha_i^{(k+1)}\bigl(\lambda^{(k+1)}(b_1, \dots, b_n)\bigr) = b_i \text{ for all } b_1, \dots, b_n \in A^{(k)};$$

$$\alpha_i^{(k+1)}(b) = \alpha_i^{(k)}(b) \text{ for all } b \in A^{(k)};$$

$$\lambda^{(k+1)}(b_1, \dots, b_n) = b \text{ if and only if } \alpha_1^{(k+1)}(b) = b_1,\ \dots,\ \alpha_n^{(k+1)}(b) = b_n,$$

$$\text{where } b, b_1, \dots, b_n \in A^{(k+1)}.$$

Finally, we put

$$F(\mathscr{A}) = \bigcup_{k<\omega} A^{(k)};$$

$$\lambda(b_1, \dots, b_n) = \lambda^{(k+1)}(b_1, \dots, b_n), \text{ where } \{b_1, \dots, b_n\} \subseteq A^{(k)};$$

$$\alpha_i(b) = \alpha_i^{(k)}(b), \text{ where } 1 \leqslant i \leqslant n \text{ and } b \in A^{(k)}.$$

It is easy to see that for all $b, b_1, \dots, b_n \in A^{(k)}$ the value $\lambda(b_1, \dots, b_n)$ is defined; moreover, $\lambda(b_1, \dots, b_n) \in A^{(k+1)}$. Therefore, the algebraic structure $\mathscr{F}(\mathscr{A}) = \langle F(\mathscr{A}); \sigma_{1n}\rangle$ is defined. It follows from the construction that $\mathscr{F}(\mathscr{A}) \in \mathbf{C}_{1n}$.

The next lemma was established in the paper of Belkin [18].

Lemma 1. *Let $\mathscr{C} \in \mathbf{C}_{1n}$ and let a set X generate $\mathscr{C}$. Let $\mathscr{A}$ denote the σ_{1n}^*-substructure in $\mathscr{C}^*$ generated by X. Then $\mathscr{C} \cong \mathscr{F}(\mathscr{A})$.*

Given a prime number p, let $\mathscr{B}_p$ denote the primary cyclic group of exponent p (with respect to addition $+$). Granted a nonempty finite subset $F \subseteq \omega\backslash\{0, \dots, n\}$ and a mapping $f : F \to \omega\backslash\{0\}$, we put

$$G = (F, f), \quad n(G) = \prod_{i\in F} p_i, \quad \mathscr{B}_G = \bigoplus_{i\in F} \bigoplus_{j<f(i)} \mathscr{B}_{p_i}.$$

If $i \in F$ and $j < f(i)$, then we can choose n generators a_{ijt}, $1 \leqslant t \leqslant n$, of the group $\mathscr{B}_{p_i}$ in such a way that a_1, ... , a_n are pairwise distinct, where $a_t = \bigl(a_{ijt} \mid i \in F,\ j < f(i)\bigr)$, $1 \leqslant t \leqslant n$. For all $t \in \{1, \dots, n\}$ and an arbitrary $b = \bigl(b_{ij} \mid i \in F,\ j < f(i)\bigr) \in B_G$, we put

$$\alpha_t(b) = \bigl(b_{ij} + a_{ijt} \mid i \in F,\ j < f(i)\bigr).$$

REMARK 1. It is not hard to see that, for every pair G, the structure $\mathscr{A}_G = \langle B_G; \alpha_1, \dots, \alpha_n \rangle$ belongs to the quasivariety $\mathbf{R}$; moreover, $\alpha_{t_0}\alpha_{t_1}(a) = \alpha_{t_1}\alpha_{t_0}(a)$ for all t_0, $t_1 \in \{1, \dots, n\}$ and $a \in A_G = B_G$.

Furthermore, let $\mathscr{A}_\varnothing$ denote the trivial structure of type σ^*_{1n} and let $\mathscr{C}_G = \mathscr{F}(\mathscr{A}_G)$.

The next statement follows from [15, Lemma 2.6].

Lemma 2. *For every pair $G = (F, f)$, the following congruence lattices are isomorphic:* Con $\mathscr{B}_G \cong$ Con $\mathscr{C}_G$. *In particular, if the abelian group $\mathscr{B}_G$ is the subdirect product of groups $\mathscr{B}_{G_0}$ and $\mathscr{B}_{G_1}$, then the Cantor algebra $\mathscr{C}_G$ is the subdirect product of Cantor algebras $\mathscr{C}_{G_0}$ and $\mathscr{C}_{G_1}$.*

Lemma 3. *Let $G = (F, f)$ and $a \in C_G$. If $i \in \{1, \dots, n\}$ and $k > 0$, then $\alpha_i^k(a) = a$ if and only if $a \in C_G^{(0)}$ and $n(G)$ divides k.*

PROOF. Let $\alpha_i^k(a) = a$ and $a \notin C_G^{(0)}$; we choose a minimal m with the property that $a \in C_G^{(m)}$. According to our assumption, $m > 0$. Therefore, $\alpha_i(a) \in C_G^{(m-1)} \neq a$, whence $\alpha_i^k(a) \in C_G^{(m-1)}$, but $a \in C_G^{(m)} \setminus C_G^{(m-1)}$ which is impossible. All other statements are trivial.

§ 4. Quasi-Identities φ_H^F

Given $\xi \in \{1, \dots, n\}^*$ and a variable x, we define the term $\xi(x)$ of type σ^*_{1n} by induction on the length of ξ:

$$\xi(x) = x, \quad \text{if } \xi \text{ is empty;}$$

$$\xi(x) = \alpha_i\big(\xi'(x)\big), \quad \text{if } \xi = i\xi', \text{ where } 1 \le i \le n.$$

Granted $i \in \{1, \dots, n\}$ and $\xi \in \{1, \dots, n\}^*$, let $\xi_i^\sharp$ denote the number of occurrences of i in the word ξ. We also put

$$\Xi_s = \big\{\xi \in \{1, \dots, n\}^* \mid \xi_1^\sharp \leqslant s, \ \dots, \ \xi_n^\sharp \leqslant s\big\}$$

for all $s < \omega$. For arbitrary finite sets F and H such that $H \subset F \subset \omega$, we consider the following quasi-identity to be denoted by φ_H^F:

$$\forall x \quad \mathop{\&}_{1 \le i \le n} \big[\alpha_i^{[F]}(x) = x\big] \ \& \mathop{\&}_{\xi \in \Xi_{[F]}} \big[\xi(x) = \alpha_1^{\xi_1^\sharp} \dots \alpha_n^{\xi_n^\sharp}(x)\big] \longrightarrow \big[\alpha_1^{[H]}(x) = x\big].$$

Lemma 4. *Let finite sets F, H and a structure $\mathscr{C} \in \mathbf{C}_{1n}$ be such that $H \subset F \subset \omega$ and $\mathscr{C} \not\models \varphi_H^F$. Then there are a nonempty finite set $F_0 \subseteq \omega \setminus \{0, \dots, n\}$ and a mapping $f : F_0 \to \omega \setminus \{0\}$ such that the following hold for the pair $G = (F_0, f)$:*

(1) $F_0 \subseteq F$ and $F_0 \nsubseteq H$;

(2) $\mathscr{C}_G \le \mathscr{C}$.

PROOF. From the condition $\mathscr{C} \not\models \varphi_H^F$ existence of $a \in C$ follows such that $\alpha_i^{[F]}(a) = a$ for all $i \in \{1, \dots, n\}$ and $\xi(a) = \alpha_1^{\xi_1^\sharp} \dots \alpha_n^{\xi_n^\sharp}(a)$ for all $\xi \in \Xi_{[F]}$, but $\alpha_1^{[H]}(a) \neq a$. We consider the substructure $\mathscr{A} \le \mathscr{C}^*$ generated by a. Since $\xi(a) = \alpha_1^{\xi_1^\sharp} \dots \alpha_n^{\xi_n^\sharp}(a)$, where $\xi \in \Xi_{[F]}$ is an arbitrary word, it is not hard to see that

$$A = \big\{\alpha_1^{k_1} \dots \alpha_n^{k_n}(a) \mid k_1, \dots, k_n < [F]\big\}.$$

Moreover, in the structure $\mathscr{A}$ we have

$$\forall x \quad \mathop{\&}_{1 \le i \le n} \big[\alpha_i^{[F]}(x) = x\big].$$

We put

$$0 = a;$$

$$\alpha_1^{k_1} \dots \alpha_n^{k_n}(a) + \alpha_1^{m_1} \dots \alpha_n^{m_n}(a) = \alpha_1^{(k_1+m_1) \bmod [F]} \dots \alpha_n^{(k_n+m_n) \bmod [F]}(a).$$

It is straightforward to verify that $\mathscr{A}^g = \langle A; +, 0\rangle$ is an abelian group with generators $\{\alpha_1(a), \dots, \alpha_n(a)\}$ which satisfies the identity $[F] \cdot x = 0$ for all x. Hence, $\mathscr{A}^g$ is isomorphic to the direct sum of finitely many primary cyclic groups whose indices belong to $\{p_i \mid i \in F\}$. In other words, $\mathscr{A}^g \cong \mathscr{B}_G$ and $\mathscr{A} \cong \mathscr{A}_G$ for some nonempty finite set $F_0 \subseteq \omega \backslash \{0, \dots, n\}$ and some mapping $f : F_0 \to \omega \backslash \{0\}$ such that $G = (F_0, f)$ and $F_0 \subseteq F$.

Consider the substructure $\mathscr{D} \leq \mathscr{C}$ generated by a. We have by Lemma 1 that

$$\mathscr{D} \cong \mathscr{F}(\mathscr{A}) \cong \mathscr{F}(\mathscr{A}_G) = \mathscr{C}_G.$$

Finally, according to Lemma 3, there is a natural $k \in F_0 \backslash H$.

Lemma 5. *Let $G = (F_0, f)$, and let finite sets F and H be such that $F \subset H \subset \omega$. Then $\mathscr{C}_G \models \varphi_H^F$ if and only if either $n(G)$ divides $[H]$ or $n(G)$ does not divide $[F]$.*

Proof. Assume first that $\mathscr{C}_G \models \varphi_H^F$. By Remark 1 and Lemma 3, this means that, for an arbitrary element $a \in C_G^{(0)}$, either $\alpha_1^{[H]}(a) = a$ or $\alpha_1^{[F]}(a) \neq a$. In the first case, $n(G)$ divides $[H]$ by Lemma 3. In the second case, $n(G)$ does not divide $[F]$ according to Remark 1 and Lemma 3.

Conversely, choose an arbitrary element $a \in C_G$. If $a \notin C_G^{(0)}$ or $n(G)$ does not divide $[F]$, then the premise of the quasi-identity φ_H^F fails at a by Lemma 3. If $a \in C_G^{(0)}$ and $n(G)$ divides $[H]$, then by Lemma 3 the conclusion of the quasi-identity φ_H^F holds at a, which was our desired conclusion.

§ 5. The Main Result

Theorem 1. *For every natural $n > 1$, there are continuum many subquasivarieties of the quasivariety $\mathbf{C}_{1n}$ of Cantor algebras which have no independent quasiequational basis, which have an ω-independent quasiequational basis, and whose intersection nevertheless has an independent quasiequational basis.*

Proof. We fix an infinite proper subset I of $\omega \backslash \{0, \dots, n\}$. Put

$$\Sigma_I = \{\varphi_{F \cap I}^F \mid F \in \mathscr{P}_{\text{fin}}(\omega \backslash \{0, \dots, n\}),\ F \neq \varnothing\}; \quad \mathbf{K}_I = \mathbf{C}_{1m} \cap \operatorname{Mod}(\Sigma_I).$$

Claim 1. *Let a finite set $F \subset \omega \backslash \{0, \dots, n\}$ be nonempty and let $G = (F, f)$. The containment $\mathscr{C}_G \in \mathbf{K}_I$ holds if and only if $F \subseteq I$.*

Proof. Indeed, if $\mathscr{C}_G \in \mathbf{K}_I$ then $\mathscr{C}_G \models \varphi_{F \cap I}^F$. Since $n(G) = [F]$, we get by Lemma 5 that $n(G)$ divides $[F \cap I]$. Therefore, $F = F \cap I$, whence $F \subseteq I$.

Conversely, suppose that $F \subseteq I$. We show that $\mathscr{C}_G \models \varphi_{H \cap I}^H$ for every nonempty finite set $H \subset \omega \backslash \{0, \dots, n\}$. Suppose that the premise of the quasi-identity $\varphi_{H \cap I}^H$ holds at some $a \in C_G$. In this case, $\alpha_1^{[H]}(a) = a$. Then $a \in C_G^{(0)}$ and $[F]$ divides $[H]$ by Lemma 3. It follows that $F \subseteq H \cap I$. In this case, we get that $[F]$ divides $[H \cap I]$, whence $\alpha_1^{[H \cap I]}(a) = a$ by Lemma 3; i.e., the conclusion of the quasi-identity $\varphi_{H \cap I}^H$ also holds at a. Thus we have proved that $\mathscr{C}_G \models \Sigma_I$.

Claim 2. *The quasivariety $\mathbf{K}_I$ consists of structures into which none of the structures $\mathscr{C}_{(F,f)}$ embeds as a substructure, where $F \subseteq \omega \backslash \{0, \dots, n\}$ is finite and $F \nsubseteq I$.*

Proof. It follows from Claim 1 that if $\mathscr{C}_{(F,f)} \leq \mathscr{A} \in \mathbf{K}_I$ then $F \subseteq I$. Suppose now that $\mathscr{A} \notin \mathbf{K}_I$. Then there is a nonempty finite set $H \subset \omega \backslash \{0, \dots, n\}$ such that $\mathscr{A} \not\models \varphi_{H \cap I}^H$. We get by Lemma 4 that $\mathscr{C}_{(F,f)} \leq \mathscr{A}$ for some finite set $F \subseteq \omega \backslash \{0, \dots, n\}$ such that $F \nsubseteq I$.

Claim 3. *From $\mathbf{K}_I = \mathbf{K}_J$ it follows that $I = J$.*

Proof. According to Claim 1, for arbitrary $i \in \omega\backslash\{0,\dots,n\}$, the containment $\mathscr{C}_{(\{p_i\},1)} \in \mathbf{K}_I$ holds if and only if $i \in I$. From here, we get the desired conclusion.

Claim 4. *The quasivariety $\mathbf{K}_I$ has no upper cover in* $\mathrm{Lq}(\mathbf{C}_{1n})$.

Proof. Assume that the inclusion $\mathbf{K}_I \subset \mathbf{B}$ is proper in the lattice $\mathrm{Lq}(\mathbf{K})$; let $\mathscr{C} \in \mathbf{B} \setminus \mathbf{K}_I$. This means that there is a nonempty finite set $F \subseteq \omega\backslash\{0,\dots,n\}$ such that $\mathscr{C} \not\models \varphi^F_{F\cap I}$. By Lemma 4, there is a pair $G = (F_0, f)$ such that $F_0 \subseteq F$, $F_0 \nsubseteq I$, and $\mathscr{C}_G \le \mathscr{C}$. Let $k \in F_0\backslash I$. Using Lemma 3, we get that $\mathscr{C}_G \in \mathbf{B}\backslash\mathbf{K}_I$. Since the set I is infinite, there is a positive integer $m \in I\backslash F$. Then $\mathscr{C}_{(p_m,1)} \in \mathbf{K}_I$ by Claim 1, whence $\mathscr{C}_{(p_m,1)} \times \mathscr{C}_G \in \mathbf{B}$. We put $G' = \big(F_0 \cup \{m\}, f_0\big)$, where $f_0(i) = f(i)$ for any $i \in F_0$ and $f_0(m) = 1$. According to Lemma 2, $\mathscr{C}_{G'} \le_s \mathscr{C}_{(p_m,1)} \times \mathscr{C}_G$, whence $\mathscr{C}_{G'} \in \mathbf{B}$. Moreover, $\mathscr{C}_{G'} \notin \mathbf{K}_I$ by Claim 1, as $F_0 \nsubseteq I$.

We put $\mathbf{B}' = \mathbf{B} \cap \mathrm{Mod}\big(\varphi^{F_0}_{F_0\backslash\{k\}}\big)$. We show first that $\mathbf{K}_I \subseteq \mathbf{B}'$. To this end, it suffices to verify that $\mathbf{K}_I \models \varphi^{F_0}_{F_0\backslash\{k\}}$. Indeed, let $\mathscr{A} \in \mathbf{K}_I$, let $a \in A$, and let the premise of the quasi-identity $\varphi^{F_0}_{F_0\backslash\{k\}}$ hold at a. This means that $\alpha_1^{[F_0]}(a) = \dots = \alpha_n^{[F_0]}(a) = a$ and $\xi(a) = \alpha_1^{\xi_1^\sharp}\dots\alpha_n^{\xi_n^\sharp}(a)$ for any word $\xi \in \Xi_{[F_0]}$. Since $\mathscr{A} \models \varphi^{F_0}_{F_0\cap I}$ and the premises of the two quasi-identities $\varphi^{F_0}_{F_0\backslash\{k\}}$ and $\varphi^{F_0}_{F_0\cap I}$ coincide, the equality $\alpha_1^{[F_0\cap I]}(a) = a$ holds. As $k \notin I$, we conclude that $F_0\cap I \subseteq F_0\backslash\{k\}$. Consequently, $[F_0\cap I]$ divides $[F_0\backslash\{k\}]$, whence $\alpha_1^{[F_0\backslash\{k\}]}(a) = a$; i.e., the conclusion of the quasi-identity $\varphi^{F_0}_{F_0\backslash\{k\}}$ holds at a, which was to prove. We have therefore that $\mathbf{K}_I \subseteq \mathbf{B}' \subseteq \mathbf{B}$.

Further, if $m \notin F_0$ and p_m divides $n(G')$, we conclude that $n(G') = p_m[F_0]$ does not divide $[F_0]$. Hence $\mathscr{C}_{G'} \models \varphi^{F_0}_{F_0\backslash\{k\}}$ by Lemma 5; i.e., $\mathscr{C}_{G'} \in \mathbf{B}'\backslash\mathbf{K}_I$. We have therefore that $\mathbf{K}_I \subset \mathbf{B}' \subseteq \mathbf{B}$. Moreover, $n(G) = [F_0]$, obviously, does not divide $[F_0\backslash\{k\}]$, whence $\mathscr{C}_G \not\models \varphi^{F_0}_{F_0\backslash\{k\}}$ by Lemma 5. Thus $\mathscr{C}_G \in \mathbf{B}\backslash\mathbf{B}'$; this means that the following proper inclusions hold: $\mathbf{K}_I \subset \mathbf{B}' \subset \mathbf{B}$. For this reason, $\mathbf{B}$ cannot be an upper cover of the quasivariety $\mathbf{K}_I$ in the lattice $\mathrm{Lq}(\mathbf{C}_{1n})$, which proves the claim.

Claim 5. *If the set $\omega\backslash I$ is infinite, then the quasivariety $\mathbf{K}_I$ has an ω-independent quasiequational basis relative to $\mathbf{C}_{1n}$.*

Proof. Consider an arbitrary bijection $g : \omega \to \omega\backslash\big(I \cup \{0,\dots,n\}\big)$. To simplify the notation, given $k < \omega$ and a finite set $F \subseteq \omega\backslash\{0,\dots,n\}$, denote the quasi-identity $\varphi_F^{F\cup\{g(k)\}}$ by ψ^k_F. Put

$$\Psi^k_I = \big\{\psi^k_F \mid F \in \mathscr{P}_{\mathrm{fin}}(\omega\backslash\{0,\dots,n\}),\ g(k) \notin F\big\}, \quad k < \omega;$$

$$\Psi_I = \bigcup_{k<\omega} \Psi_k.$$

We establish first that Ψ_I is a quasiequational basis of $\mathbf{K}_I$.

Indeed, let $k < \omega$ and let $F \in \mathscr{P}_{\mathrm{fin}}\big(\omega\backslash\{0,\dots,n\}\big)$ be such that $g(k) \notin F$. Let also $\mathscr{A} \in \mathbf{K}_I$; i.e., $\mathscr{A} \models \Sigma_I$. If the premise of the quasi-identity ψ^k_F holds at some $a \in A$, then the premise of the quasi-identity $\varphi^{F\cup\{g(k)\}}_{F\cap I} \in \Sigma_I$ holds at a. Since $\mathscr{A} \models \Sigma_I$, we conclude that $\alpha_1^{[F\cap I]}(a) = a$. As $F \cap I \subseteq F$, the integer $[F \cap I]$ divides $[F]$, whence $\alpha_1^{[F]}(a) = a$. This means also that the conclusion of the quasi-identity ψ^k_F holds at a. Therefore, $\mathscr{A} \models \Psi_I$ and thus $\mathbf{K}_I \models \Psi_I$.

Conversely, suppose that $\mathscr{A} \models \Psi_I$ and consider an arbitrary set $F' \in \mathscr{P}_{\mathrm{fin}}\big(\omega\backslash\{0,\dots,n\}\big)$ such that $F' \nsubseteq I$; let $k \in F'\backslash I$ and $F = F'\backslash\{k\}$. Consider also an arbitrary mapping $f : F' \to \omega\backslash\{0\}$ and the Cantor algebra $\mathscr{C}_{(F',f)}$. Choose any element $a \in C^{(0)}_{(F',f)} = B_{(F',f)}$. It is not hard to see that the premise of the quasi-identity ψ^k_F holds at a, but its conclusion, however, fails at a. Therefore, $\mathscr{C}_{(F',f)} \not\models \psi^k_F$ and thus $\mathscr{C}_{(F',f)} \nleq \mathscr{A}$. According to Claim 2, $\mathscr{A} \in \mathbf{K}_I$. So Ψ_I is a quasiequational basis for $\mathbf{K}_I$.

We will demonstrate now that the quasiequational basis $\Psi_I = \bigcup_{k<\omega} \Psi_I^k$ of the quasivariety $\mathbf{K}_I$ is ω-independent. Indeed, fix $k < \omega$ and consider the Cantor algebra $\mathscr{C}_{(\{g(k)\},1)}$. Since $g(k) \notin I \cup \{0, \dots, n\}$, we conclude by Claim 2 that $\mathscr{C}_{(\{g(k)\},1)} \notin \mathbf{K}_I$. It is not hard to see that the premise of the quasi-identity $\psi_\varnothing^k$ holds at $a \in C^{(0)}_{(\{g(k)\},1)} = B_{(\{g(k)\},1)}$. However, its conclusion fails at a. Therefore, $\mathscr{C}_{(\{g(k)\},1)} \not\models \psi_\varnothing^k$.

For any natural $m \neq k$, we show that the Cantor algebra $\mathscr{C}_{(\{g(k)\},1)}$ satisfies any quasi-identity from Ψ_I^m. Indeed, consider an arbitrary set $F \in \mathscr{P}_{\text{fin}}\big(\omega \backslash \{0, \dots, n\}\big)$ and suppose that the premise of the quasi-identity ψ_F^m holds at some $a \in C_{(\{g(k)\},1)}$. This means by Lemma 3 that $a \in C^{(0)}_{(\{g(k)\},1)}$, and the prime $p_{g(k)} = n(\{g(k)\}, 1)$ divides $p_{g(m)} \cdot [F]$. As $m \neq k$ and the mapping g is one-to-one, we conclude that $p_{g(k)}$ divides $[F]$. Since $\alpha_1^{p_{g(k)}}(a) = a$, we get that $\alpha_1^{[F]}(a) = a$; i.e., also the conclusion of the quasi-identity ψ_F^m holds at a. Therefore, $\mathscr{C}_{(\{g(k)\},1)} \models \Psi_I \backslash \Psi_I^k$ and $\mathscr{C}_{(\{g(k)\},1)} \not\models \psi_\varnothing^k \in \Psi_I^k$, which was the desired conclusion.

Let $\mathbf{K}$ denote the intersection of the quasivarieties $\mathbf{K}_I$, where I ranges over the set of all infinite proper subsets of $\omega \backslash \{0, \dots, n\}$. The next statement follows from Claim 2.

Claim 6. *The quasivariety $\mathbf{K}$ consists of the structure in which none of the structures $\mathscr{C}_{(F,f)}$ embeds as a substructure, where the finite set $F \subseteq \omega \backslash \{0, \dots, n\}$ is nonempty.*

Claim 7. *$\mathbf{K}$ has an independent quasiequational basis relative to $\mathbf{C}_{1n}$.*

Proof. Put $F_m = \{k \in \omega \mid n < k \leqslant n + m\}$ for all $m \in \omega$. Show that $\Phi = \big\{\varphi_{F_m}^{F_{m+1}} \mid m \in \omega\big\}$ is an independent quasiequational basis for $\mathbf{K}$ relative to $\mathbf{C}_{1n}$. Indeed, let $\mathscr{A} \not\models \varphi_{F_m}^{F_{m+1}}$ for some $m < \omega$. By Lemma 4, there are a nonempty finite set $F \subseteq \omega \backslash \{0, \dots, n\}$ and a mapping $f : F \to \omega \backslash \{0\}$ such that $F \subseteq F_{m+1}$, $F \nsubseteq F_m$, and $\mathscr{C}_{(F,f)} \leq \mathscr{A}$. By Claim 6 $\mathscr{A} \notin \mathbf{K}$. Conversely, if $\mathscr{A} \notin \mathbf{K}$ then, by Claim 6, $\mathscr{C}_{(F,f)} \leq \mathscr{A}$ for some nonempty finite set $F \subseteq \omega \backslash \{0, \dots, n\}$ and some mapping $f : F \to \omega \backslash \{0\}$. Let m denote the number of the biggest prime belonging to F. According to our choice of F, $m > n$. In this case, $F \subseteq F_{m-n}$. Choose an arbitrary $a \in C^{(0)}_{(F,f)} = B_{(F,f)}$. It is not hard to see that the premise of the quasi-identity $\varphi_{F_{m-n-1}}^{F_{m-n}}$ holds at a; however, its conclusion fails at a. Therefore, $\mathscr{A} \not\models \varphi_{F_{m-n-1}}^{F_{m-n}}$. Thus we have shown that Φ is a quasiequational basis of $\mathbf{K}$.

We show that this basis is independent. Indeed, let $k \in \omega$. We note that the premise of the quasi-identity $\varphi_{F_k}^{F_{k+1}}$ holds at every $a \in C^{(0)}_{(F_{k+1},1)} = B_{(F_{k+1},1)}$, whereas its conclusion fails. Therefore, $\mathscr{C}_{(F_{k+1},1)} \not\models \varphi_{F_k}^{F_{k+1}}$. To complete the proof, it suffices to show in view of Claim 6 that the Cantor algebra $\mathscr{C}_{(F_{k+1},1)}$ satisfies the quasi-identity $\varphi_{F_m}^{F_{m+1}}$ for all $m \neq k$. If $m < k$, then by Lemma 3, the premise of the quasi-identity $\varphi_{F_m}^{F_{m+1}}$ fails at each element of $C_{(F_{k+1},1)}$. If $m > k$ then $F_{k+1} \subseteq F_m$. Let the premise of the quasi-identity $\varphi_{F_m}^{F_{m+1}}$ hold at some $a \in C_{(F_{k+1},1)}$. This means in view of Lemma 3 that $a \in C^{(0)}_{(F_{k+1},1)} = B_{(F_{k+1},1)}$. Since $F_{k+1} \subseteq F_m$, also the conclusion of the quasi-identity $\varphi_{F_m}^{F_{m+1}}$ holds at a. Therefore, $\mathscr{C}_{(F_{k+1},1)} \models \varphi_{F_m}^{F_{m+1}}$ for all $m \neq k$, which was to be proved.

Referring to Proposition 1 and recalling that there are continuum many infinite subsets of the set $\omega \backslash \{0, \dots, n\}$, whose complements are also infinite, complete the proof of the theorem.

Corollary 1. *For all m and n such that $0 < m < n < \omega$, there are continuum many subquasivarieties of the variety $\mathbf{C}_{mn}$ of Cantor algebras which do not have an independent quasiequational basis, but possess an ω-independent quasiequational basis, and whose intersection does have an independent quasiequational basis.*

Proof. Put $k = n - m + 1$. According to [15, Lemma 3.1], the variety $\mathbf{C}_{mn}$ contains a subvariety $\mathbf{D}_{mn}$ which has a finite equational basis relative to $\mathbf{C}_{mn}$ and which is term-equivalent to the variety $\mathbf{C}_{1k}$. Since $k > 1$, the claim follows from Theorem 1 and Proposition 1.

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