

DEGENERATE LINEAR EVOLUTION EQUATIONS WITH THE RIEMANN–LIOUVILLE FRACTIONAL DERIVATIVE

© V. E. Fedorov, M. V. Plekhanova, and R. R. Nazhimov

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Abstract: We study the unique solvability of the Cauchy and Schowalter–Sidorov type problems in a Banach space for an evolution equation with a degenerate operator at the fractional derivative under the assumption that the operator acting on the unknown function in the equation is p -bounded with respect to the operator at the fractional derivative. The conditions are found ensuring existence of a unique solution representable by means of the Mittag-Leffler type functions. Some abstract results are illustrated by an example of a finite-dimensional degenerate system of equations of a fractional order and employed in the study of unique solvability of an initial-boundary value problem for the linearized Scott-Blair system of dynamics of a medium.

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1. Introduction

We study the unique solvability of the Cauchy and Schowalter–Sidorov type problems for the degenerate evolution equation

$$D_t^\alpha Lu(t) = Mu(t) + f(t) \quad (1.1)$$

with the Riemann–Liouville fractional derivative of order $\alpha > 0$ and linear operators L and M , taking a Banach space \mathfrak{U} into a Banach space \mathfrak{V} . The assumption $\ker L \neq \{0\}$ is essential as implying the insolvability of the equation with respect to the fractional derivative. Moreover, M is assumed to (L, p) -bounded [1]. It is demonstrated that under certain conditions on the smoothness of f the equation has a unique solution representable through the Mittag-Leffler type functions. We compare the results that are obtained with those for a similar equation with a right-hand side of the form $LD_t^\alpha u(t)$. These abstract results are illustrated by an example of a finite-dimensional degenerate system of equations of a fractional order and employed in the study of unique solvability of an initial-boundary value problem for the linearized Scott-Blair system of dynamics of a medium [2].

Linear and nonlinear fractional order equations (mainly with the Riemann–Liouville or Gerasimov–Caputo derivative) in the finite-dimensional case are studied by many authors; and we can refer to the surveys [3, 4] and the monographs [5–7]. The solvability questions for linear fractional order equations in the infinite-dimensional case are studied by Kostin [8], Bazhlekova [9], and Glushak [10, 11]. Apart from fractional differential equations, more general evolution equations are studied by Prüss [12] and Kostić [13]. However, all above but some Kostić’s articles concern equations that are solved with respect to the time derivative. Solvability of initial problems for linear equations of the form (1.1) with a degenerate operator acting on the Gerasimov–Caputo derivative under different conditions on L and M is studied in [14–18]. Probably, there are no similar studies for the case of the fractional Riemann–Liouville derivative.

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2. The Initial Problem for Inhomogeneous Nondegenerate Fractional Order Equations

We now state the main definitions and properties of fractional integrals of use below. All integrals of functions with values in a Banach space \mathfrak{Z} are understood in the Bochner sense. The detailed proofs of the statements given here can be found, for example, in [9].

Let $g_\beta(t) = t^{\beta-1}/\Gamma(\beta)$, where $\beta > 0$ and $t > 0$. The fractional Riemann–Liouville integral of order $\alpha > 0$ of $f : (0, T) \rightarrow \mathfrak{Z}$ is defined as follows:

$$J_t^\alpha f(t) = (g_\alpha * f)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds, \quad t > 0, \quad J_t^0 f(t) = f(t).$$

The operators of fractional differentiation meet the semigroup property; i.e., $J_t^\alpha J_t^\beta = J_t^{\alpha+\beta}$, $\alpha, \beta \geq 0$.

Let $\alpha > 0$ and let m be the least integer greater or equal to α . The fractional Riemann–Liouville derivative of a function f is of the form

$${}^R D_t^\alpha f(t) = D_t^m (g_{m-\alpha} * f)(t) = D_t^m J_t^{m-\alpha} f(t),$$

where $D_t^m = \frac{d^m}{dt^m}$ is the usual integer order derivative. In much the same way as in the case of differentiation and integration of integer order, the symbol ${}^R D_t^\alpha$ stands for the inversion of J_t^α from the left. For brevity, for $\beta < 0$, the fractional Riemann–Liouville integral is denoted by ${}^R D_t^\beta \equiv J_t^{-\beta}$.

Since we use below only the Riemann–Liouville derivative, we abbreviate its notation as follows: $D_t^\alpha \equiv {}^R D_t^\alpha$.

Assume that $\alpha > 0$, $m-1 < \alpha \leq m \in \mathbb{N}$, and D_t^α is the fractional Riemann–Liouville derivative. Examine the so-called Cauchy type problem (see [6])

$$\lim_{t \rightarrow 0+} D_t^{\alpha-m+k} z(t) = z_k, \quad k = 0, 1, \dots, m-1, \quad (2.1)$$

for the fractional differential equation

$$D_t^\alpha z(t) = Az(t) + f(t), \quad t \in [0, T], \quad (2.2)$$

where $A \in \mathcal{L}(\mathfrak{Z})$ (i.e., A is a bounded operator in \mathfrak{Z}), $f : [0, T] \rightarrow \mathfrak{Z}$, $T \in (0, +\infty]$. For $k = 0$, we have $\alpha - m < 0$, and, hence, $D_t^{\alpha-m} z(t)$ is a fractional Riemann–Liouville integral. A function $z \in C([0, T]; \mathfrak{Z})$ such that $g_{m-\alpha} * z \in C^m([0, T]; \mathfrak{Z})$ and (2.1), (2.2) hold for all $t \in [0, T]$ is called a *solution* to (2.1), (2.2). The Mittag-Leffler function is denoted by

$$E_{\alpha, \beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)}.$$

Theorem 2.1 [19]. *Let $A \in \mathcal{L}(\mathfrak{Z})$ and $f \in C([0, T]; \mathfrak{Z})$. Then, for every $z_k \in \mathfrak{Z}$, $k = 0, 1, \dots, m-1$, there exists a unique solution to (2.1), (2.2) representable as*

$$\sum_{k=0}^{m-1} t^{\alpha-m+k} E_{\alpha, \alpha-m+k+1}(At^\alpha) z_k + \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(A(t-s)^\alpha) f(s) ds. \quad (2.3)$$

REMARK 2.1. In the one-dimensional case a solution of the form (2.3) is obtained in [20]. For equations with the Gerasimov–Caputo derivative a result similar to that in Theorem 2.1 is established in [15].

3. Relatively σ -Bounded Operators

Let \mathfrak{U} and \mathfrak{V} be Banach spaces. Denote by $\mathcal{L}(\mathfrak{U}; \mathfrak{V})$ the Banach space of continuous linear operators from \mathfrak{U} into \mathfrak{V} and by $\mathcal{C}l(\mathfrak{U}; \mathfrak{V})$, the set of closed densely defined linear operators on \mathfrak{U} with values in \mathfrak{V} . Put $\mathcal{L}(\mathfrak{U}; \mathfrak{U}) \equiv \mathcal{L}(\mathfrak{U})$ and $\mathcal{C}l(\mathfrak{U}; \mathfrak{U}) \equiv \mathcal{C}l(\mathfrak{U})$.

Let D_M be the domain of $M \in \mathcal{C}l(\mathfrak{U}; \mathfrak{V})$ and $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{V})$. Denote $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{V}; \mathfrak{U})\}$, $R_\mu^L(M) = (\mu L - M)^{-1}L$, $L_\mu^L(M) = L(\mu L - M)^{-1}$. It is easy to verify that the operator-functions $(\mu L - M)^{-1}$ and $R_\mu^L(M)$, $L_\mu^L(M)$ are analytic on $\rho^L(M)$.

An operator M is referred to as (L, σ) -bounded whenever

$$\exists a > 0 \forall \mu \in \mathbb{C} (|\mu| > a) \Rightarrow (\mu \in \rho^L(M)).$$

Given an (L, σ) -bounded operator M , choose a closed contour $\gamma = \{\mu \in \mathbb{C} : |\mu| = R > a\}$ on the complex plane \mathbb{C} . In this case the following integrals over this closed contour make sense:

$$P = \frac{1}{2\pi i} \int_{\gamma} R_\mu^L(M) d\mu, \quad Q = \frac{1}{2\pi i} \int_{\gamma} L_\mu^L(M) d\mu.$$

Using the pseudoresolvent identity which is valid for the operators $R_\mu^L(M)$ (or for $L_\mu^L(M)$), it is easy to show that $P \in \mathcal{L}(\mathfrak{U})$ and $Q \in \mathcal{L}(\mathfrak{V})$ are projections. Put $\mathfrak{U}^0 = \ker P$, $\mathfrak{U}^1 = \operatorname{im} P$, $\mathfrak{V}^0 = \ker Q$, and $\mathfrak{V}^1 = \operatorname{im} Q$. In this case $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$ and $\mathfrak{V} = \mathfrak{V}^0 \oplus \mathfrak{V}^1$. Denote by L_k (M_k) the restriction of L (M) to \mathfrak{U}^k ($D_{M_k} = D_M \cap \mathfrak{U}^k$), $k = 0, 1$.

Theorem 3.1 [1]. *Let M be an (L, σ) -bounded operator. Then*

- (i) $L_k \in \mathcal{L}(\mathfrak{U}^k; \mathfrak{V}^k)$, $k = 0, 1$;
- (ii) $M_0 \in \mathcal{C}l(\mathfrak{U}^0; \mathfrak{V}^0)$ and $M_1 \in \mathcal{L}(\mathfrak{U}^1; \mathfrak{V}^1)$;
- (iii) *there exist operators $L_1^{-1} \in \mathcal{L}(\mathfrak{V}^1; \mathfrak{U}^1)$ and $M_0^{-1} \in \mathcal{L}(\mathfrak{V}^0; \mathfrak{U}^0)$.*

An (L, σ) -bounded operator M is called (L, p) -bounded for $p \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$ if $H^p \neq \mathbb{O}$ and $H^{p+1} = \mathbb{O}$, where $H \equiv M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$.

REMARK 3.1. Assume that $\ker L \neq \{0\}$ and $\varphi_0 \in \ker L \setminus \{0\}$. An ordered collection of vectors $\{\varphi_1, \varphi_2, \dots\}$ is called a *chain of M -associated vectors of L corresponding to the vector φ_0* if

$$L\varphi_q = M\varphi_{q-1}, \quad \varphi_q \notin \ker L, \quad q = 1, 2, \dots$$

The index of a vector in this definition is called its *height* in a chain of M -associated vectors. A chain is finite if there exists an M -associated vector φ_p such that either $\varphi_p \notin D_M$ or $M\varphi_p \notin \operatorname{im} L$. The height of the last vector in a chain is called the *length of a chain*. A chain can be of infinite length.

It is possible to demonstrate that (see [1]) for an $(L, 0)$ -bounded operator M , an operator L has no M -associated vectors and $\mathfrak{U}^0 = \ker L$. If M is (L, p) -bounded for $p \in \mathbb{N}$ then the maximal length of a chain of M -associated vectors of L is equal to p and the subspace \mathfrak{U}^0 comprises M -associated vectors of height at most p .

4. The Phase Space of a Homogeneous Degenerate Equation

Assume that $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{V})$, $M \in \mathcal{C}l(\mathfrak{U}; \mathfrak{V})$, the domain D_M is endowed with the graph norm of M , $\overline{\mathbb{R}}_+ \equiv \{t \in \mathbb{R} : t \geq 0\}$. By a *solution to the equation*

$$D_t^\alpha Lu(t) = Mu(t), \quad t \geq 0, \tag{4.1}$$

we mean a function $u \in C(\overline{\mathbb{R}}_+; D_M)$ such that $g_{m-\alpha} * Lu \in C^m(\overline{\mathbb{R}}_+; \mathfrak{V})$ and equality (4.1) holds for all $t \geq 0$. By a *solution to the Cauchy type initial problem*

$$\lim_{t \rightarrow 0+} D_t^{\alpha-m+k} u(t) = u_k, \quad k = 0, 1, \dots, m-1, \tag{4.2}$$

for (4.1) we mean its solution such that $g_{m-\alpha} * u \in C^{m-1}(\overline{\mathbb{R}}_+; \mathfrak{U})$ and (4.2) holds.

A closed set $\mathfrak{P} \subset \mathfrak{U}$ is called the *phase space* of (4.1) if

- (i) for every solution $u = u(t)$ to (4.1), $u(t) \in \mathfrak{P}$ for all $t \geq 0$;
- (ii) for all $u_k \in \mathfrak{P}$, $k = 0, 1, \dots, m-1$, there exists a unique solution to (4.1), (4.2).

Lemma 4.1. *The phase space to (4.1) is unique (if existent).*

PROOF. Let \mathfrak{P}_1 and \mathfrak{P}_2 be the phase spaces of (4.1) and $\mathfrak{P}_1 \neq \mathfrak{P}_2$. In this case the symmetric difference $\mathfrak{P}_1 \Delta \mathfrak{P}_2$ is nonempty. Take a point u_0 in this set; it belongs only to one of the phase spaces \mathfrak{P}_1 or \mathfrak{P}_2 and, hence, for u_1, u_2, \dots, u_{m-1} in this phase space there exists a solution $u(\cdot)$ to (4.1), (4.2). The values of a solution must lie in $\mathfrak{P}_1 \cap \mathfrak{P}_2$ for every $t \geq 0$ and since the phase space must be closed, $J_t^{m-\alpha} u(t) \in \mathfrak{P}_1 \cap \mathfrak{P}_2$ and $u_0 = \lim_{t \rightarrow 0+} J_t^{m-\alpha} u(t) \in \mathfrak{P}_1 \cap \mathfrak{P}_2$; a contradiction. \square

If $\rho^L(M) \neq \emptyset$ then we consider (4.1) on the space \mathfrak{V} together with the equivalent equation

$$D_t^\alpha L_\beta^L(M)v(t) = M(\beta L - M)^{-1}v(t), \quad (4.3)$$

where $\beta \in \rho^L(M)$. The connection between the solutions to equations (4.1) and (4.3) is specified by the equality $u(t) = (\beta L - M)^{-1}v(t)$.

Theorem 4.1. *Let M be an (L, σ) -bounded operator. Then the phase space of (4.1) ((4.3)) agrees with the subspace \mathfrak{U}^1 (\mathfrak{V}^1).*

PROOF. Let $u = u(t)$ be a solution to (4.1). Put

$$u^0(t) \equiv (I - P)u(t), \quad u^1(t) \equiv Pu(t).$$

By Theorem 3.1, we have

$$D_t^\alpha H u^0(t) = u^0(t), \quad H \equiv M_0^{-1}L_0, \quad (4.4)$$

$$D_t^\alpha u^1(t) = S u^1(t), \quad S \equiv L_1^{-1}M_1. \quad (4.5)$$

If $p = 0$ then $H = \mathbb{O}$ and $u^0 \equiv 0$. If $p = \mathbb{N}$ then applying $D_t^\alpha H$ to both sides of (4.4), we obtain

$$u^0(t) = D_t^\alpha H u^0(t) = \dots = (D_t^\alpha H)^{p+1} u^0(t) = (D_t^\alpha)^{p+1} H^{p+1} u^0(t) \equiv 0$$

in view of the continuity and nilpotency of H . Hence, $u(t) = u^1(t) \in \mathfrak{U}^1$ for every $t \geq 0$.

Next, $S \in \mathcal{L}(\mathfrak{U}^1)$ in view of Theorem 3.1. By Theorem 2.1, for every $u_0, u_1, \dots, u_{m-1} \in \mathfrak{U}^1$, there exists a unique solution to (4.2), (4.5) and so to (4.1), (4.2) of the form

$$u(t) = \sum_{k=0}^{m-1} t^{\alpha-m+k} E_{\alpha, \alpha-m+k+1}(St^\alpha) u_k.$$

The statement concerning with the phase space of (4.3) is proven by analogy. \square

By a solution to the equation

$$LD_t^\alpha u(t) = Mu(t), \quad t \geq 0, \quad (4.6)$$

we mean a vector-function $u \in C(\overline{\mathbb{R}}_+; D_M)$ whose convolution $g_{m-\alpha} * u$ belongs to $C^m(\overline{\mathbb{R}}_+; \mathfrak{V})$ and (4.6) holds for all $t \geq 0$. A solution to the initial Cauchy type problem (4.2) for (4.6) and its phase space are defined in the same way as for (4.1). An equivalent equation on \mathfrak{V} in this case is of the form

$$L_\beta^L(M)D_t^\alpha v(t) = M(\beta L - M)^{-1}v(t), \quad t \geq 0. \quad (4.7)$$

Theorem 4.2. *Let M be an (L, σ) -bounded operator. Then the phase space of (4.6) ((4.7)) agrees with the subspace \mathfrak{U}^1 (\mathfrak{V}^1).*

PROOF. In contrast to the previous theorem we arrive at the equation $HD_t^\alpha u^0(t) = u^0(t)$, on the subspace \mathfrak{U}^0 whose solution satisfies the relations

$$u^0(t) = HD_t^\alpha u^0(t) = \dots = (HD_t^\alpha)^{p+1} u^0(t) = (D_t^\alpha)^{p+1} H^{p+1} u^0(t) \equiv 0.$$

Hence, $u(t) = u^1(t) \in \mathfrak{U}^1$ for all $t \geq 0$. The remaining arguments are the same as those in the proof of Theorem 4.1. \square

Corollary 4.1. *Assume that the operator M is (L, σ) -bounded and $u_0, u_1, \dots, u_{m-1} \in \mathfrak{U}^1$. The solutions to problem (4.1), (4.2), problem (4.2), (4.5), and problem (4.2), (4.6) coincide.*

5. Mittag-Leffler Type Operator-Functions

We are given an (L, σ) -bounded operator M , a contour $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$, and $\alpha, \beta > 0$. Put

$$U_{\alpha, \beta}(t) = \frac{1}{2\pi i} \int_{\gamma} R_{\mu}^L(M) E_{\alpha, \beta}(\mu t^{\alpha}) d\mu, \quad t \in \overline{\mathbb{R}}_+,$$

$$V_{\alpha, \beta}(t) = \frac{1}{2\pi i} \int_{\gamma} L_{\mu}^L(M) E_{\alpha, \beta}(\mu t^{\alpha}) d\mu, \quad t \in \overline{\mathbb{R}}_+.$$

Note that these operator families are continuous in t in the topology of the operator norm.

Consider the following restrictions of these operators:

$$U_{\alpha, \beta, 1}(t) \equiv U_{\alpha, \beta}(t)|_{\mathfrak{U}^1}, \quad V_{\alpha, \beta, 1}(t) \equiv V_{\alpha, \beta}(t)|_{\mathfrak{U}^1}.$$

By Theorem 3.1, we infer

$$U_{\alpha, \beta, 1}(t) = E_{\alpha, \beta}(St^{\alpha}) \in \mathcal{L}(\mathfrak{U}^1), \quad V_{\alpha, \beta, 1}(t) = E_{\alpha, \beta}(Tt^{\alpha}) \in \mathcal{L}(\mathfrak{V}^1),$$

where $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{U}^1)$ and $T = M_1L_1^{-1} \in \mathcal{L}(\mathfrak{V}^1)$. Indeed, for example,

$$\begin{aligned} V_{\alpha, \beta, 1}(t) &= \frac{1}{2\pi i} \int_{\gamma} L_{\mu}^{L_1}(M_1) E_{\alpha, \beta}(\mu t^{\alpha}) d\mu = \frac{1}{2\pi i} \int_{\gamma} (\mu I - M_1L_1^{-1})^{-1} E_{\alpha, \beta}(\mu t^{\alpha}) d\mu \\ &= \frac{1}{2\pi i} \int_{\gamma} \sum_{k=0}^{\infty} \mu^{-k-1} T^k \sum_{n=0}^{\infty} \frac{\mu^n t^{\alpha n}}{\Gamma(\alpha n + \beta)} d\mu = E_{\alpha, \beta}(Tt^{\alpha}). \end{aligned}$$

Similar arguments imply that for a continuously invertible operator L

$$U_{\alpha, \beta}(t) = E_{\alpha, \beta}(L^{-1}Mt^{\alpha}), \quad V_{\alpha, \beta}(t) = E_{\alpha, \beta}(ML^{-1}t^{\alpha}).$$

Lemma 5.1 [15]. *Let M be an (L, σ) -bounded operator. Then, for every $t \geq 0$, we have*

$$\begin{aligned} U_{\alpha, \beta}(t)P &= PU_{\alpha, \beta}(t) = U_{\alpha, \beta}(t), \quad V_{\alpha, \beta}(t)Q = QU_{\alpha, \beta}(t) = V_{\alpha, \beta}(t), \\ U_{\alpha, \beta}(t) &= E_{\alpha, \beta}(L_1^{-1}M_1t^{\alpha})P, \quad V_{\alpha, \beta}(t) = E_{\alpha, \beta}(M_1L_1^{-1}t^{\alpha})Q. \end{aligned}$$

As an obvious consequence, we have

Corollary 5.1 [15]. *Let an operator M be (L, σ) -bounded. Then, for $t \geq 0$,*

$$\mathfrak{U}^0 \subset \ker U_{\alpha, \beta}(t), \quad \text{im } U_{\alpha, \beta}(t) \subset \mathfrak{U}^1, \quad \mathfrak{V}^0 \subset \ker V_{\alpha, \beta}(t), \quad \text{im } V_{\alpha, \beta}(t) \subset \mathfrak{V}^1.$$

Corollary 5.2. *For $\alpha, \beta > 0$, $u_0 \in \mathfrak{U}$ ($v_0 \in \mathfrak{V}$), the vector-function $u(t) = t^{\alpha-1}U_{\alpha, \beta}(t)u_0$ ($v(t) = t^{\alpha-1}V_{\alpha, \beta}(t)v_0$) satisfies (4.1) ((4.3)).*

PROOF. By Lemma 5.1, we infer

$$\begin{aligned} D_t^{\alpha} t^{\alpha-1} U_{\alpha, \beta} u_0 &= D_t^{\alpha} t^{\alpha-1} E_{\alpha, \beta}(St^{\alpha}) P u_0 = D_t^m J_t^{m-\alpha} \sum_{n=0}^{\infty} \frac{S^n t^{\alpha n + \alpha - 1}}{\Gamma(\alpha n + \beta)} P u_0 \\ &= D_t^m \sum_{n=0}^{\infty} \frac{t^{m+\alpha n-1} S^n P u_0}{\Gamma(m-\alpha+\alpha n+\beta)} = \sum_{n=1}^{\infty} \frac{t^{\alpha n-1} S^n P u_0}{\Gamma(\alpha n - \alpha + \beta)} = St^{\alpha-1} E_{\alpha, \beta}(St^{\alpha}) P u_0. \end{aligned}$$

Hence, $t^{\alpha-1}U_{\alpha, \beta}u_0$ is a solution to (4.5) and so to (4.1). \square

6. Initial Problems for an Inhomogeneous Degenerate Equation

Assume that \mathfrak{U} and \mathfrak{V} are Banach spaces, $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{V})$, $M \in \mathcal{C}l(\mathfrak{U}; \mathfrak{V})$, and $f : [0, T] \rightarrow \mathfrak{V}$, where $T \in (0, +\infty]$. By a *solution to the equation*

$$D_t^\alpha Lu(t) = Mu(t) + f(t), \quad t \in [0, T], \quad (6.1)$$

we mean $u \in C([0, T]; D_M)$ such that $g_{m-\alpha} * Lu \in C^m([0, T]; \mathfrak{V})$ and (6.1) holds for all $t \in [0, T]$. By a *solution to the Cauchy type problem*

$$\lim_{t \rightarrow 0+} D_t^{\alpha-m+k} u(t) = u_k, \quad k = 0, 1, \dots, m-1, \quad (6.2)$$

for (6.1) we mean a solution such that $g_{m-\alpha} * u \in C^{m-1}([0, T]; \mathfrak{U})$ and (6.2) hold.

Lemma 6.1. *Assume that $G \in \mathcal{L}(\mathfrak{U})$ is nilpotent of degree $p \in \mathbb{N}_0$ and there exist the fractional derivatives $(D_t^\alpha G)^k g \in C([0, T]; \mathfrak{U})$, $k = 0, 1, \dots, p$. Then there exists a unique solution to the equation*

$$D_t^\alpha Gz(t) = z(t) + g(t), \quad t \in [0, T], \quad (6.3)$$

representable as

$$z(t) = - \sum_{k=0}^p (D_t^\alpha G)^k g(t). \quad (6.4)$$

PROOF. Let $z = z(t)$ be a solution to (6.3). Applying G to both sides of (6.3), we arrive at the equality

$$GD_t^\alpha Gz(t) = Gz(t) + Gg(t).$$

In this case there exists the fractional derivative of the right side and thus on the left side. Applying the operator D_t^α to both sides, we infer

$$(D_t^\alpha G)^2 z = D_t^\alpha Gz + D_t^\alpha Gg = z + g + D_t^\alpha Gg.$$

Repeating the procedure, at the p th step we derive the equality

$$(D_t^\alpha G)^{p+1} z = z + \sum_{k=0}^p (D_t^\alpha G)^k g.$$

In view of continuity and nilpotency of G , we have

$$(D_t^\alpha G)^{p+1} z = (D_t^\alpha)^{p+1} G^{p+1} z \equiv 0,$$

and so (6.4) is valid. This equality implies the existence of a solution to (6.3) (it can be checked by inserting this function into the equation) and its uniqueness. Indeed, the difference of two solutions is a solution to (6.3) with $g \equiv 0$. Formula (6.4) implies that this solution is identically zero. \square

Theorem 6.1. *Assume that the operator M is (L, p) -bounded, $Qf \in C([0, T]; \mathfrak{V})$, there exist the fractional derivatives $(D_t^\alpha H)^k M_0^{-1}(I - Q)f \in C([0, T]; \mathfrak{U})$ for $k = 0, 1, \dots, p$, and*

$$\lim_{t \rightarrow 0+} D_t^{\alpha-m+l} \sum_{k=0}^p (D_t^\alpha H)^k M_0^{-1}(I - Q)f(t) = -(I - P)u_l, \quad l = 0, 1, \dots, m-1. \quad (6.5)$$

Then there exists a unique solution to (6.1), (6.2) representable as

$$\begin{aligned} u(t) = & \sum_{k=0}^{m-1} t^{\alpha-m+k} U_{\alpha, \alpha-m+k+1}(t) u_k + \int_0^t (t-s)^{\alpha-1} U_{\alpha, \alpha}(t-s) L_1^{-1} Qf(s) ds \\ & - \sum_{k=0}^p (D_t^\alpha H)^k M_0^{-1}(I - Q)f(t). \end{aligned} \quad (6.6)$$

PROOF. Put $u^0(t) \equiv (I - P)u(t)$, $u^1(t) \equiv Pu(t)$. Theorem 3.1 yields

$$D_t^\alpha H u^0(t) = u^0(t) + M_0^{-1}(I - Q)f(t), \quad H \equiv M_0^{-1}L_0, \quad (6.7)$$

$$D_t^\alpha u^1(t) = S u^1(t) + h(t), \quad S \equiv L_1^{-1}M_1, \quad h(t) = L_1^{-1}Qf(t). \quad (6.8)$$

By Lemma 6.1 there exists a unique solution to (6.7) of the form

$$u^0 = - \sum_{k=0}^p (D_t^\alpha H)^k M_0^{-1}(I - Q)f.$$

Hence, for the initial conditions to be satisfied, it is necessary to require the fulfillment of the consistency conditions (6.5).

Theorem 3.1 yields $S \in \mathcal{L}(\mathfrak{U}^1)$. By Theorem 2.1 there exists a unique solution to the problem

$$\lim_{t \rightarrow 0+} J_t^{m-\alpha} u^1(t) = P u_0, \quad \lim_{t \rightarrow 0+} D_t^{\alpha-m+k} u^1(t) = P u_k, \quad k = 1, 2, \dots, m-1,$$

for equation (6.8) which is representable as

$$u^1(t) = \sum_{k=0}^{m-1} t^{\alpha-m+k} E_{\alpha, \alpha-m+k+1}(t^\alpha S) P u_k + \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(S(t-s)^\alpha) h(s) ds.$$

Taking Lemma 5.1 into account, we obtain (6.6). \square

Consider the Schowalter–Sidorov problem often arising in applications when we study degenerate evolution equations. In this case the initial data are posed only for the projection of the unknown function onto the subspace \mathfrak{U}^1 , i.e.,

$$\lim_{t \rightarrow 0+} D_t^{\alpha-m+k} P u(t) = u_k, \quad k = 0, 1, \dots, m-1. \quad (6.9)$$

Theorem 6.2. Assume that the operator M is (L, p) -bounded, $Qf \in C([0, T]; \mathfrak{V})$, and there exist the fractional derivatives $(D_t^\alpha H)^k M_0^{-1}(I - Q)f \in C([0, T]; \mathfrak{U})$ for $k = 0, 1, \dots, p$. Then, for every $u_k \in \mathfrak{U}^1$, $k = 0, 1, \dots, m-1$, there exists a unique solution to (6.1), (6.9) of the form (6.6).

PROOF. Demonstration to that above. The peculiarities of the initial Schowalter–Sidorov type condition are such that it does not imply constraints on the projection of a solution u^0 to (6.1) at the initial time. Hence, we do not need the consistency conditions (6.5). \square

By a solution to the equation

$$L D_t^\alpha u(t) = M u(t) + f(t), \quad t \in [0, T], \quad (6.10)$$

we mean $u \in C([0, T]; D_M) \cap C^{m-1}([0, T]; \mathfrak{U})$, such that $g_{m-\alpha} * u \in C^m([0, T]; \mathfrak{U})$ and equality (6.10) holds for all $t \in [0, T]$. A solution satisfying (6.2) is called a *solution* to (6.2), (6.10).

Lemma 6.2. Assume that $G \in \mathcal{L}(\mathfrak{Z})$ is nilpotent of degree $p \in \mathbb{N}_0$ and we have the fractional derivatives $D_t^\alpha (G D_t^\alpha)^k g \in C([0, T]; \mathfrak{Z})$, $k = 0, 1, \dots, p$. Then there exists a unique solution to the equation

$$G D_t^\alpha z(t) = z(t) + g(t), \quad t \in [0, T],$$

representable as

$$z(t) = - \sum_{k=0}^p (G D_t^\alpha)^k g(t).$$

PROOF. Demonstration is similar to that of Lemma 6.1. The difference is in stronger smoothness requirements for the function g which can be explained by another structure of the equation. In contrast to the previous case, we first involve the fractional differentiation and then apply the degenerate operator. \square

By analogy with Theorems 6.1 and 6.2, using Lemma 6.2 rather than Lemma 6.1, we can prove the following two statements about the initial problems for (6.10).

Theorem 6.3. Assume that the operator M is (L, p) -bounded, $Qf \in C([0, T]; \mathfrak{V})$, and we have the fractional derivatives $D_t^\alpha (HD_t^\alpha)^k M_0^{-1}(I - Q)f \in C([0, T]; \mathfrak{U})$ for $k = 0, 1, \dots, p$. Then

(i) if

$$\lim_{t \rightarrow 0+} D_t^{\alpha-m+l} \sum_{k=0}^p (HD_t^\alpha)^k M_0^{-1}(I - Q)f(t) = -(I - P)u_l, \quad l = 0, 1, \dots, m-1,$$

then there exists a unique solution to (6.2), (6.10) representable as

$$\begin{aligned} u(t) = & \sum_{k=0}^{m-1} t^{\alpha-m+k} U_{\alpha, \alpha-m+k+1}(t) u_k + \int_0^t (t-s)^{\alpha-1} U_{\alpha, \alpha}(t-s) L_1^{-1} Qf(s) ds \\ & - \sum_{k=0}^p (HD_t^\alpha)^k M_0^{-1}(I - Q)f(t); \end{aligned} \quad (6.11)$$

(ii) for every $u_k \in \mathfrak{U}^1$, $k = 0, 1, \dots, m-1$, there exists a unique solution to (6.9), (6.10) of the form (6.11).

REMARK 6.1. Clearly, the conditions on u_k , $k = 0, 1, \dots, m-1$, in the claims of Theorem 6.1–6.3 are necessary and sufficient for solvability of the problem.

REMARK 6.2. As is seen from the results obtained, solutions (6.6) and (6.11) to the initial problems for (6.1) and (6.10), respectively, differ from each other only by the form a solution on the subspace \mathfrak{U}^0 . This observation agrees with the claim of Corollary 4.1 that solutions to the initial problems for analogous homogeneous equations coincide. Indeed, in this case solutions to the equations on the subspace \mathfrak{U}^0 vanish.

REMARK 6.3. It is just what was to be expected, solutions to initial problems for (6.1) and (6.10) with the Riemann–Liouville derivative differ from solutions to similar problems for the same equations with the Caputo derivative [15] only in the part concerning with the homogeneous equation by the powers of t and second indices of the Mittag-Leffler functions which corresponds to the difference in the initial conditions of these problems.

7. A Degenerate System of Fractional Differential Equations for Functions of One Variable

Assume that $\alpha > 0$, $u, f : \overline{\mathbb{R}}_+ \rightarrow \mathbb{R}^n$, B and C is a square matrices of order $n \in \mathbb{N}$, $\text{rang } B = r \in \{0, 1, \dots, n-1\}$. We consider the initial problem

$$\lim_{t \rightarrow 0+} D_t^{\alpha-m+k} u(t) = u_k, \quad k = 0, 1, \dots, m-1, \quad (7.1)$$

for the following system of fractional differential equations of one variable:

$$D_t^\alpha B u(t) = C u(t) + f(t). \quad (7.2)$$

Problem (7.1), (7.2) is reduced to (6.1), (6.2) if we put $\mathfrak{U} = \mathfrak{V} = \mathbb{R}^n$, and identify L and M with the matrices B and C , respectively.

As is known [1, Section 5.1, Lemma 5.1.1], if $\det(\beta B - C) \neq 0$ for some $\beta \in \mathbb{C}$ then B is (C, p) -bounded, where $p \leq n-1$. In this case we can take

$$a = \max\{|\beta| : \beta \in \mathbb{C}, \det(\beta B - C) = 0\}$$

in the definition of (B, σ) -boundedness of C .

8. The Linearized System of the Scott-Blair Equations

Consider the initial-boundary value problem for the system of equations of fractional order with respect to time:

$$\lim_{t \rightarrow 0+} D_t^{\alpha-m+k} v(x, t) = v_k(x), \quad x \in \Omega, \quad k = 0, 1, \dots, m-1, \quad (8.1)$$

$$v(x, t) = 0, \quad (x, t) \in \partial\Omega \times \overline{\mathbb{R}}_+, \quad (8.2)$$

$$D_t^\alpha (1 - \chi \Delta) v(x, t) = -(\tilde{v} \cdot \nabla) v(x, t) - (v \cdot \nabla) \tilde{v}(x, t) - r(x, t), \quad (x, t) \in \Omega \times \overline{\mathbb{R}}_+, \quad (8.3)$$

$$\nabla \cdot v(x, t) = 0, \quad (x, t) \in \Omega \times \overline{\mathbb{R}}_+. \quad (8.4)$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega$ of class C^∞ , $\chi \in \mathbb{R}$, and \tilde{v} is a given vector-function. The velocity vector-function $v = (v_1, v_2, \dots, v_n)$ of a fluid and its pressure gradient $r = (r_1, r_2, \dots, r_n) = \nabla p$ are unknown. This system can be obtained under the assumption that the dynamics of a Scott-Blair medium [2] is described with the help of the time derivative of the same order as that in the rheological relation for this medium, with subsequent linearization.

Denote $\mathbb{L}_2 = (L_2(\Omega))^n$, $\mathbb{H}^1 = (W_2^1(\Omega))^n$, $\mathbb{H}^2 = (W_2^2(\Omega))^n$. The closure of the subspace $\mathcal{L} = \{v \in (C_0^\infty(\Omega))^n : \nabla \cdot v = 0\}$ in the norm of \mathbb{L}_2 is denoted by \mathbb{H}_σ and in the norm of \mathbb{H}^1 , by \mathbb{H}_σ^1 . We have the representation $\mathbb{L}_2 = \mathbb{H}_\sigma \oplus \mathbb{H}_\pi$, where \mathbb{H}_π is the orthogonal complement to \mathbb{H}_σ . Denote by $\Pi : \mathbb{L}_2 \rightarrow \mathbb{H}_\pi$ the orthogonal projection associated with this decomposition, $\Sigma = I - \Pi$, $\mathbb{H}_\sigma^2 = \mathbb{H}_\sigma^1 \cap \mathbb{H}^2$.

The operator $A = \Sigma \Delta$ with the domain \mathbb{H}_σ^2 in \mathbb{H}_σ has a real negative discrete spectrum of finite multiplicity which condenses only at $-\infty$ [21]. Denote by $\{\lambda_k\}$ its eigenvalues that are enumerated in nonincreasing order with multiplicity counted, and by $\{\varphi_k\}$, the corresponding system of orthonormal eigenfunctions which constitute a basis for \mathbb{H}_σ [21].

Take $\tilde{v} \in \mathbb{H}^1$. In this case the formula $Dw = -(\tilde{v} \cdot \nabla)w - (w \cdot \nabla)\tilde{v}$ defines $D \in \mathcal{L}(\mathbb{H}_\sigma^2; \mathbb{L}_2)$.

Put

$$\mathfrak{U} = \mathbb{H}_\sigma^2 \times \mathbb{H}_\pi, \quad \mathfrak{V} = \mathbb{L}_2 = \mathbb{H}_\sigma \times \mathbb{H}_\pi, \quad (8.5)$$

$$L = \begin{pmatrix} I - \chi A & \mathbb{O} \\ -\chi \Pi \Delta & \mathbb{O} \end{pmatrix} \in \mathcal{L}(\mathfrak{U}; \mathfrak{V}), \quad M = \begin{pmatrix} \Sigma D & \mathbb{O} \\ \Pi D & -I \end{pmatrix} \in \mathcal{L}(\mathfrak{U}; \mathfrak{V}). \quad (8.6)$$

In this case the choice of \mathfrak{U} , i.e., the membership of $v(\cdot, t)$ in the subspace \mathbb{H}_σ^2 as $t \geq 0$, accounts for (8.4) and (8.2). The function $r(\cdot, t)$ is a gradient since it belongs to \mathbb{H}_π for $t \geq 0$.

Lemma 8.1. *Assume that $\chi \neq 0$, $\chi^{-1} \notin \sigma(A)$ and the subspaces \mathfrak{U} and \mathfrak{V} and the operators L and M are given by (8.5) and (8.6), respectively. Then M is $(L, 0)$ -bounded and the projections are of the form*

$$P = \begin{pmatrix} I & \mathbb{O} \\ \chi \Pi \Delta (I - \chi A)^{-1} \Sigma D + \Pi D & \mathbb{O} \end{pmatrix}, \quad Q = \begin{pmatrix} I & \mathbb{O} \\ -\chi \Pi \Delta (I - \chi A)^{-1} & \mathbb{O} \end{pmatrix}. \quad (8.7)$$

PROOF. Given $v \in \mathbb{H}_\sigma$, we have

$$\begin{aligned} \|(I - \chi A)^{-1} v\|_{\mathbb{H}_\sigma^2}^2 &= \|(I - \chi A)^{-1} v\|_{\mathbb{H}_\sigma}^2 + \|A(I - \chi A)^{-1} v\|_{\mathbb{H}_\sigma}^2 \\ &= \sum_{k=1}^{\infty} \frac{(1 + \lambda_k^2) |\langle v, \varphi_k \rangle|^2}{|1 - \chi \lambda_k|^2} \leq C_1 \sum_{k=1}^{\infty} |\langle v, \varphi_k \rangle|^2 = C_1 \|v\|_{\mathbb{H}_\sigma}^2; \end{aligned}$$

hence, $(I - \chi A)^{-1} \in \mathcal{L}(\mathbb{H}_\sigma; \mathbb{H}_\sigma^2)$. Therefore, $D(I - \chi A)^{-1} \in \mathcal{L}(\mathbb{H}_\sigma; \mathbb{L}_2)$.

Thus, the inverse operator

$$(\mu(I - \chi A) - \Sigma D)^{-1} = \frac{1}{\mu} (I - \chi A)^{-1} \left(I - \frac{1}{\mu} \Sigma D (I - \chi A)^{-1} \right)^{-1}$$

acts continuously from \mathbb{H}_σ into \mathbb{H}_σ^2 for $|\mu| > \|\Sigma D (I - \chi A)^{-1}\|_{\mathcal{L}(\mathbb{H}_\sigma)}$. For these $\mu \in \mathbb{C}$, we have

$$(\mu L - M)^{-1} = \begin{pmatrix} (\mu(I - \chi A) - \Sigma D)^{-1} & \mathbb{O} \\ (\mu \chi \Pi \Delta + \Pi D)(\mu(I - \chi A) - \Sigma D)^{-1} & I \end{pmatrix}.$$

Since

$$\begin{aligned} & (\mu\chi\Pi\Delta + \Pi D)(\mu(I - \chi A) - \Sigma D)^{-1} \\ &= \left(\chi\Pi\Delta + \frac{1}{\mu}\Pi D \right) (I - \chi A)^{-1} \left(I - \frac{1}{\mu}\Sigma D(I - \chi A)^{-1} \right)^{-1} \end{aligned}$$

acts continuously from \mathbb{H}_σ into \mathbb{H}_π for $|\mu|$ sufficiently large, the operator $(\mu L - M)^{-1} : \mathfrak{V} \rightarrow \mathfrak{U}$ is continuous.

Since

$$\begin{aligned} & \chi\Pi\Delta \left(I - \frac{1}{\mu}(I - \chi A)^{-1}\Sigma D \right)^{-1} - \chi\Pi\Delta \\ &= \chi\Pi\Delta \frac{1}{\mu}(I - \chi A)^{-1}\Sigma D \left(I - \frac{1}{\mu}(I - \chi A)^{-1}\Sigma D \right)^{-1}, \end{aligned}$$

we infer

$$\begin{aligned} R_\mu^L(M) &= \begin{pmatrix} (\mu I - (I - \chi A)^{-1}\Sigma D)^{-1} & \mathbb{O} \\ (\mu\chi\Pi\Delta + \Pi D)(\mu - (I - \chi A)^{-1}\Sigma D)^{-1} - \chi\Pi\Delta & \mathbb{O} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\mu}(I - \frac{1}{\mu}(I - \chi A)^{-1}\Sigma D)^{-1} & \mathbb{O} \\ (\chi\Pi\Delta \frac{1}{\mu}(I - \chi A)^{-1}\Sigma D + \frac{1}{\mu}\Pi D)(I - \frac{1}{\mu}(I - \chi A)^{-1}\Sigma D)^{-1} & \mathbb{O} \end{pmatrix}, \\ L_\mu^L(M) &= \begin{pmatrix} \frac{1}{\mu}(I - \frac{1}{\mu}\Sigma D(I - \chi A)^{-1})^{-1} & \mathbb{O} \\ -\frac{1}{\mu}\chi\Pi\Delta(I - \chi A)^{-1}(I - \frac{1}{\mu}\Sigma D(I - \chi A)^{-1})^{-1} & \mathbb{O} \end{pmatrix}. \end{aligned}$$

Integrating these functions over a circle of sufficiently large radius and center zero and applying the Residue Theorem, we obtain (8.6). Obviously, $\ker L = \ker P$ and so $L_0 = \mathbb{O}$ and the operator M is $(L, 0)$ -bounded. \square

The form of the projections P and Q implies that $\mathfrak{U}^0 = \{0\} \times \mathbb{H}_\pi$, $\mathfrak{U}^1 = \{(w_1, w_2) \in \mathbb{H}_\sigma^2 \times \mathbb{H}_\pi : w_2 = (\chi\Pi\Delta(I - \chi A)^{-1}\Sigma D + \Pi D)w_1\}$, $\mathfrak{V}^0 = \{0\} \times \mathbb{H}_\pi$, and $\mathfrak{V}^1 = \{(w_1, w_2) \in \mathbb{H}_\sigma \times \mathbb{H}_\pi : w_2 = -\chi\Pi\Delta(I - \chi A)^{-1}w_1\}$.

Theorem 8.1. *For all $v_k \in \mathbb{H}_\sigma^2$, $k = 0, 1, \dots, m-1$, there exists a unique solution to (8.1)–(8.4) representable as*

$$\begin{aligned} v(x, t) &= \sum_{l=0}^{m-1} t^{\alpha-m+l} E_{\alpha, \alpha-m+l+1}(t^\alpha(I - \chi A)^{-1}\Sigma D), \\ r(x, t) &= (\chi\Pi\Delta(I - \chi A)^{-1}\Sigma D + \Pi D) \sum_{l=0}^{m-1} t^{\alpha-m+l} E_{\alpha, \alpha-m+l+1}(t^\alpha(I - \chi A)^{-1}\Sigma D). \end{aligned}$$

PROOF. The form of the projection P implies that (8.1) is equivalent to a Schowalter–Sidorov type problem. To describe the form of a solution for $\alpha, \beta > 0$, we calculate

$$\begin{aligned} & \int_{|\mu|=a+1} \frac{1}{\mu} \left(I - \frac{1}{\mu}(I - \chi A)^{-1}\Sigma D \right)^{-1} E_{\alpha, \beta}(\mu t^\alpha) d\mu \\ &= \int_{|\mu|=a+1} \sum_{k=1}^{\infty} \frac{[(I - \chi A)^{-1}\Sigma D]^k}{\mu^{k+1}} \sum_{l=0}^{\infty} \frac{\mu^l t^{\alpha l}}{\Gamma(\alpha l + \beta)} d\mu = \sum_{k=1}^{\infty} \frac{[(I - \chi A)^{-1}\Sigma D]^k t^{\alpha k}}{\Gamma(\alpha k + \beta)} \\ &= E_{\alpha, \beta}(t^\alpha(I - \chi A)^{-1}\Sigma D). \end{aligned}$$

Hence,

$$U_{\alpha, \beta}(t) = \begin{pmatrix} E_{\alpha, \beta}(t^\alpha(I - \chi A)^{-1}\Sigma D) & \mathbb{O} \\ (\chi\Pi\Delta(I - \chi A)^{-1}\Sigma D + \Pi D)E_{\alpha, \beta}(t^\alpha(I - \chi A)^{-1}\Sigma D) & \mathbb{O} \end{pmatrix}.$$

Theorem 6.2 ensures the claim. \square

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V. E. FEDOROV; M. V. PLEKHANOVA
 CHELYABINSK STATE UNIVERSITY
 SOUTH URAL STATE UNIVERSITY, CHELYABINSK, RUSSIA
E-mail address: kar@csu.ru; mariner79@mail.ru

R. R. NAZHIMOV
 CHELYABINSK STATE UNIVERSITY, CHELYABINSK, RUSSIA
E-mail address: goldenboy454@mail.ru