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On contraction of vertices of the circuits in coset diagrams for $PSL(2, \mathbb{Z})$

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Abstract. Coset diagrams for the action of $PSL(2, \mathbb{Z})$ on real quadratic irrational numbers are infinite graphs. These graphs are composed of circuits. When modular group acts on projective line over the finite field F_q, denoted by $PL(F_q)$, vertices of the circuits in these infinite graphs are contracted and ultimately a finite coset diagram emerges. Thus the coset diagrams for $PL(F_q)$ is composed of homomorphic images of the circuits in infinite coset diagrams. In this paper, we consider a circuit in which one vertex is fixed by $(xy)^{m_1}(xy^{-1})^{m_2}$, that is, (m_1, m_2). Let α be the homomorphic image of (m_1, m_2) obtained by contracting a pair of vertices v, u of (m_1, m_2). If we change the pair of vertices and contract them, it is not necessary that we get a homomorphic image different from α. In this paper, we answer the question: how many distinct homomorphic images are obtained, if we contract all the pairs of vertices of (m_1, m_2)? We also mention those pairs of vertices, which are 'important'. There is no need to contract the pairs, which are not mentioned as 'important'. Because, if we contract those, we obtain a homomorphic image, which we have already obtained by contracting 'important' pairs.

Keywords. Modular group; coset diagrams; homomorphic images; projective line over finite field.

2010 Mathematics Subject Classification. Primary: 05C25, Secondary: 20G40.

1. Introduction

The modular group $PSL(2, \mathbb{Z})$ [1,5] has a finite presentation $\langle x, y : x^2 = y^3 = 1 \rangle$, where $x : z \to \frac{-1}{z}$ and $y : z \to \frac{z-1}{z}$ are linear fractional transformations. The extended modular group $PGL(2, \mathbb{Z})$ has a finite presentation $\langle x, y, t : x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1 \rangle$, where t is a linear fractional transformation which maps z to $\frac{1}{z}$. Let q be a power of a prime p, and $PL(F_q)$ denote the projective line over the finite field F_q, that is, $PL(F_q) = F_q \cup \{\infty\}$.

In 1978, Higman [5] introduced a new type of graph called coset diagram for the action of $PGL(2, \mathbb{Z})$ on different objects, and in 1983, Mushtaq [8] laid its foundation. The three-cycles of y are denoted by small triangles whose vertices are permuted counter-clockwise by y and any two vertices which are interchanged by x are joined by an edge. The fixed

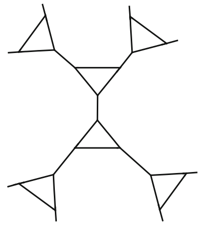

Figure 1. A segment of infinite coset diagram.

points of x and y are denoted by heavy dots. Since $(yt)^2 = 1$ is equivalent to $tyt = y^{-1}$, therefore t reverses the orientation of the triangles representing the three-cycles of y. Thus, there is no need to make the diagram complicated by introducing t-edges.

DEFINITION 1 [7]

A coset graph (subgraph) D' is a homomorphic image of the coset graph (subgraph) D if and only if

(i) the number of vertices in D' are less than that in D,
(ii) for each vertex v in D, which is fixed by an element $g \in PSL(2, \mathbb{Z})$, there is vertex v' in D' such that $(v')g = v'$.

The real quadratic irrational numbers can be expressed in the form $\frac{a+\sqrt{n}}{c}$, where n is a non-square positive integer and a, $\frac{(a^2-n)}{c}$ and c do not have any common factor. Coset diagrams for the action of modular group on real quadratic irrational numbers $\frac{a+\sqrt{n}}{c}$ are infinite graphs [9]. A portion of these graphs is shown in figure 1.

The action of the modular group on real quadratic irrational numbers through coset diagrams is very difficult to study, as the diagrams are infinite. Therefore, the action of the modular group on $PL(F_q)$, where q is the power of some prime p, becomes important, as its coset diagrams are finite. These coset diagrams are homomorphic images of the coset diagrams for $\frac{a+\sqrt{n}}{c}$, where $n \equiv z^2 \bmod p$ for some $z \in \mathbb{N}$. For instance, consider the action of $PGL(2, \mathbb{Z})$ on $PL(F_{19})$. We can calculate the permutation representations x, y and t by $(z)x = \frac{-1}{z}$, $(z)y = \frac{z-1}{z}$ and $(z)t = \frac{1}{z}$ respectively. So

$$
\begin{aligned}
x &: (0\ \infty)(1\ 18)(2\ 9)(3\ 6)(4\ 14)(5\ 15)(7\ 8)(10\ 17)(11\ 12)(13\ 16),\\
y &: (0\ \infty\ 1)(2\ 10\ 18)(3\ 7\ 9)(4\ 15\ 6)(5\ 16\ 14)(13\ 17\ 11)(8)(12),\\
t &: (0\ \infty)(2\ 10)(3\ 13)(4\ 5)(6\ 16)(7\ 11)(8\ 12)(9\ 17)(14\ 15)(1)(18).
\end{aligned}
$$

Of course, the above coset diagram (figure 2) is a homomorphic image of the coset diagram for $\frac{a+\sqrt{17}}{c}$ as $17 \equiv 6^2 \mod 19$.

For more on coset diagrams, we suggest references [2,3,6,10,11].

DEFINITION 2

By a circuit of length k, denoted by $(m_1, m_2, \ldots, m_k)$, we mean the circuit containing one vertex fixed by $(xy)^{m_1}(xy^{-1})^{m_2} \cdots (xy)^{m_{k-1}}(xy^{-1})^{m_k} \in PSL(2, \mathbb{Z})$. In other words, it is the circuit in which m_1 triangles have one vertex inside the circuit and m_2 triangles

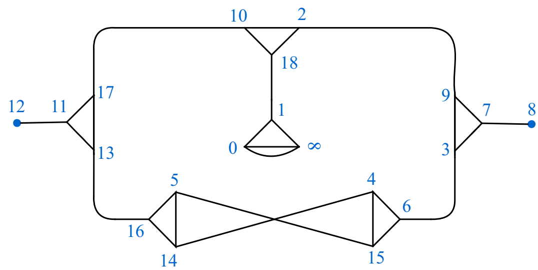

Figure 2. Coset diagram for $PL(F_{19})$.

have one vertex outside the circuit and so on. Since it is a cycle $(m_1, m_2, \ldots, m_k)$, so it does not make any difference if m_1 triangles have one vertex outside the circuit and m_2 triangles have one vertex inside the circuit and so on.

Note that, the length of a circuit is always even. Suppose there is a circuit $(m_1, m_2, \ldots, m_k)$ of odd length, that is, k is odd. By definition, $(m_1, m_2, \ldots, m_k)$ means the circuit in which m_1 triangles have one vertex inside the circuit and m_2 triangles have one vertex outside the circuit and continuing in this way, m_k triangles have one vertex inside the circuit. Since it is a closed path, therefore, the first m_1 and the last m_k triangles are adjacent and have one vertex inside the circuit. Therefore the circuit is in fact $(m_1 + m_k, m_2, \ldots, m_{k-1})$, which is of even length.

Remark 1. If v is a fixed point of an element $g_i = xy^{\kappa_1}xy^{\kappa_2}\ldots xy^{\kappa_n}$ ($\kappa = 1$ or -1) of the modular group, then $(v)g$ is a fixed point of $g^{-1}g_ig$.

Let v_i and v_j be any vertices in the circuit (m_1, m_2) such that $(v_i)g_i = v_i$ and $(v_j)g_j = v_j$. Suppose $(v_i)g_k = v_j$, then $g_i^{-1}g_k$ also maps v_i to v_j. Note that g_k and $g_i^{-1}g_k$ are the only two paths to reach v_j from v_i. Now by contraction of vertices v_i and v_j, we mean that v_i and v_j melt together to become one node $v = v_i = v_j$ in such a way that $v = v_i = v_j$ is fixed by both the elements g_k and $g_i^{-1}g_k$, which are the paths from v_i to v_j. This is achieved by creating a circuit (closed path) C so that the vertex v in C is fixed by g_k, then by applying $g_i^{-1}g_k$ on v such that that $g_i^{-1}g_k$ ends at v. As a result of this type of contraction of pairs of vertices in (m_1, m_2), we obtain a graph α which is a homomorphic image of (m_1, m_2). Note that other than v_i, v_j, there are so many pairs of vertices in (m_1, m_2) which form α by contraction. How many such pairs are there? The following theorem helps us to find that number.

PROPOSITION 1

Let the vertices v_i and v_j in (m_1, m_2) be contracted and a homomorphic image α of (m_1, m_2) evolves. Then α is obtainable if the vertices $(v_i)g$ and $(v_j)g$ in (m_1, m_2) are contracted.

Proof. Clearly, the vertex $v = v_i = v_j$ of α is fixed by g_k and $g_i^{-1}g_k$. Now let us contract $(v_i)g$ and $(v_j)g$ to become one node $(v')g$ so that a homomorphic image β of (m_1, m_2) evolves. Then $(v')g$ in β is a fixed point of $g^{-1}g_kg$ and $g^{-1}g_i^{-1}g_kg$, whereas $((v')g)g^{-1}$

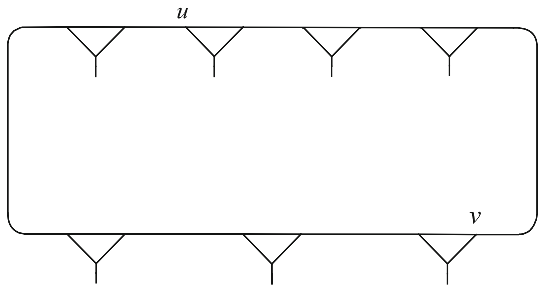

Figure 3. Graph of the circuit (4, 3).

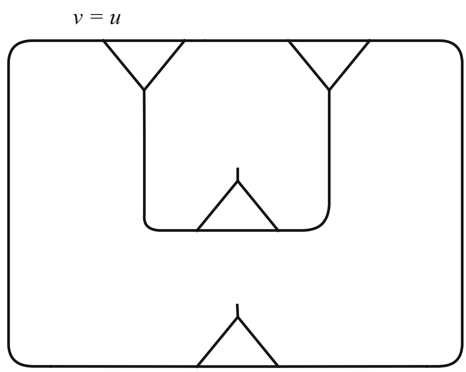

Figure 4. A homomorphic image of the circuit (4, 3).

in β is a fixed point of $g(g^{-1}g_k g)g^{-1} = g_k$ and $g(g^{-1}g_i^{-1}g_k g)g^{-1} = g_i^{-1}g_k$. Thus β and α are the same graphs. $\square$

Let E be the set of elements g in $PSL(2, \mathbb{Z})$ so that $(v_i)g$ and $(v_j)g$ lie in (m_1, m_2). Then by Proposition 1, we have the following result.

COROLLARY 1

If v_i and v_j are contracted to obtain α, then during this process $|E|$ number of pairs of vertices are contracted all together.

Note that $|E|$ is not the total number of pairs of vertices to create α. In next section, a method to count all the pairs for α is given.

Example 1. Consider a circuit in which there is a vertex v, fixed by $(xy)^4(xy^{-1})^3$. Thus, it is a circuit of length two and is denoted by (4, 3) (figure 3). Let us contract the vertices v and u of (4, 3) and a homomorphic image of (4, 3) is evolved. (figure 4).

By α^* we mean the mirror image of α. If $g = xy^{\kappa_1}xy^{\kappa_2}\ldots xy^{\kappa_n}$ ($\kappa_i = 1$ or -1) is a word, then let $g^* = xy^{-\kappa_1}xy^{-\kappa_2}\ldots xy^{-\kappa_n}$. If $(v)g = v$, then g^* fixed v^*.

We define a vertical axis of the symmetry of α as follows.

DEFINITION 3

A homomorphic image α has a vertical axis of symmetry if and only if by contracting v_i and v_j, the vertices v_i^* and v_j^* are also contracted.

Remark 2. In coset diagrams, t reverses the orientation of the triangles representing the three cycles of y (as reflection does). So corresponding to each vertex v fixed by the pair g_i, g_j in α, there is a vertex v^* in α^* (mirror image of α) such that v^* is a fixed point of $g_i{}^*, g_j{}^*$. In other words, it is created by contracting v_i^* and v_j^*. There are certain α's which have a vertical symmetry and so have the same orientations as those of their mirror images. The homomorphic image α of a circuit (m_1, m_2), which has a vertex v fixed by the pair g_i, g_j, has the same orientation as that of its mirror image if and only if there is a vertex v^* in α such that $(v^*)g_i{}^* = v^*$, $(v^*)g_j{}^* = v^*$.

2. Counting of the number of pairs of vertices for a homomorphic image

Let a homomorphic image α be obtained by contracting v_i and v_j in (m_1, m_2). The by Proposition 1, α has $|E|$ number of pairs of vertices. Note that $|E|$ is not the total number of pairs of vertices to create α. To find the total number of pairs of vertices, one should follow the following steps:

Step (*i*). If by contracting v_i and v_j to create α, the vertices v_i^* and v_j^* are not contracted. Then α has a different orientation from its mirror image α^*. So there are $|E|$ number of more pairs of vertices for the mirror image of α.

But if v_i, v_j and v_i^*, v_j^* are contracted all together, then α has a vertical symmetry. So in this case, α has $|E|$ number of pairs of vertices.

Step (*ii*). Now we check whether $m_1 = m_2$. If it is, then in addition to v_i, there is another vertex u_i which is fixed by the same word g_i in (m_1, m_2). It means that α is obtainable, if we contract u_i and v_j. If by contracting v_i and v_j to create α, and v_1^* and v_2^* to create α^*, the vertices u_i and v_j are not contracted. Then as many number of pairs of vertices for α are increased as obtained at the end of Step (i).

But if by contracting v_i and v_j or v_1^* and v_2^*, u_i and v_j are also contracted, or $m_1 \neq m_2$. Then there is no extra pairs for α. Thus we are left with as many pairs of vertices as obtained at the end of Step (i).

Remark 3. Consider a circuit (m_1, m_2) for convenience. Let $m_1 \geq m_2$. Let $i = 1, 2, 3, \ldots, 3m_1$ and $j = 1, 2, 3, \ldots, 3m_2$. In figure 5, one can see that the vertex $u_i^* = u_{3m_1-(i-1)}$ and $v_j^* = v_{3m_2-(j-1)}$. So in (m_1, m_2) corresponding to each vertex v, fixed by g, there is a vertex v^* such that $(v^*)g^* = v^*$.

The coset diagrams are composed of circuits. The vertices of the circuits in infinite diagrams are contracted in a certain way, and a finite coset diagram evolves. It is therefore necessary to ask, how many distinct homomorphic images are obtained if we contract all the pairs of vertices of a circuit? We not only give the answer to this question for a circuit of length two (m_1, m_2), but also mention those pairs of vertices which are 'important'. There is no need to contract the pairs which are not mentioned as 'important'. If we contract those, we obtain a homomorphic image, which we have already obtained by contracting 'important' pairs.

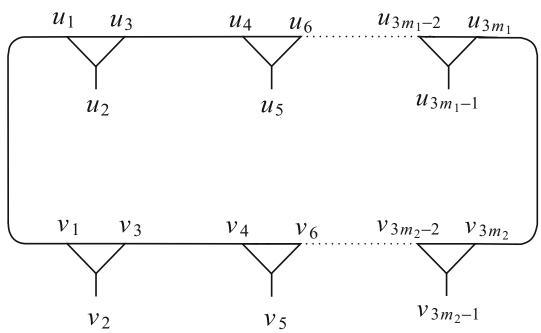

Figure 5. Graph of the circuit (m_1, m_2).

Theorem 1. *If the vertices* u_{3m_1} *and* $u_{3i_1+1} : i_1 = 0, 1, 2, \ldots, m_2 - 1$ *in* (m_1, m_2) *are contracted* (*melted together to become one node*), *then* m_2 *distinct homomorphic images of* (m_1, m_2) *are created. Moreover, the total number of pairs of vertices for contraction to create* α_{i_1} *are* $3(m_2^2 + 3m_2 - 2)$.

Proof. Let us contract u_{3m_1} and u_{3i_1+1} to obtain a family of homomorphic images of (m_1, m_2), denoted by α_{i_1}. Graphically, α_{i_1} can be classified into two families:

(i) $\alpha_{i_1} : i_1 < m_2 - 1$ (figure 6),
(ii) $\alpha_{i_1} : i_1 = m_2 - 1$ (figure 7).

In figure 5, one can see that $(xy)^{m_2}(xy^{-1})^{i_1}x$ and $y^{-1}(xy^{-1})^{m_1-i_1-1}$ are the two possible paths between u_{3m_1} and u_{3i_1+1}. Then for each i_1, there is a vertex v in α_{i_1} fixed by $(xy)^{m_2}(xy^{-1})^{i_1}x$ and $y^{-1}(xy^{-1})^{m_1-i_1-1}$. Now

$$E_1 = \{x, xy^{-1}, xy, xyx, xyxy^{-1}, (xy)^2, \ldots, (xy)^{i_1}x, \\ (xy)^{i_1}xy^{-1}, (xy)^{i_1+1}, e, y, y^{-1}\}$$

is the set of elements in $PSL(2, \mathbb{Z})$ so that $(u_{3m_1})g$ and $(u_{3i_1+1})g$ lies in (m_1, m_2) for all $g \in E_1$. Since $|E_1| = 3(i_1 + 2)$, by Proposition 1, the number of pairs of vertices to create α_{i_1} by contraction is $3(i_1 + 2)$. In figures 6 and 7, one can see that for $k \neq l$, α_k and α_l have different number of triangles. Therefore, all $\alpha_{i_1} : i_1 = 0, 1, 2, \ldots, m_2 - 1$ are different and none of them is a mirror image of the other.

Hence $|\alpha_{i_1}| = m_2$, so there are $3\sum_{i_1=0}^{m_2-1}(i_1 + 2)$ pairs of vertices to create $\{\alpha_{i_1} : i_1 = 0, 1, 2, \ldots, m_2 - 1\}$. Also from figures 6 and 7, only α_0 has a vertical axis of symmetry, that is, α_0 has the same orientation as that of its mirror image and all other $m_2 - 1$ homomorphic images of (m_1, m_2) do not possess a vertical axis of symmetry. Thus there are

$$6\sum_{i_2=1}^{m_2-1}(i_2 + 2) + 6 = 6\left\{\sum_{i_2=1}^{m_2-1}(i_2 + 2) + 1\right\} = 3(m_2^2 + 3m_2 - 2)$$

pairs of vertices for contraction to create $\{\alpha_{i_1} : i_1 = 0, 1, 2, \ldots, m_2 - 1\}$. □

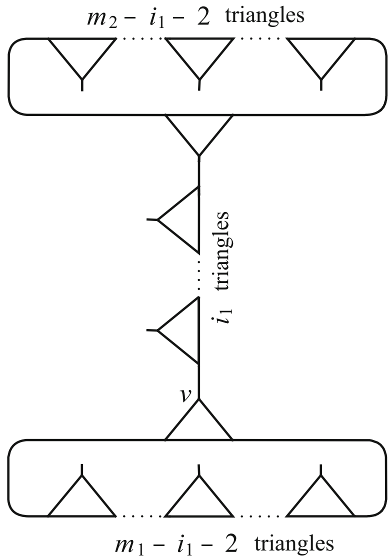

Figure 6. Graph of the homomorphic image $\alpha_{i_1} : i_1 < m_2 - 1$.

Theorem 2. *If the vertices u_{3m_1} and $v_{3i_1+1} : i_1 = 0, 1, 2, \ldots, m_2 - 1$ in (m_1, m_2) are contracted (melted together to become one node), then m_2 distinct homomorphic images of (m_1, m_2) are created. Moreover, the total number of pairs of vertices for contraction to create β_{i_1} are $\frac{3}{2}(m_2^2 + 3m_2)$.*

Proof. Let us contract u_{3m_1} and v_{3i_1+1} to obtain a family of homomorphic images of (m_1, m_2) denoted by β_{i_1}. Suppose r is the remainder of $\frac{i_1}{m_2-i_1}$. Graphically, β_{i_1} can be classified into four families:

(i) $\beta_{i_1} : m_2 - 2i_1 > 1$ (figure 8),
(ii) $\beta_{i_1} : m_2 - 2i_1 = 1$ (figure 9),
(iii) $\beta_{i_1} : m_2 - 2i_1 < 1$ and $m_2 - i_1 > r + 1$ (figure 10),
(iv) $\beta_{i_1} : m_2 - 2i_1 < 1$ and $m_2 - i_1 = r + 1$ (figure 11).

In figure 5, one can see that $(xy)^{m_2-i_1}$ and $(xy)^{i_1}(xy^{-1})^{m_1}$ are the two possible paths between u_{3m_1} and v_{3i_1+1}. Then for each i_1, β_{i_1} has a vertex v fixed by $(xy)^{m_2-i_1}$ and $(xy)^{i_1}(xy^{-1})^{m_1}$. Now $E_2 = \{x, xy^{-1}, xy, xyx, xyxy^{-1}, (xy)^2, \ldots, (xy)^{i_1}x, (xy)^{i_1}xy^{-1}, (xy)^{i_1+1}, e, y, y^{-1}\}$ is the set of elements in $PSL(2, \mathbb{Z})$ so that $(u_{3m_1})g$ and $(v_{3i_1+1})g$ lies in (m_1, m_2) for all $g \in E_2$. Since $|E_2| = 3(i_1 + 2)$, by Proposition 1, the number of pairs of vertices to create β_{i_1} by contraction is $3(i_1 + 2)$. In figures 8, 9, 10 and 11, one can see that for $k \neq l$, β_k and β_l have different number of triangles. Therefore, all $\beta_{i_1} : i_1 = 0, 1, 2, \ldots, m_2 - 1$ are different and none of them is a mirror image of the other.

Hence $|\beta_{i_1}| = m_2$, so there are $3\sum_{i_1=0}^{m_2-1}(i_1 + 2)$ pairs of vertices to create $\{\beta_{i_1} : i_1 = 0, 1, 2, \ldots, m_2 - 1\}$. Also from figures 8–11, all β_{i_1} have a vertical axis of symmetry. In other words, these diagrams have the same orientations as those of their mirror images. Thus there are $3\sum_{i_1=0}^{m_2-1}(i_1+2) = \frac{3}{2}(m_2^2+3m_2)$ pairs of vertices for contraction to create $\{\beta_{i_1} : i_1 = 0, 1, 2, \ldots, m_2 - 1\}$. $\square$

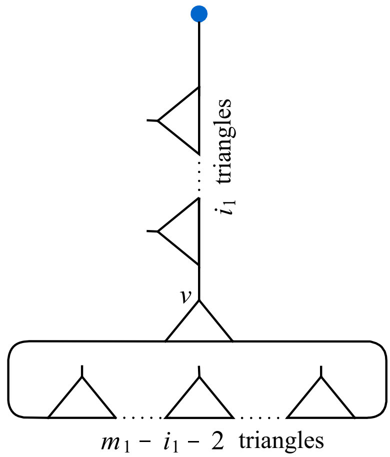

Figure 7. Graph of the homomorphic image $\alpha_{i_1} : i_1 = m_2 - 1$.

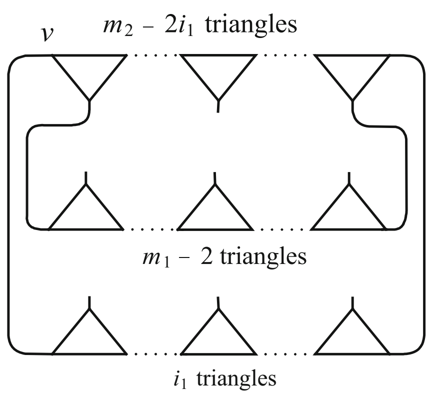

Figure 8. Graph of the homomorphic image $\beta_{i_1} : m_2 - 2i_1 > 1$.

Theorem 3. *If the vertices v_{3m_2} and $u_{3i_3+1} : i_3 = 1, 2, \ldots, m_1 - 1$ in (m_1, m_2) are contracted (melted together to become one node), then $m_1 - 1$ distinct homomorphic images of (m_1, m_2) are created. Moreover, the total number of pairs of vertices for contraction to create β'_{i_3} are $\frac{3}{2}(m_1^2 + 3m_1 - 4)$.*

The above theorem can be proved along the same lines as that of Theorem 2, by interchanging m_1, m_2, E_2, i_1 and β_{i_1} by m_2, m_1, E_3, i_3 and β'_{i_3} respectively. Suppose r' is the remainder of $\frac{i_3}{m_1 - i_3}$. Graphically, β'_{i_3} can be classified into four families:

(i) $\beta'_{i_3} : m_1 - 2i_3 > 1$ (figure 12),
(ii) $\beta'_{i_3} : m_1 - 2i_3 = 1$ (figure 13),
(iii) $\beta'_{i_3} : m_1 - 2i_3 < 0$ and $m_1 - i_3 > r' + 1$ (figure 14),
(iv) $\beta'_{i_3} : m_1 - 2i_3 < 0$ and $m_1 - i_3 = r' + 1$ (figure 15).

Let $\epsilon_1 = \begin{cases} 0, & \text{if } m_1 + m_2 \equiv 0 \bmod 2 \\ 1, & \text{if } m_1 + m_2 \equiv 1 \bmod 2 \end{cases}$.

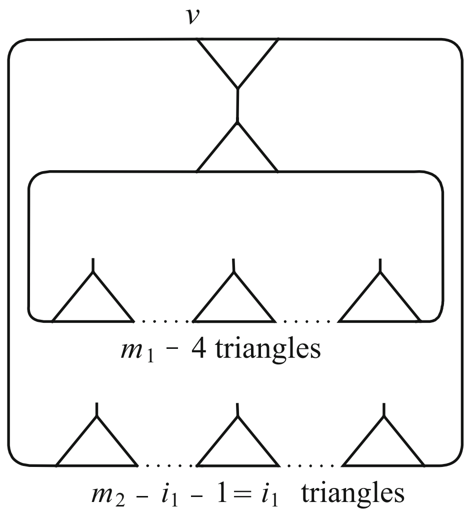

Figure 9. Graph of the homomorphic image $\beta_{i_1} : m_2 - 2i_1 = 1$.

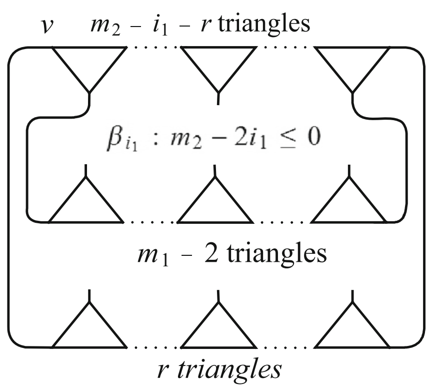

Figure 10. Graph of the homomorphic image $\beta_{i_1} : m_2 - 2i_1 < 1, m_2 - i_1 > r + 1$.

Theorem 4. *If the vertices u_{3m_1} and $u_{3j_1+1} : j_1 = m_2 + 1, m_2 + 2, \ldots, \frac{m_1+m_2-\epsilon_1}{2}$ in (m_1, m_2) are contracted (melted together to become one node), then $\frac{1}{2}(m_1 - m_2 - \epsilon_1)$ distinct homomorphic images of (m_1, m_2) are created. Moreover, the total number of pairs of vertices for contraction to create γ_{j_1} are $3(m_2 + 2)(m_1 - m_2 - 1)$.*

Proof. Let us contract u_{3m_1} and u_{3j_1+1} to obtain a family of homomorphic images of (m_1, m_2) denoted by γ_{j_1}. Diagrammatically by γ_{j_1}, we mean figure 16.

In figure 5, one can see that $(xy)^{m_2}(xy^{-1})^{j_1}x$ and $y^{-1}(xy^{-1})^{m_1-j_1-1}$ are the two possible paths between u_{3m_1} and u_{3j_1+1}. Then for each j_1, γ_{j_1} contains a vertex v fixed by $(xy)^{m_2}(xy^{-1})^{j_1}x$ and $y^{-1}(xy^{-1})^{m_1-j_1-1}$. Now

$$E_4 = \left\{ \begin{array}{c} x, xy^{-1}, xy, xyx, xyxy^{-1}, (xy)^2, \ldots, (xy)^{m_2}, \\ (xy)^{m_2}x, (xy)^{m_2}xy^{-1}, (xy)^{m_2+1}, e, y, y^{-1} \end{array} \right\}$$

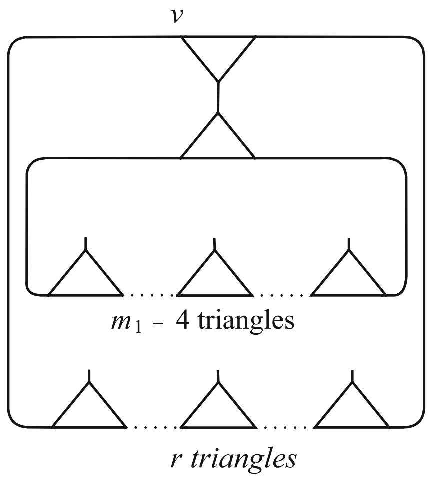

Figure 11. Graph of the homomorphic image $\beta_{i_1} : m_2 - 2i_1 < 1, m_2 - i_1 = r + 1$.

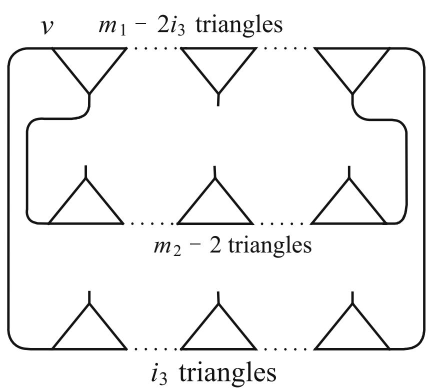

Figure 12. Graph of the homomorphic image $\beta'_{i_3} : m_1 - 2i_3 > 1$.

is the set of elements in $PSL(2, \mathbb{Z})$ so that $(u_{3m_1})g$ and $(u_{3j_1+1})g$ lies in (m_1, m_2) for all $g \in E_4$. Since $| E_4 | = 3(m_2 + 2)$, therefore by Proposition 1, the number of pairs of vertices to create γ_{j_1} by contraction is $3(m_2 + 2)$.

Consider two homomorphic images γ_k and γ_l of (m_1, m_2). Then γ_k and γ_l are created by contraction of u_{3m_1}, v_{3k+1} and u_{3m_1}, v_{3l+1} respectively. Let γ_k and γ_l be the same homomorphic images of (m_1, m_2). Then there is an element g in E_4 such that $(u_{3m_1})g = u_{3m_1}$ and $(v_{3k+1})g = v_{3l+1}$. There is only one element $e \in E_4$ which maps u_{3m_1} to u_{3m_1}, but $(v_{3k+1})e \neq v_{3l+1}$. Thus all homomorphic images in $\left\{\gamma_{j_1} : j_1 = m_2 + 1, m_2 + 2, \ldots, \frac{m_1+m_2-\epsilon_1}{2}\right\}$ are distinct. Thus $|\gamma_{j_1}| = \frac{1}{2}(m_1 - m_2 - \epsilon_1)$.

From figure 16, one can see that γ_k and γ_l are mirror images of each other if and only if $k - m_2 - 1 = m_1 - l - 1$ and $l - m_2 - 1 = m_1 - k - 1$. It means that $k = \frac{m_1+m_2}{2} = h$, implying that for $k \neq l$, γ_k and γ_l are not mirror images of each other and $\gamma_{\frac{m_1+m_2}{2}}$ is the mirror image of itself, that is, it has the same orientation as that of its mirror image. Now

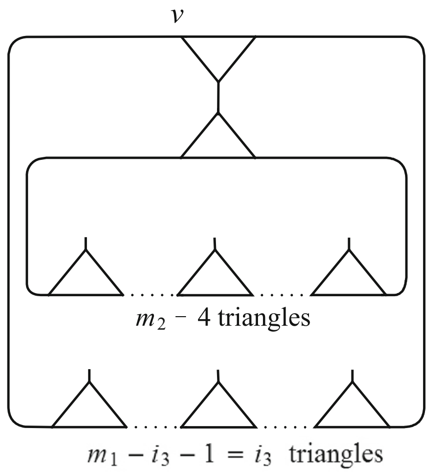

Figure 13. Graph of the homomorphic image β'_{i_3} : $m_1 - 2i_3 = 1$.

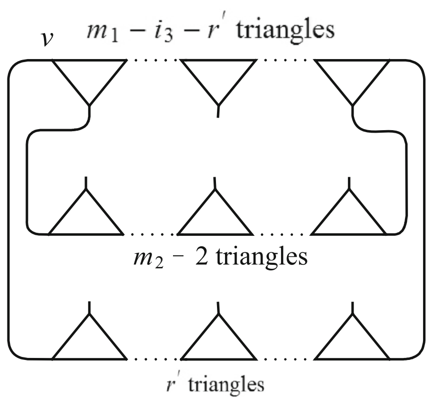

Figure 14. Graph of the homomorphic image β'_{i_3} : $m_1 - 2i_3 < 0, m_1 - i_3 > r' + 1$.

(i) If $m_1 + m_2 \equiv 0 \mod 2$, then only $\mu_{\frac{m_1+m_2}{2}} \in \{\gamma_{j_1}\}$ has the same orientation as that of its mirror image, and all other $\frac{1}{2}(m_1 - m_2 - 2)$ homomorphic images have different orientations from their mirror images. Hence there are

$$2\,|\,E_4\,|\left(\frac{m_1 - m_2 - 2}{2}\right) + |\,E_4\,| = 3(m_2 + 2)(m_1 - m_2 - 1)$$

pairs of vertices to create γ_{j_1}.

(ii) If $m_1 + m_2 \equiv 1 \mod 2$, then all γ_{j_1} have different orientations from their mirror images. Hence there are

$$2\,|\,E_4\,|\left(\frac{m_1 - m_2 - 1}{2}\right) = 3(m_2 + 2)(m_1 - m_2 - 1)$$

pairs of vertices to compose γ_{j_1}.

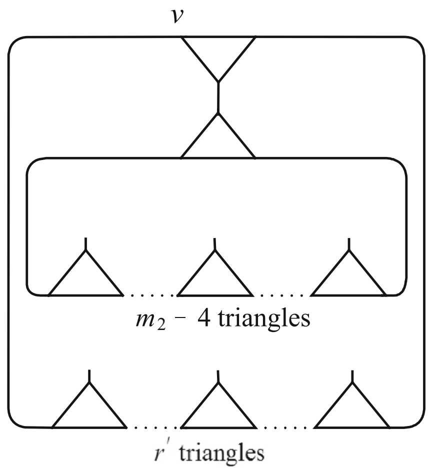

Figure 15. Graph of the homomorphic image $\beta'_{i_3}: m_1 - 2i_3 < 0, m_1 - i_3 = r' + 1$.

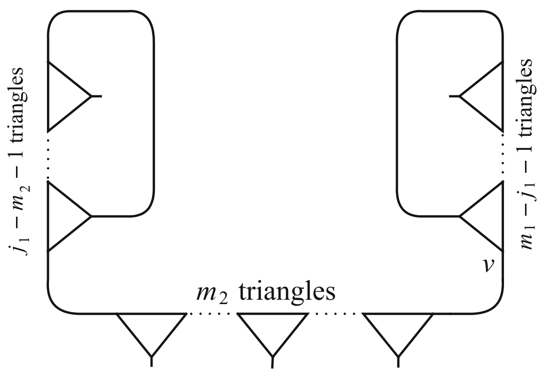

Figure 16. Graph of the homomorphic image γ_{j_1}.

Let $\epsilon_2 = \begin{cases} 0, \text{ if } m_2 \equiv 0 \bmod 2 \\ 1, \text{ if } m_2 \equiv 1 \bmod 2 \end{cases}$. □

Theorem 5. *If the vertices* $u_{3i_3+1} : i_3 = 1, 2, \ldots, m_1 - 1$ *and* $v_{3j_2} : j_2 = 1, 2, \ldots, \frac{m_2-(\epsilon_2+2)}{2}$ *in* (m_1, m_2) *are contracted* (*melted together to become one node*), *then* $\frac{1}{2}(m_1 - 1)(m_2 - (\epsilon_2 + 2))$ *distinct homomorphic images of* (m_1, m_2) *are created. Moreover, the total number of pairs of vertices for contraction to create* $\lambda_{(i_3,j_2)}$ *are* $6(m_1 - 1)(m_2 - (\epsilon_2 + 2))$.

Proof. Let us contract u_{3i_3+1} and v_{3j_2} to obtain a family of homomorphic images of (m_1, m_2) denoted by $\lambda_{(i_3,j_2)}$. Diagrammatically by $\lambda_{(i_3,j_2)}$, we mean figure 17.

In figure 5, one can see that $(xy)^{i_3}(xy^{-1})^{j_2}$ and $(xy^{-1})^{m_2-j_2}(xy)^{m_1-i_3}$ are the two possible paths between u_{3i_3+1} and v_{3j_2}. Then for each i_3, j_2, $\lambda_{(i_3,j_2)}$ contains a vertex v fixed by $(xy)^{i_3}(xy^{-1})^{j_2}$ and $(xy^{-1})^{m_2-j_2}(xy)^{m_1-i_3}$. Now $E_5 = \{x, xy^{-1}, xy, e, y, y^{-1}\}$ is the set of elements in $PSL(2, \mathbb{Z})$ so that $(u_{3i_3+1})g$ and $(v_{3j_2})g$ lies in (m_1, m_2) for all

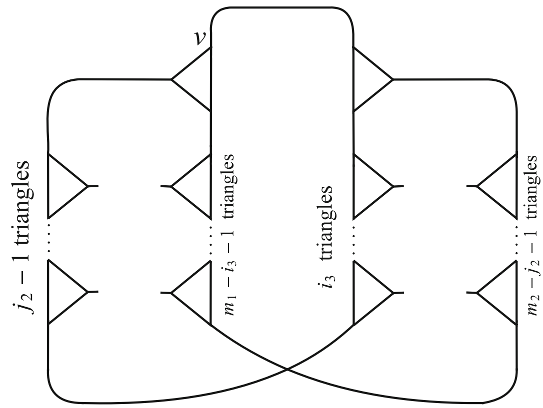

Figure 17. Graph of the homomorphic image $\lambda_{(i_3,\ j_2)}$.

$g \in E_5$. Since $|\,E_5\,| = 6$, therefore by Proposition 1, the number of pairs of vertices to create each $\lambda_{(i_3,j_2)}$ by contraction is 6.

(i) Consider two homomorphic images, $\lambda_{(h_1,k_1)}, \lambda_{(h_2,k_2)} \in \{\lambda_{(i_3,j_2)}\}$. Then $\lambda_{(h_1,k_1)}$ and $\lambda_{(h_2,k_2)}$ are formed by contraction of u_{3h_1+1}, v_{3k_1} and u_{3h_2+1}, v_{3k_2} respectively. Let $\lambda_{(h_1,k_1)}$ and $\lambda_{(h_2,k_2)}$ be the same homomorphic image of (m_1, m_2). Then there is an element g in E_5 such that $(u_{3h_1+1})g = u_{3h_2+1}$ and $(v_{3k_1})g = v_{3k_2}$. There is only one element $xy^{-1} \in E_5$ which maps u_{3h_1+1} to u_{3h_2+1} $(h_2 = h_1+1)$, but $(v_{3k_1})xy^{-1} \neq v_{3k_2}$. Therefore $\lambda_{(h_1,k_1)}, \lambda_{(h_2,k_2)}$ are distinct.
(ii) From figure 17, one can see that $\lambda_{(h_1,k_1)}$ and $\lambda_{(h_2,k_2)}$ are mirror images of each other if and only if $h_1-1 = m_1-h_2-1$, $m_1-h_2-1 = h_1-1$ and $m_2-k_1-1 = k_2-1$, $k_1-1 = m_2-k_2-1$ which means $h_2 = m_1-h_1$, $k_2 = m_2-k_1$.

So $\lambda_{(h_1,k_1)}$ and $\lambda_{(m_1-h_1,m_2-k_1)}$ are mirror images of each other, but for each $k_1 \in \{1, 2, \ldots, \frac{m_2-(\epsilon_2+2)}{2}\}$, $m_2 - k_1 \notin \{1, 2, \ldots, \frac{m_2-(\epsilon_2+2)}{2}\}$ consequently, $\lambda_{(m_1-h_1,m_2-k_1)} \notin \{\lambda_{(i_3,j_2)}\}$. Therefore $\lambda_{(h_1,k_1)}, \lambda_{(h_2,k_2)} \in \{\lambda_{(i_3,j_2)}\}$ are not mirror images of each other.
(iii) Let $\lambda_{(h,k)} \in \left\{\lambda_{(i_3,j_2)}\right\}$ be the mirror image of itself, that is, has the same orientation as that of its mirror image. Then from figure 17, $h-1 = m_1-h-1$ and $k-1 = m_2-k-1$, which means $h = \frac{m_1}{2}$, $k = \frac{m_2}{2}$. But $\lambda_{(h,k)} \notin \{\lambda_{(\frac{m_1}{2},\frac{m_2}{2})}\}$.

From (i), (ii) and (iii), we have all homomorphic images in

$$\left\{\lambda_{(i_3,j_2)} : i_3 = 1, 2, \ldots, m_1-1,\ j_2 = 1, 2, \ldots, \frac{m_2-(\epsilon_2+2)}{2}\right\}$$

which are distinct and none of them is a mirror image of the other or itself. Thus, $|\,\lambda_{(i_3,j_2)}\,| = \frac{1}{2}(m_1-1)(m_2-(\epsilon_2+2))$, and there are $2 \times |\,E_5\,| \times |\,\lambda_{(i_3,j_2)}\,| = 6(m_1-1)(m_2-(\epsilon_2+2))$ pairs of vertices to compose $\lambda_{(i_3,j_2)}$. $\square$

Recall $\epsilon_2 = \begin{cases} 0, \text{ if } m_2 \equiv 0 \bmod 2 \\ 1, \text{ if } m_2 \equiv 1 \bmod 2 \end{cases}$ and let

$$i_4 = \begin{cases} 1, 2, \ldots, \frac{m_1-\epsilon_3}{2} & \text{if } m_2 \equiv 0 \bmod 2 \\ 1, 2, \ldots, m_1-1 & \text{if } m_2 \equiv 1 \bmod 2 \end{cases}, \quad \epsilon_3 = \begin{cases} 0 & \text{if } m_1 \equiv 0 \bmod 2 \\ 1 & \text{if } m_1 \equiv 1 \bmod 2 \end{cases}.$$

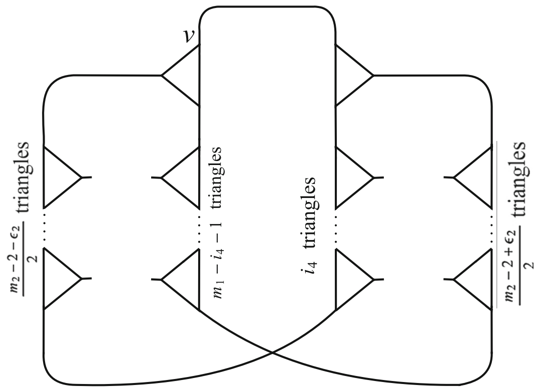

Figure 18. Graph of the homomorphic image $\lambda_{\left(i_4,\frac{m_2-\epsilon_2}{2}\right)}$.

Theorem 6. *If the vertices u_{3i_4+1} and $v_{3(\frac{m_2-\epsilon_2}{2})}$ in (m_1,m_2) are contracted (melted together to become one node), then* $\begin{cases} \frac{m_1-\epsilon_3}{2}, & \text{if } m_2 \equiv 0 \bmod 2 \\ m_1-1, & \text{if } m_2 \equiv 1 \bmod 2 \end{cases}$ *distinct homomorphic images of (m_1,m_2) are created. Moreover, the total number of pairs of vertices for contraction to create $\lambda_{(i_4,\frac{m_2-\epsilon_2}{2})}$ are* $\begin{cases} 6(m_1-1), & \text{if } m_2 \equiv 0 \bmod 2 \\ 12(m_1-1), & \text{if } m_2 \equiv 1 \bmod 2 \end{cases}$.

Proof. Let us contract u_{3i_4+1} and $v_{3(\frac{m_2-\epsilon_2}{2})}$ to obtain a family of homomorphic images of (m_1,m_2) denoted by $\lambda_{(i_4,\frac{m_2-\epsilon_2}{2})}$. Diagrammatically by $\lambda_{(i_4,\frac{m_2-\epsilon_2}{2})}$, we mean figure 18.

In figure 5, one can see that $(xy)^{i_4}(xy^{-1})^{\frac{m_2-\epsilon_2}{2}}$ and $(xy^{-1})^{\frac{m_2+\epsilon_2}{2}}(xy)^{m_1-i_4}$ are the two possible paths between u_{3i_4+1} and $v_{3(\frac{m_2-\epsilon_2}{2})}$. Then for each i_4, $\lambda_{(i_4,\frac{m_2-\epsilon_2}{2})}$ contains a vertex v fixed by $(xy)^{i_4}(xy^{-1})^{\frac{m_2-\epsilon_2}{2}}$ and $(xy^{-1})^{\frac{m_2+\epsilon_2}{2}}(xy)^{m_1-i_4}$. Now $E_6 = \{x, xy^{-1}, xy, e, y, y^{-1}\}$ is the set of elements in $PSL(2,\mathbb{Z})$ so that $(u_{3i_4+1})g$ and $(v_{3(\frac{m_2-\epsilon_2}{2})})g$ lies in (m_1,m_2) for all $g \in E_6$. Since $|E_6| = 6$, therefore, by Proposition 1, the number of pairs of vertices to create each $\lambda_{(i_4,\frac{m_2-\epsilon_2}{2})}$ by contraction is 6.

We prove in Theorem 5, that all fragments in $\{\lambda_{(i_3,j_2)}\}$ are distinct, and that the mirror image of $\lambda_{(h_1,k_1)} \in \{\lambda_{(i_3,j_2)}\}$ is $\lambda_{(m_1-h_1,m_2-k_1)}$. Similarly, we have that all the fragments in $\{\lambda_{(i_4,\frac{m_2-\epsilon_2}{2})}\}$ are different, and the mirror image of $\lambda_{(h_1,\frac{m_2-\epsilon_2}{2})} \in \{\lambda_{(i_4,\frac{m_2-\epsilon_2}{2})}\}$ is $\lambda_{(m_1-h_1,\frac{m_2+\epsilon_2}{2})}$. Now

(i) If $m_2 \equiv 1 \bmod 2$, then $\lambda_{(m_1-h_1,\frac{m_2+1}{2})} \notin \{\lambda_{(i_4,\frac{m_2-1}{2})}\}$. This shows that none of the fragments in $\{\lambda_{(i_4,\frac{m_2-1}{2})}\}$ is the mirror image of the other. Hence $|\{\lambda_{(i_4,\frac{m_2-1}{2})}\}| = m_1-1$, and so there are $2 \times |E_6| \times |\{\lambda_{(i_4,\frac{m_2-1}{2})}\}| = 12(m-1)$ pairs of vertices to compose $\lambda_{(i_4,\frac{m_2-\epsilon_2}{2})}$.

(ii) If $m_2 \equiv 0 \bmod 2$ and $m_1 \equiv 1 \bmod 2$, then for all $h_1 \in \{1,2,\ldots,\frac{m_1-1}{2}\}$, we have $m_1-h_1 > \frac{m-1}{2}$ implying that $\lambda_{(m_1-h_1,\frac{m_2}{2})} \notin \{\lambda_{(i_4,\frac{m_2}{2})}\}$. So none of the fragments in $\{\lambda_{(i_4,\frac{m_2}{2})}\}$ is the mirror image of the other, which implies that $|\{\lambda_{(i_4,\frac{m_2}{2})}\}| = \frac{m-1}{2}$. Hence the total number of pairs of vertices for $\lambda_{(i_4,\frac{m_2}{2})}$ are $2 \times |E_6| \times |\{\lambda_{(i_4,\frac{m_2}{2})}\}| = 6(m-1)$.

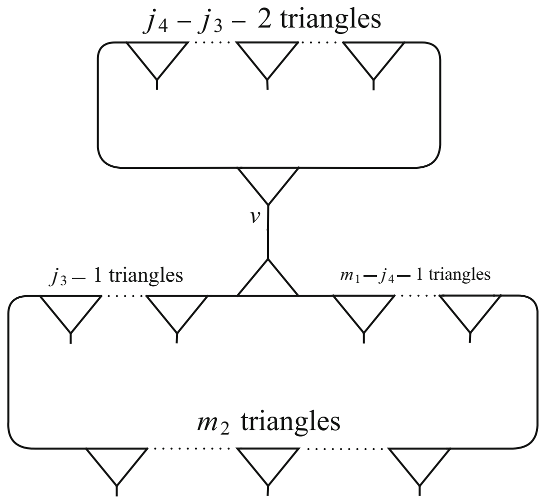

Figure 19. Graph of the homomorphic image $\chi_{(j_3,\ j_4)}$.

(iii) If m_1 and m_2 are both even, then for all $h_1 \in \left\{1, 2, \ldots, \frac{m_1}{2}\right\} \setminus \frac{m_1}{2}$, we have $m_1 - h_1 > \frac{m_1}{2}$ implying that $\lambda_{(m_1-h_1, \frac{m_2}{2})} \notin \{\lambda_{(i_4, \frac{m_2}{2})}\}$, and for $h_1 = \frac{m_1}{2}$, we have $\lambda_{(m_1-h_1, \frac{m_2}{2})} = \lambda_{(\frac{m_1}{2}, \frac{m_2}{2})} \in \{\lambda_{(i_4, \frac{m_2}{2})}\}$. This shows that for $i_4 < \frac{m_1}{2}$, none of the fragments in $\{\lambda_{(i_4, \frac{m_2}{2})}\}$ is the mirror image of the other and $\lambda_{(\frac{m_1}{2}, \frac{m_2}{2})}$ is the mirror image of itself, which implies that $|\{\lambda_{(i_4, \frac{m_2}{2})}\}| = \frac{m_1}{2}$. Hence there are $2 \times |E_6| \times (\frac{m_1-2}{2}) + |E_6| = 6(m_1 - 1)$ pairs of vertices for $\lambda_{(i_4, \frac{m_2}{2})}$. $\square$

Recall $\epsilon_3 = \begin{cases} 0, & \text{if } m_1 \equiv 0 \bmod 2 \\ 1, & \text{if } m_1 \equiv 1 \bmod 2 \end{cases}$ and let

$$j_3 = 1, 2, 3, \ldots, \frac{m_1 - 2 + \epsilon_3}{2}, \quad j_4 = j_3 + 1, j_3 + 2, \ldots, m_1 - j_3.$$

Theorem 7. *If the vertices $u_{3_{j_3+1}}$ and $u_{3_{j_4}}$ in (m_1, m_2) are contracted (melted together to become one node), then $\frac{1}{4}(m_1^2 - 2m_1 + \epsilon_3)$ distinct homomorphic images of (m_1, m_2) are created. Moreover, the total number of pairs of vertices for contraction to create $\chi_{(j_3, j_4)}$ are $3(m_1^2 - 3m_1 + 2)$.*

Proof. Let us contract $u_{3_{j_3+1}}$ and $u_{3_{j_4}}$ to obtain a family of homomorphic images of (m_1, m_2) denoted by $\chi_{(j_3, j_4)}$. Diagrammatically by $\chi_{(j_3, j_4)}$, we mean figure 19.

In figure 5, one can see that $y^{-1}(xy^{-1})^{j_4-j_3-1}$ and $(xy)^{j_3}(xy^{-1})^{m_2}(xy)^{m_1-j_4}x$ are the two possible paths between $u_{3_{j_3+1}}$ and $u_{3_{j_4}}$. Then for each j_3, j_4, there is a vertex v in $\chi_{(j_3, j_4)}$ such that

$$(v)y^{-1}(xy^{-1})^{j_4-j_3-1} = v = (v)(xy)^{j_3}(xy^{-1})^{m_2}(xy)^{m_1-j_4}x.$$

Now $E_7 = \{x, xy^{-1}, xy, e, y, y^{-1}\}$ is the set of elements in $PSL(2, \mathbb{Z})$ so that $(u_{3_{j_3+1}})g$ and $(u_{3_{j_4}})g$ lies in (m_1, m_2) for all $g \in E_7$. Since $|E_7| = 6$, therefore, by Proposition 1, the number of pairs of vertices to create each $\chi_{(j_3,j_4)}$ by contraction is 6. Consider two homomorphic images $\chi_{(h_1,k_1)}, \chi_{(h_2,k_2)} \in \{\chi_{(j_3,j_4)}\}$. Then $\chi_{(h_1,k_1)}$ and $\chi_{(h_2,k_2)}$ are formed by contraction of u_{3h_1+1}, u_{3k_1} and u_{3h_2+1}, u_{3k_2} respectively. Let $\chi_{(h_1,k_1)}$ and $\chi_{(h_2,k_2)}$ be the same homomorphic images of (m_1, m_2). Then there is an element g in E_7 such that $(u_{3h_1+1})g = u_{3h_2+1}$ and $(u_{3k_1})g = u_{3k_2}$. There is only two elements $xy, xy^{-1} \in E_7$ such that xy maps u_{3h_1+1} to u_{3h_2+1} $(h_2 = h_1 - 1)$ and xy^{-1} maps u_{3h_1} to u_{3h_2} $(h_2 = h_1 + 1)$, but in both the cases neither $(u_{3k_1})xy^{-1} = u_{3k_2}$ nor $(u_{3k_1})xy^{-1} = u_{3k_2}$. Therefore $\chi_{(h_1,k_1)}, \chi_{(h_2,k_2)}$ are distinct.

Let $\chi_{(h_1,k_1)}$ and $\chi_{(h_2,k_2)}$ be mirror images of each other, that is, $\chi_{(h_1,k_1)} = \chi^*_{(h_2,k_2)}$. Then by Remark 2, $\chi_{(h_1,k_1)}$ can be obtained by contracting $u^*_{3h_2+1}$ and $u^*_{3k_2}$. In other words, $u^*_{3h_2+1} \longleftrightarrow u^*_{3k_2}$ is one of the 6 pairs of vertices which create $\chi_{(h_1,k_1)}$ by contraction. So there is an element g in E_7 such that $(u_{3h_1+1})g = u^*_{3h_2+1}$ and $(u_{3k_1})g = u^*_{3k_2}$. There is only one element $x \in E_7$ which maps u_{3h_1+1} to $u_{3h_1} = u^*_{3(m_1-h_1)+1}$ and u_{3k_1} to $u_{3k_1+1} = u^*_{3(m_1-k_1)}$. This implies that for $h_2 = m_1 - h_1$ and $k_2 = m_1 - k_1$, $\chi_{(h_1,k_1)}$ and $\chi_{(h_2,k_2)}$ are mirror images of each other. But for each $h_1 \in \left\{1, 2, 3, \ldots, \frac{m_1-2+\epsilon_3}{2}\right\}$, clearly $h_2 = m_1 - h_1 > \frac{m_1-2+\epsilon_3}{2}$, which means that $\chi_{(h_2,k_2)} = \chi_{(m_1-h_1,m_1-k_1)} \notin \left\{\begin{array}{c}\chi_{(j_3,j_4)} : j_3 = 1, 2, 3, \ldots, \frac{m_1-2+\epsilon_3}{2} \text{ and} \\ j_4 = j_3+1, j_3+2, \ldots, m_1 - j_3\end{array}\right\}$. Therefore, $\chi_{(h_1,k_1)}, \chi_{(h_2,k_2)} \in \{\chi_{(j_3,j_4)}\}$ are not mirror images of each other. Hence,

$$\begin{aligned}|\chi_{(i_3,j_2)}| &= (m_1 - 2) + (m_1 - 4) + (m_1 - 6) + \cdots + (4 - \epsilon_3) + (2 - \epsilon_3)\\ &= \frac{1}{4}(m_1^2 - 2m_1 + \epsilon_3).\end{aligned}$$

Let $\varkappa_{(h,k)} \in \{\chi_{(j_3,j_4)}\}$ be the mirror image of itself, that is, has the same orientation as that of its mirror image. Then from figure 19, $h-1 = m_1-k-1$ which means $k = m_1-h$. Now for all $h \in \{1, 2, 3, \ldots, \frac{m_1-2+\epsilon_3}{2}\}$, we have $k = m_1-h \in \{h+1, h+2, \ldots, m_1-h\}$ implying that for all $h \in \{1, 2, 3, \ldots, \frac{m_1-2+\epsilon_3}{2}\}$, $\varkappa_{(h,m_1-h)} \in \{\chi_{(j_3,j_4)}\}$ which is the mirror image of itself.

So out of $\frac{1}{4}(m_1^2 - 2m_1 + \epsilon_3)$ homomorphic images in $\{\chi_{(j_3,j_4)}\}$, $\frac{m_1-2+\epsilon_3}{2}$ are the mirror images of itself, and hence there are

$$\begin{aligned}&2\times |E_7| \times \left(\frac{1}{4}(m_1^2 - 2m_1 + \epsilon_3) - \frac{m_1 - 2 + \epsilon_3}{2}\right)\\ &\quad + |E_7| \times \left(\frac{m_1 - 2 + \epsilon_3}{2}\right) = 3(m_1^2 - 3m_1 + 2)\end{aligned}$$

pairs of vertices to compose $\chi_{(j_3,j_4)}$. □

Recall $\epsilon_2 = \begin{cases}0, & \text{if } m_2 \equiv 0 \bmod 2\\ 1, & \text{if } m_2 \equiv 1 \bmod 2\end{cases}$ and let

$$j_5 = 1, 2, 3, \ldots, \frac{m_2 - 2 + \epsilon_2}{2}, \quad j_6 = j_5 + 1, j_5 + 2, \ldots, m_2 - j_5.$$

Theorem 8. *If the vertices v_{3j_5+1} and v_{3j_6} in (m_1, m_2) are contracted (melted together to become one node), then $\frac{1}{4}(m_2^2 - 2m_2 + \epsilon_2)$ distinct homomorphic images of (m_1, m_2) are*

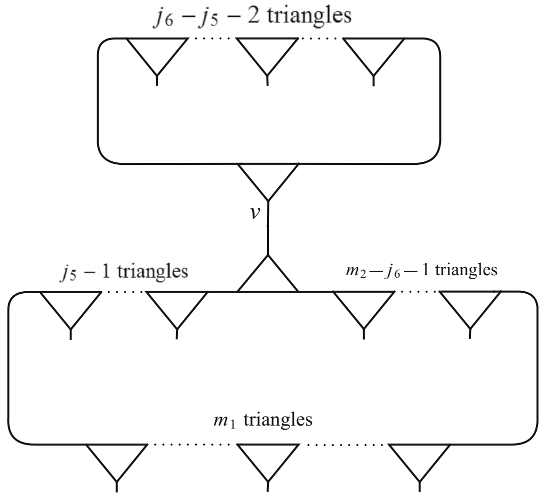

Figure 20. Graph of the homomorphic image $\chi'_{(j_5,\, j_6)}$.

created. Moreover, the total number of pairs of vertices for contraction to create $\chi'_{(j_5,j_6)}$ *are* $3(m_2^2 - 3m_2 + 2)$.

The above theorem can be proved along the same lines as those of Theorem 7, by interchanging m_1, m_2, E_7, j_3, j_4 and $\chi_{(j_3,j_4)}$ by m_2, m_1, E_8, j_5, j_6 and $\chi'_{(j_5,j_6)}$ respectively. Diagrammatically by $\chi'_{(j_5,j_6)}$, we mean figure 20.

Recall, $\epsilon_3 = \begin{cases} 0, & \text{if } m_1 \equiv 0 \bmod 2 \\ 1, & \text{if } m_1 \equiv 1 \bmod 2 \end{cases}$ and let $j_7 = \left\{1, 2, 3, \ldots, \frac{m_1 - \epsilon_3}{2}\right\}$.

Theorem 9. *If the vertices* u_{3j_7+1} *and* u_{3j_7} *in* (m_1, m_2) *are contracted* (*melted together to become one node*), *then* $\frac{1}{2}(m_1 - \epsilon_3)$ *distinct homomorphic images of* (m_1, m_2) *are created. Moreover, the total number of pairs of vertices for contraction to create* $\chi_{(j_7,j_7)}$ *are* $3(m_1 - 1)$.

Proof. Let us contract u_{3j_7+1} and u_{3j_7} to obtain a family of homomorphic images of (m_1, m_2) denoted by $\chi_{(j_7,j_7)}$. Diagrammatically by $\chi_{(j_7,j_7)}$, we mean figure 21.

In figure 5, one can see that x and $y(xy)^{j_7-1}(xy^{-1})^{m_2}(xy)^{m_1-j_7}$ are the two possible paths between u_{3j_7+1} and u_{3j_7}. Then for each j_7, $\chi_{(j_7,j_7)}$ contains a vertex v fixed by x and $y(xy)^{j_7-1}(xy^{-1})^{m_2}(xy)^{m_1-j_7}$. Now $E_9 = \{x, xy, xy^{-1}, e, y, y^{-1}\}$ is the set of elements in $PSL(2, \mathbb{Z})$ so that $(u_{3j_7+1})g$ and $(u_{3j_7})g$ lies in (m_1, m_2) for all $g \in E_9$. But one can see that '$(u_{3j_7+1})x$, $(u_{3j_7})x$ and $(u_{3j_7})e$, $(u_{3j_7+1})e$', '$(u_{3j_7+1})xy$, $(u_{3j_7})xy$ and $(u_{3j_7})y$, $(u_{3j_7+1})y$' and '$(u_{3j_7+1})xy^{-1}$, $(u_{3j_7})xy^{-1}$ and $(u_{3j_7})y^{-1}$, $(u_{3j_7+1})y^{-1}$' are the same pairs of vertices. Therefore, the number of pairs of vertices to create each $\chi_{(j_7,j_7)}$ by contraction is 3.

In Theorem 7, we prove that all homomorphic images of (m_1, m_2) in $\{\chi_{(j_3,j_4)}\}$ are distinct, and that the mirror image of $\chi_{(h_1,k_1)} \in \{\chi_{(j_3,j_4)}\}$ is $\chi_{(m_1-h_1,m_1-k_1)}$. Similarly, we have that all homomorphic images of (m_1, m_2) in $\chi_{(j_7,j_7)}$ are different, and the mirror image of $\chi_{(h_1,h_1)} \in \chi_{(j_7,j_7)}$ is $\chi_{(m_1-h_1,m_1-h_1)}$. Now for all $h_1 \in \left\{1, 2, 3, \ldots, \frac{m_1-\epsilon_3}{2}\right\} \setminus \frac{m_1}{2}$,

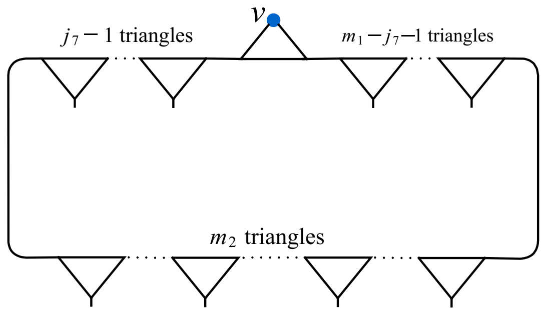

Figure 21. Graph of the homomorphic image $\chi_{(j_7,\ j_7)}$.

we have $m_1 - h_1$ is greater than all the members in $\left\{1, 2, 3, \ldots, \frac{m_1-\epsilon_3}{2}\right\} \diagdown \frac{m_1}{2}$, implying that $\chi_{(m_1-h_1, m_1-h_1)} \notin \{\chi_{(j_7,j_7)}\}$. But for $h_1 = \frac{m_1}{2}$, we have $m_1 - h_1 = \frac{m_1}{2}$, so $\chi_{\frac{m_1}{2}}$ is the mirror image of itself. Thus all homomorphic images of (m_1, m_2) in $\{\chi_{(j_7,j_7)}\}$ are different, none of them is a mirror image of the other, implying that $|\chi_{(j_7,j_7)}| = \frac{1}{2}(m_1 - \epsilon_3)$. Now we have two cases:

(i) If $m_1 \equiv 0 \mod 2$, then none of the homomorphic image (m_1, m_2) in $\{\chi_{(j_7,j_7)}\}$ is the mirror image of itself. Hence, there are $2 \times |E_9| \times \frac{1}{2}(m_1 - 1) = 3(m_1 - 1)$ pairs of vertices to compose $\chi_{(j_7,j_7)}$.
(ii) If $m_1 \equiv 0 \mod 2$, then only $\chi_{\frac{m_1}{2}} \in \{\chi_{(j_7,j_7)}\}$ is the mirror image of itself. Hence, there are $2 \times |E_9| \times \frac{1}{2}(m_1 - 2) + |E_9| = 3(m_1 - 1)$ pairs of vertices to compose $\chi_{(j_7,j_7)}$. □

Recall $\epsilon_2 = \begin{cases} 0, & \text{if } m_2 \equiv 0 \bmod 2 \\ 1, & \text{if } m_2 \equiv 1 \bmod 2 \end{cases}$ and let $j_8 = 1, 2, 3, \ldots, \frac{m_2-\epsilon_2}{2}$.

Theorem 10. *If the vertices v_{3j_8+1} and v_{3j_8} in (m_1, m_2) are contracted (melted together to become one node), then $\frac{1}{2}(m_2 - \epsilon_2)$ distinct homomorphic images of (m_1, m_2) are created. Moreover, the total number of pairs of vertices for contraction to create $\chi_{(j_8,j_8)}$ are $3(m_2 - 1)$.*

The above theorem can be proved along the same lines as those of Theorem 9, by interchanging m_1, m_2, E_9, j_7 and $\chi_{(j_7,j_7)}$ by m_2, m_1, E_{10}, j_8 and $\chi'_{(j_8,j_8)}$ respectively. Diagrammatically by $\chi'_{(j_8,j_8)}$, we mean figure 22.

Theorem 11. *Let η be the homomorphic image of (m_1, m_2) composed by contracting the vertices u_{3m_1} and v_{3m_2} in (m_1, m_2). Then there are*

$$\begin{cases} 6(m_2 + 1), & \text{if } m_1 > m_2 \\ 3m_1, & \text{if } m_1 = m_2 \end{cases}.$$

pairs of vertices to compose η and their mirror images.

Proof. Let us contract the vertices u_{3m_1} and v_{3m_2} and a homomorphic image of (m_1, m_2) denoted by η is created. Diagrammatically by η, we mean

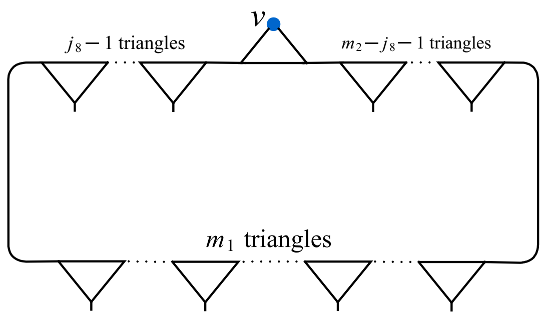

Figure 22. Graph of the homomorphic image $\chi_{(j_8,\ j_8)}$.

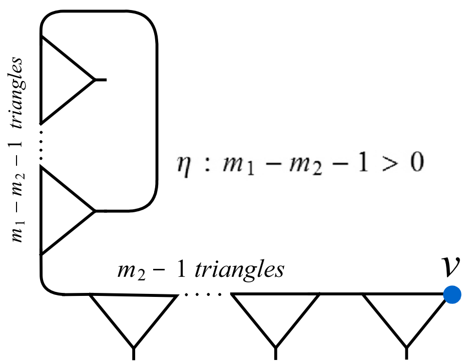

Figure 23. Graph of the homomorphic image $\eta : m_1 - m_2 - 1 > 0$.

In figure 5, we have x and $y(xy)^{m_2-1}(xy^{-1})^{m_1}$ are the two possible paths between u_{3m_1} and v_{3m_2}. Then η contains a vertex v fixed by x and $y(xy)^{m_2-1}(xy^{-1})^{m_1}$. Now if $m_1 > m_2$, then

$$E_{11} = \left\{ \begin{array}{c} x, xy^{-1}, xy, xyx, xyxy^{-1}, (xy)^2, x, xy^{-1}, (xy)^3, \ldots, \\ (xy)^{m_2-1}x, (xy)^{m_2-1}xy^{-1}, (xy)^{m_2}, (xy)^{m_2}x, (xy)^{m_2}xy^{-1}, (xy)^{m_2+1} \end{array} \right\}$$

and for $m_1 = m_2$,

$$E_{11} = \left\{ \begin{array}{c} x, xy^{-1}, xy, xyx, xyxy^{-1}, (xy)^2, x, xy^{-1}, (xy)^3, \ldots, \\ (xy)^{m_2-1}x, (xy)^{m_2-1}xy^{-1}, (xy)^{m_2}, (xy)^{m_2}x, (xy)^{m_2+1}, (xy)^{m_2}xy^{-1}, \\ (xy)^{m_2}xy^{-1}x, (xy)^{m_2}xy^{-1}xy, (xy)^{m_2}(xy^{-1})^2, \ldots, \\ (xy)^{m_2}(xy^{-1})^{m_1-1}x, (xy)^{m_2}(xy^{-1})^{m_1-1}xy, (xy)^{m_2}(xy^{-1})^{m_1} \end{array} \right\}$$

is the set of elements in $PSL(2, \mathbb{Z})$ so that $(u_{3m_1})g$ and $(v_{3m_2})g$ lies in (m_1, m_2) for all $g \in E_{11}$. One can see that if $m_1 = m_2$, then for each $g \in E_{11}$ there is a word g' in E_{11} such that $(u_{3m_1})g$, $(v_{3m_2})g$ and $(u_{3m_1})g'$, $(v_{3m_2})g'$ are the same pairs of vertices. Therefore by Proposition 1, the number of pairs of vertices to create each η, by contraction, is $\begin{cases} 3(m_2+1) & \text{if } m_1 > m_2 \\ 3m_1 & \text{if } m_1 = m_2 \end{cases}$. From figures 23 and 24, one can see that $\eta : m_1 > m_2$ has

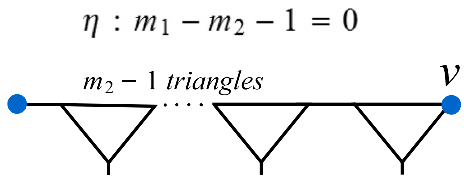

Figure 24. Graph of the homomorphic image $\eta : m_1 - m_2 - 1 = 0$.

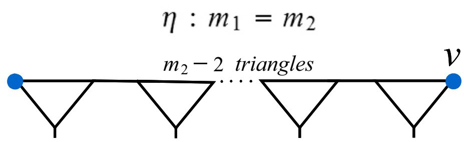

Figure 25. Graph of the homomorphic image $\eta : m_1 = m_2$.

distinct orientation from its mirror image, whereas figure 25 shows that $\eta : m_1 = m_2$ is the mirror image of itself. Thus there are $\begin{cases} 6(m_2+1), & \text{if } m_1 > m_2 \\ 3m_1, & \text{if } m_1 = m_2 \end{cases}$ pairs of vertices to compose η. $\square$

Remark 4. Now we show that if $m_1 \neq m_2$, all the sets of homomorphic images $\{\alpha_{i_1}\}$, $\{\beta_{i_1}\}$, $\{\beta_{i_3}{}'\}$, $\{\gamma_{j_1}\}$, $\{\lambda_{(i_3,j_2)}\}$, $\{\lambda_{(i_4,\frac{m_2-\epsilon_2}{2})}\}$, $\{\chi_{(j_3,j_4)}\}$, $\{\chi'_{(j_5,j_6)}\}$, $\{\chi_{(j_7,j_7)}\}$, $\{\chi'_{(j_8,j_8)}\}$ and $\{\eta\}$ are mutually disjoint.

Since each α_{i_1} contain a vertex, fixed by $(xy)^{m_2}(xy^{-1})^{i_1}x$ and $y^{-1}(xy^{-1})^{m_1-i_1-1}$, it is clear from figures 8 to 25 that none of the diagrams contains a vertex fixed by $(xy)^{m_2}(xy^{-1})^{i_1}x$ and $y^{-1}(xy^{-1})^{m_1-i_1-1}$. This implies that

$$\begin{aligned}\{\alpha_{i_1}\} \cap \{\beta_{i_1}\} &= \{\alpha_{i_1}\} \cap \{\beta'_{i_3}\} = \{\alpha_{i_1}\} \cap \{\gamma_{j_1}\} = \{\alpha_{i_1}\} \cap \{\lambda_{(i_3,j_2)}\} \\ &= \{\alpha_{i_1}\} \cap \left\{\lambda_{\left(i_4,\frac{m_2-\epsilon_2}{2}\right)}\right\} = \{\alpha_{i_1}\} \cap \{\chi_{(j_3,j_4)}\} \\ &= \{\alpha_{i_1}\} \cap \{\chi'_{(j_5,j_6)}\} = \{\alpha_{i_1}\} \cap \{\chi_{(j_7,j_7)}\} \\ &= \{\alpha_{i_1}\} \cap \{\chi'_{(j_8,j_8)}\} = \{\alpha_{i_1}\} \cap \{\eta\} = \phi. \end{aligned} \tag{2.1}$$

Since each β_{i_1} has a vertex, fixed by $(xy)^{m_2-i_1}$ and $(xy)^{i_1}(xy^{-1})^{m_1}$, from figures 12 to 25, one can see that none of the homomorphic images contains a vertex fixed by $(xy)^{m_2-i_1}$ and $(xy)^{i_1}(xy^{-1})^{m_1}$. This implies that

$$\begin{aligned}\{\beta_{i_1}\} \cap \{\beta'_{i_3}\} &= \{\beta_{i_1}\} \cap \{\gamma_{j_1}\} = \{\beta_{i_1}\} \cap \{\lambda_{(i_3,j_2)}\} \\ &= \{\beta_{i_1}\} \cap \left\{\lambda_{\left(i_4,\frac{m_2-\epsilon_2}{2}\right)}\right\} = \{\beta_{i_1}\} \cap \{\chi_{(j_3,j_4)}\} \\ &= \{\beta_{i_1}\} \cap \{\chi'_{(j_5,j_6)}\} = \{\beta_{i_1}\} \cap \{\chi_{(j_7,j_7)}\} \\ &= \{\beta_{i_1}\} \cap \{\chi'_{(j_8,j_8)}\} = \{\beta_{i_1}\} \cap \{\eta\} = \phi. \end{aligned} \tag{2.2}$$

Similarly, it is easy to show that

$$\begin{aligned}
\{\beta'_{i_3}\}\cap\{\gamma_{j_1}\} &= \{\beta'_{i_3}\}\cap\{\lambda_{(i_3,j_2)}\} = \{\beta'_{i_3}\}\cap\left\{\lambda_{\left(i_4,\frac{m_2-\epsilon_2}{2}\right)}\right\}\\
&= \{\beta'_{i_3}\}\cap\{\chi_{(j_3,j_4)}\} = \{\beta'_{i_3}\}\cap\{\chi'_{(j_5,j_6)}\}\\
&= \{\beta'_{i_3}\}\cap\{\chi_{(j_7,j_7)}\}\\
&= \{\beta'_{i_3}\}\cap\{\chi'_{(j_8,j_8)}\} = \{\beta'_{i_3}\}\cap\{\eta\} = \phi, \qquad (2.3)\\
\{\gamma_{j_1}\}\cap\left\{\lambda_{(i_3,j_2)}\right\} &= \{\gamma_{j_1}\}\cap\left\{\lambda_{\left(i_4,\frac{m_2-\epsilon_2}{2}\right)}\right\} = \{\gamma_{j_1}\}\cap\{\chi_{(j_3,j_4)}\}\\
&= \{\gamma_{j_1}\}\cap\{\chi'_{(j_5,j_6)}\} = \{\gamma_{j_1}\}\cap\{\chi_{(j_7,j_7)}\}\\
&= \{\gamma_{j_1}\}\cap\{\chi'_{(j_8,j_8)}\} = \{\gamma_{j_1}\}\cap\{\eta\} = \phi, \qquad (2.4)\\
\{\lambda_{(i_3,j_2)}\}\cap\left\{\lambda_{\left(i_4,\frac{m_2-\epsilon_2}{2}\right)}\right\} &= \{\lambda_{(i_3,j_2)}\}\cap\{\chi_{(j_3,j_4)}\} = \{\lambda_{(i_3,j_2)}\}\cap\{\chi'_{(j_5,j_6)}\}\\
&= \{\lambda_{(i_3,j_2)}\}\cap\{\chi_{(j_7,j_7)}\} = \{\lambda_{(i_3,j_2)}\}\cap\{\chi'_{(j_8,j_8)}\}\\
&= \{\lambda_{(i_3,j_2)}\}\cap\{\eta\} = \phi, \qquad (2.5)\\
\left\{\lambda_{\left(i_4,\frac{m_2-\epsilon_2}{2}\right)}\right\}\cap\{\chi_{(j_3,j_4)}\} &= \left\{\lambda_{\left(i_4,\frac{m_2-\epsilon_2}{2}\right)}\right\}\cap\{\chi'_{(j_5,j_6)}\}\\
&= \left\{\lambda_{\left(i_4,\frac{m_2-\epsilon_2}{2}\right)}\right\}\cap\{\chi_{(j_7,j_7)}\}\\
&= \left\{\lambda_{\left(i_4,\frac{m_2-\epsilon_2}{2}\right)}\right\}\cap\{\chi'_{(j_8,j_8)}\}\\
&= \left\{\lambda_{\left(i_4,\frac{m_2-\epsilon_2}{2}\right)}\right\}\cap\{\eta\} = \phi, \qquad (2.6)\\
\{\chi_{(j_3,j_4)}\}\cap\{\chi'_{(j_5,j_6)}\} &= \{\chi_{(j_3,j_4)}\}\cap\{\chi_{(j_7,j_7)}\}\\
&= \{\chi_{(j_3,j_4)}\}\cap\{\chi'_{(j_8,j_8)}\} = \{\chi_{(j_3,j_4)}\}\cap\{\eta\} = \phi, \qquad (2.7)\\
\{\chi'_{(j_5,j_6)}\}\cap\{\chi_{(j_7,j_7)}\} &= \{\chi'_{(j_5,j_6)}\}\cap\{\chi'_{(j_8,j_8)}\} = \{\chi'_{(j_5,j_6)}\}\cap\{\eta\} = \phi, \qquad (2.8)\\
\{\chi_{(j_7,j_7)}\}\cap\{\chi'_{(j_8,j_8)}\} &= \{\chi_{(j_7,j_7)}\}\cap\{\eta\} = \phi, \qquad (2.9)\\
\{\chi'_{(j_8,j_8)}\}\cap\{\eta\} &= \phi. \qquad (2.10)
\end{aligned}$$

From equations (2.1) to (2.10), it is clear that if $m_1 \neq m_2$, $\{\alpha_{i_1}\}, \{\beta_{i_1}\}, \{\beta'_{i_3}\}, \{\gamma_{j_1}\}$, $\{\lambda_{(i_3,j_2)}\}, \{\lambda_{(i_4,\frac{m_2-\epsilon_2}{2})}\}, \{\chi_{(j_3,j_4)}\}, \{\chi'_{(j_5,j_6)}\}, \{\chi_{(j_7,j_7)}\}, \{\chi'_{(j_8,j_8)}\}$ and $\{\eta\}$ are mutually disjoint.

Remark 5. Let $m_1 = m_2$. Then from figures 8 to 15, one can see that β_{i_i} and β'_{i_1} are the same sets of homomorphic images of (m_1, m_2). Similarly, figures 19 to 22 show that $\chi_{(j_3,j_4)}$ and $\chi'_{(j_5,j_6)}$, $\chi_{(j_7,j_7)}$ and $\chi'_{(j_8,j_8)}$ are pair-wise the same sets of homomorphic images of (m_1, m_2).

Let

$$\Gamma_1 = \begin{cases} 0 & \text{if } m_1 \text{ and } m_2 \text{ are odd} \\ 4 & \text{if } m_1 \text{ and } m_2 \text{ are even} \\ -1 & \text{if } m_1 \text{ is even and } m_2 \text{ is odd} \\ -1 & \text{if } m_1 \text{ is odd and } m_2 \text{ is even} \end{cases}, \qquad \Gamma_2 = \begin{cases} 5 & \text{if } m_1 \text{ is odd} \\ 8 & \text{if } m_1 \text{ is even} \end{cases}.$$

Now we are in a position to prove our main results.

Theorem 12. *There are* $\begin{cases} \frac{1}{4}\{(m_1+m_2)^2+4(m_1+m_2)+\Gamma_1\} & \text{if } m_1 > m_2 \\ \frac{1}{4}\{3m_1^2+4m_1+\Gamma_2\} & \text{if } m_1 = m_2 \end{cases}$ *distinct homomorphic images of* (m_1, m_2) *obtained by contracting all the pairs of vertices in* (m_1, m_2).

Proof. Let us contract the following pairs vertices:

(i) u_{3m_1} and u_{3i_1+1}.
(ii) u_{3m_1} and v_{3i_1+1}.
(iii) v_{3m_2} and u_{3i_3+1}.
(iv) u_{3m_1} and u_{3j_1+1}.
(v) u_{3i_3+1} and v_{3j_2}.
(vi) u_{3i_4+1} and $v_{3(\frac{m_2-\epsilon_2}{2})}$.
(vii) u_{3j_3+1} and u_{3j_4}.
(viii) v_{3j_5+1} and v_{3j_6}.
(ix) u_{3j_7+1} and u_{3j_7}.
(x) u_{3j_8+1} and u_{3j_8}.
(xi) u_{3m_1} and v_{3m_2}.

Then by Theorems 1 to 11 and Remarks 4 and 5, we obtain the set

$$F = \begin{cases} \left\{ \begin{array}{c} \alpha_{i_1}, \beta_{i_1}, \beta'_{i_3}, \gamma_{j_1}, \lambda_{(i_3, j_2)}, \lambda_{\left(i_4, \frac{m_2-\epsilon_2}{2}\right)}, \chi_{(j_3, j_4)}, \\ \chi'_{(j_5, j_6)}, \chi_{(j_7, j_7)}, \chi'_{(j_8, j_8)}, \eta \end{array} \right\} & \text{if } m_1 > m_2 \\ \\ \{\alpha_{i_1}, \beta_{i_1}, \lambda_{(i_3, j_2)}, \lambda_{\left(i_4, \frac{m_2-\epsilon_2}{2}\right)}, \chi_{(j_3, j_4)}, \chi_{(j_7, j_7)}, \eta\} & \text{if } m_1 = m_2 \end{cases}$$

of homomorphic images of (m_1, m_2), and there are

$$\begin{aligned} S = {} & 3(m_2^2 + 3m_2 - 2) + \frac{3}{2}(m_2^2 + 3m_2) + \frac{3}{2}(m_1^2 + 3m_1 - 4) \\ & +3(m_2 + 2)(m_1 - m_2 - 1) + 6(m_1 - 1)(m_2 - (\epsilon_2 + 2)) \\ & + \begin{cases} 6(m_1 - 1) & \text{if } m_2 \equiv 0 \bmod 2 \\ 12(m_1 - 1) & \text{if } m_2 \equiv 1 \bmod 2 \end{cases} + 3(m_1^2 - 3m_1 + 2) \\ & +3(m_2^2 - 3m_2 + 2) + 3(m_1 - 1) + 3(m_2 - 1) \\ & + \begin{cases} 6(m_2 + 1) & \text{if } m_1 > m_2 \\ 3m_1 & \text{if } m_1 = m_2 \end{cases} \end{aligned}$$

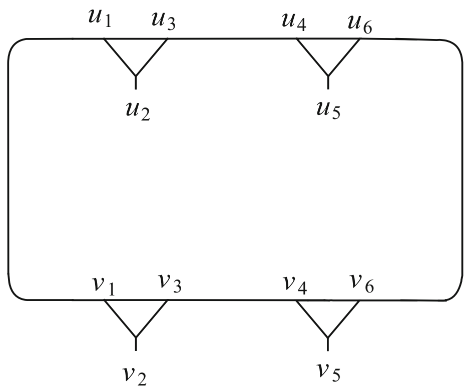

Figure 26. Graph of the circuit (2, 2).

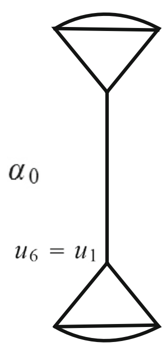

Figure 27. Graph of the homomorphic image α_0.

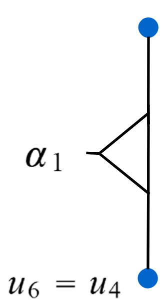

Figure 28. Graph of the homomorphic image α_1.

pairs of vertices to compose homomorphic images in F. Since $S = \binom{3(m_1+m_2)}{2}$ is the total number of pairs of vertices in (m_1, m_2), it implies that F contains all the homomorphic images of (m_1, m_2). Also,

$$|F| = \begin{cases} \frac{1}{4}\{(m_1+m_2)^2 + 4(m_1+m_2) + \Gamma_1\} & \text{if } m_1 > m_2 \\ \frac{1}{4}\{3m_1^2 + 4m_1 + \Gamma_2\} & \text{if } m_1 = m_2 \end{cases}.$$

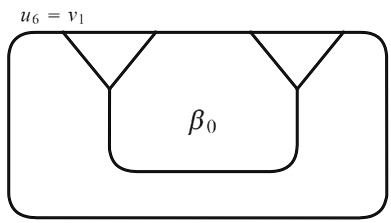

Figure 29. Graph of the homomorphic image β_0.

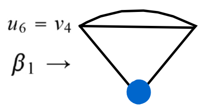

Figure 30. Graph of the homomorphic image β_1.

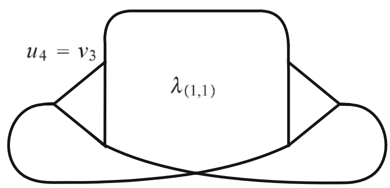

Figure 31. Graph of the homomorphic image $\lambda_{(1,\,1)}$.

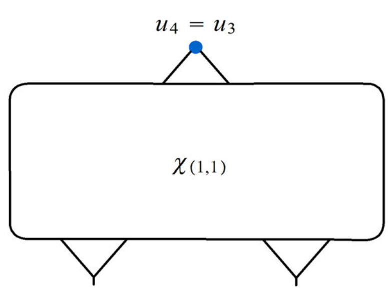

Figure 32. Graph of the homomorphic image $\chi_{(1,\,1)}$.

Thus, there are

$$\begin{cases} \frac{1}{4}\{(m_1+m_2)^2+4(m_1+m_2)+\Gamma_1\} & \text{if } m_1 > m_2 \\ \frac{1}{4}\{3m_1^2+4m_1+\Gamma_2\} & \text{if } m_1 = m_2 \end{cases}$$

distinct homomorphic images of (m_1, m_2) obtained by contracting all the pairs of vertices in (m_1, m_2). $\square$

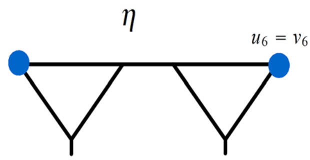

Figure 33. Graph of the homomorphic image η.

3. Conclusion

There are $\binom{3(m_1+m_2)}{2}$ pairs of vertices in (m_1, m_2), in order to compose all the homomorphic images of (m_1, m_2). Out of those only

$$\begin{cases} \frac{1}{4}\{(m_1+m_2)^2 + 4(m_1+m_2) + \Gamma_1\} & \text{if } m_1 > m_2 \\ \frac{1}{4}\{3m_1^2 + 4m_1 + \Gamma_2\} & \text{if } m_1 = m_2 \end{cases}$$

pairs of vertices are important. There is no need to contract the pairs which are not mentioned in Theorem 12. Because, if we contract those, we obtain a homomorphic image, which we have already obtained by contracting ‘important’ pairs.

Example 2. Consider a circuit (2, 2) (figure 26).

By Theorem 12, the set of homomorphic images of (2, 2), evolved by contracting all the pairs of vertices in (2, 2), is $\{\alpha_0, \alpha_1, \beta_0, \beta_1, \lambda_{(1,1)}, \chi_{(1,1)}, \eta\}$.

The homomorphic images $\alpha_0, \alpha_1, \beta_0, \beta_1, \lambda_{(1,1)}, \chi_{(1,1)}$ and η of (2, 2) are obtained by contracting u_6 and u_1 (figure 27), u_6 and u_4 (figure 28), u_6 and v_1 (figure 29), u_6 and v_4 (figure 30), u_4 and v_3 (figure 31), u_4 and u_3 (figure 32), and, u_6 and v_6 (figure 33) respectively.

The total number of vertices in (2, 2) are 12, implying that there are 66 pairs of vertices in (2, 2). Theorem 12 assures us that, in order to create all homomorphic images of (2, 2), we just have to contract 7 pairs (mentioned in Theorem 12). There is no need to contract the remaining 59 pairs.

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