



Accurate multistep multi-derivative collocation methods applied to chaotic systems

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Abstract

Accurate multistep multi-derivative collocation methods are derived for the numerical integration of chaotic systems. These methods have high order of accuracy, with A-stable regions of absolute stability. Although the calculation of the third derivative terms in the methods is relatively high compared to the first and second derivative terms, the advantage gained makes them suitable for solving chaotic system of equations with large eigenvalues. The stability characteristic properties and order of accuracy of the methods are studied. Figurative comparisons of the solution curves obtained are in good agreement with the exact solutions which demonstrate practically the effectiveness of the proposed methods.

Keywords: Chaotic system, Chen system, Continuous scheme, multistep collocation formula, multiderivative method.

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1. Introduction

In this paper, we present accurate multistep multi-derivative collocation methods (MMDCMs) derived for continuous numerical integration of chaotic system in ordinary differential equations (ODEs) of the form

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0, \quad x \in [x_0, T]. \quad (1.1)$$

Here the unknown function y is a mapping $[x_0, T] \rightarrow \mathbb{R}^d$ and the right-hand side f is $[x_0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ which is assumed to be sufficiently smooth and the initial vector y_0 is given in \mathbb{R}^d . Without loss of generality we assume that the constant step-size is greater than zero ($h > 0$) and along the x -axis we introduce a set of grid points x_n defined by $x_n = x_0 + nh$, $n = 0, 1, 2, \dots, N$ where $Nh = T - x_0$, and a set of equally spaced points on the integration interval is defined by $x_0 < x_1 < x_2 < \dots < x_{n+1} = T$.

Recently several new methods have been derived for the numerical integration of ordinary differential equations with good stability properties and evaluation of high order derivatives, see for example [10, 1, 13,

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[14]. A great number of these methods considered for the solution of ODEs are based on the modification of the classical formulae. Amongst the classical formulae we may mention the formula of Obrechhoff, [21] which uses the first m derivatives of $y(x)$. Later, Urabe [25], introduced an implicit single-step discrete variable method which utilizes second derivatives terms. One of the fascinating features of the PC pair of Urabe was the introduction of the second derivative terms of $y(x)$ into the numerical scheme, defined by

$$y''(x) = g(x, y) = f_x + f_y y' = f_x + f f_y.$$

In Urabe's case the predictor is of order 5, while the corrector is of order 6. Indeed, the author placed much emphasis on using such a single-step method only to get starting values required for his predictor-corrector (PC) scheme, though he applied the scheme successfully to the computations of nonlinear oscillator analysis. Shortly thereafter, Cash [4] followed a similar procedure for constructing high-order implicit integration formulae with large regions of absolute stability for the approximate numerical integration of both stiff and non-stiff system of ODEs. In 1987 Mitsui [19] modified Urabe's implicit single-step method and studied the convergence, stability and the a posteriori error estimate of the method. Many individuals have derived methods with second derivative evaluations in the literature, see for example [7, 8, 11].

In this paper, we should point out that multi-derivative methods form the bridge that unifies the two disparate families of the traditional methods (Runge-Kutta methods and linear multistep methods) by defining a framework that includes each of them as special cases. The introduction of the higher-order derivatives allows one to derive methods with high-order of accuracy and good stability properties for the solution of system of ordinary differential equations with large eigenvalues. Hence in this paper we consider methods that generate continuous solutions simultaneously within the interval of integration. These methods are derived by introducing second and third derivative terms into the numerical schemes. In this way, we obtain methods which give high order of accuracy within the interval of integration, with very low error constants, large regions of absolute stability and essentially converge to the exact solution rapidly.

Definition 1.1. [6] A numerical method is said to be *A-stable* if its region of absolute stability contains the whole of the complex left hand-half plane $\text{Re } \lambda < 0$. Alternatively, a numerical method is called *A-stable* if all the solution of (1.1) tend to zero as $n \rightarrow \infty$, when the method is applied with a fixed positive h to any differential equation of the form $dy/dx = \lambda y$ where λ is a complex constant with negative real part.

Definition 1.2. A solution $y(x)$ is said to be *stable* if given any $\epsilon > 0$ there is a $\delta > 0$ such that any other solution $\hat{y}(x)$ of (1.1) satisfying

$$|y(a) - \hat{y}(a)| \leq \delta$$

also satisfies

$$|y(x) - \hat{y}(x)| \leq \epsilon$$

for all $x > a$. The solution $y(x)$ is asymptotically stable if in addition to the above $|y(x) - \hat{y}(x)| \rightarrow 0$, as $x \rightarrow \infty$.

Theorem 1.3. [3] If f satisfies Lipschitz condition with constant L then the initial value problem

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0,$$

possesses a unique solution on the interval $[x_0, T]$.

Assumption 1.1: In the ODEs (1.1), the function f belongs to C^1 -class and therefore satisfies the Lipschitz condition with the constant L . That is, if the estimation

$$\|f(x, y(x)) - f(x, \tilde{y}(x))\| \leq L \|y - \tilde{y}\|$$

holds, L is called the Lipschitz constant.

2. Description of the methods

In this section we describe the derivation of the special class of multistep multi-derivative collocation methods for continuous numerical integration of chaotic systems. In this regard, we consider an approximate solution to the exact solution of (1.1), an interpolant of the form,

$$y(x) = \sum_{i=0}^{p-1} \phi_i x^i \quad (2.1)$$

which is continuously differentiable of any order required. We set the sum $p = q + r + s + t$ so as to be able to determine $\{\phi_i\}$ in (2.1). In this formulation q denotes the number of interpolation points used and $r > 0, s > 0, t > 0$ are distinct collocation points. Interpolating $y(x)$ in (2.1) at the points $\{x_{n+j}\}$ and collocating $y'(x)$, $y''(x)$ and $y'''(x)$ at the points $\{x_{n+j}\}$ we have the following system of equations

$$y(x_{n+j}) = y_{n+j}, \quad (j = 0, 1, 2, \dots, q-1), \quad (2.2)$$

$$y'(x_{n+j}) = f_{n+j}, \quad (j = 0, 1, 2, \dots, r-1), \quad (2.3)$$

$$y''(x_{n+j}) = g_{n+j}, \quad (j = 0, 1, 2, \dots, s-1), \quad (2.4)$$

$$y'''(x_{n+j}) = l_{n+j}, \quad (j = 0, 1, 2, \dots, t-1). \quad (2.5)$$

In the sense of [20] the system of equations (2.2)-(2.5) can be expressed in the matrix form as follows:

$$D\phi = y \quad (2.6)$$

where the p -square matrix D , the p -vectors ϕ and y are defined as follows:

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & \cdots & x_n^{p-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n+q-1} & x_{n+q-1}^2 & x_{n+q-1}^3 & x_{n+q-1}^4 & \cdots & x_{n+q-1}^{p-1} \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & \cdots & D'x_n^{p-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2x_{n+r-1} & 3x_{n+r-1}^2 & 4x_{n+r-1}^3 & \cdots & D'x_{n+r-1}^{p-2} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & \cdots & D''x_n^{p-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 2 & 6x_{n+s-1} & 12x_{n+s-1}^2 & \cdots & D''x_{n+s-1}^{p-3} \\ 0 & 0 & 0 & 6 & 24x_n & \cdots & D'''x_n^{p-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 6 & 24x_{n+t-1} & \cdots & D'''x_{n+t-1}^{p-4} \end{pmatrix}. \quad (2.7)$$

$$\phi = (\phi_0, \phi_1, \dots, \phi_{p-1})^T, y = (y_n, \dots, y_{n+q-1}, y'_n, \dots, y'_{n+r-1}, y''_n, \dots, y''_{n+s-1}, y'''_n, \dots, y'''_{n+t-1})^T,$$

where $D' = (p-1)$, $D'' = (p-1)(p-2)$ and $D''' = (p-1)(p-2)(p-3)$ in (2.7) represent the first, second and third derivatives respectively and correspond to the differentiation with respect to x . Similar to the Vander-monde matrix, the matrix D in (2.6) is non-singular. A closed form of the solution for the system

in (2.6) is presented which has been obtained by considering the inverse of the Vander-monde matrix, that is,

$$\phi = Cy \text{ where } C = D^{-1}. \quad (2.8)$$

Re-arranging equations (2.6) -(2.8) to have the multistep collocation formula of the type in [23] which was a generalization of [17] and here we extend the collocation formula to second and third derivatives as follows,

$$y(x) = \sum_{j=0}^{q-1} \phi_j(x) y_{n+j} + h \sum_{j=0}^{r-1} \psi_j(x) f_{n+j} + h^2 \sum_{j=0}^{s-1} \gamma_j(x) g_{n+j} + h^3 \sum_{j=0}^{t-1} \omega_j(x) l_{n+j}, \quad (2.9)$$

where

$$y_{n+j} = y(x_n + jh), \quad f_{n+j} = f(x_n + jh, y(x_n + jh)),$$

$$g_{n+j} = \left. \frac{df(x, y(x))}{dx} \right|_{\substack{x=x_{n+j} \\ y=y_{n+j}}}, \quad l_{n+j} = \left. \frac{d^2 f(x, y(x))}{dx^2} \right|_{\substack{x=x_{n+j} \\ y=y_{n+j}}}.$$

The parameters $\phi_j(x)$, $\psi_j(x)$, $\gamma_j(x)$ and $\omega_j(x)$ in (2.9) are the continuous coefficients of the formula which are to be determined. They are polynomials of the form

$$\begin{aligned} \phi_j(x) &= \sum_{i=0}^{p-1} \phi_{j,i+1} x^i, & h\psi_j(x) &= h \sum_{i=0}^{p-1} \psi_{j,i+1} x^i, \\ h^2\gamma_j(x) &= h^2 \sum_{i=0}^{p-1} \gamma_{j,i+1} x^i, & h^3\omega_j(x) &= h^3 \sum_{i=0}^{p-1} \omega_{j,i+1} x^i. \end{aligned} \quad (2.10)$$

The numerical constant coefficients $\phi_{j,i+1}$, $\psi_{j,i+1}$, $\gamma_{j,i+1}$ and $\omega_{j,i+1}$ in (2.10) are to be determined. They are obtained from the components of the matrix D^{-1} . That is, if the identity in (2.11) holds,

$$C = \begin{pmatrix} \phi_{1,0} & \cdots & \phi_{1,r-1} & h\psi_{1,0} & \cdots & h\psi_{1,s-1} & h^2\gamma_{1,0} & \cdots & h^2\gamma_{1,t-1} \\ \phi_{2,0} & \cdots & \phi_{2,r-1} & h\psi_{2,0} & \cdots & h\psi_{2,s-1} & h^2\gamma_{2,0} & \cdots & h^2\gamma_{2,t-1} \\ \phi_{3,0} & \cdots & \phi_{3,r-1} & h\psi_{3,0} & \cdots & h\psi_{3,s-1} & h^2\gamma_{3,0} & \cdots & h^2\gamma_{3,t-1} \\ \phi_{4,0} & \cdots & \phi_{4,r-1} & h\psi_{4,0} & \cdots & h\psi_{4,s-1} & h^2\gamma_{4,0} & \cdots & h^2\gamma_{4,t-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_{p-1,0} & \cdots & \phi_{p-1,r-1} & h\psi_{p-1,0} & \cdots & h\psi_{p-1,s-1} & h^2\gamma_{p-1,0} & \cdots & h^2\gamma_{p-1,t-1} \end{pmatrix} \equiv D^{-1}. \quad (2.11)$$

The choice $C = D^{-1}$ leads to the determination of the numerical constant coefficients $\phi_{j,i+1}$, $\psi_{j,i+1}$, $\gamma_{j,i+1}$ and $\omega_{j,i+1}$ in (2.10). Actual evaluations of the matrices C and D are carried out with a computer algebra system, for example, Maple to determine the constant coefficients. In the multistep multi-derivative methods, we see that not only the function $f(x,y)$ is evaluated at some internal intermediate points, but in addition the functions Df , D^2f , \dots , where D is the differential operator. Hence, in addition to the computation of the f -values at the internal stages in the standard linear multistep methods, the multi-derivative methods involve computing g -values and l -values where f , g and l are as defined in (2.9).

3. The multistep multi-derivative collocation methods

3.1 A sixth order accurate multistep multi-derivative collocation method

In this section, we develop the general form of conditions for obtaining the coefficients of the accurate multistep multi-derivative collocation methods. To obtain the coefficients of the first method we consider the multistep multi-derivative collocation formula in (2.9) which takes the form,

$$y(x) = \phi_0(x)y_n + h \sum_{j=0}^1 \psi_j(x)f_{n+j} + h^2 \sum_{j=0}^1 \gamma_j(x)g_{n+j} + h^3 \sum_{j=0}^1 \omega_j(x)l_{n+j}. \quad (3.1)$$

Expanding (3.1) to obtain the continuous scheme for the first method, we have,

$$y(x) = \phi_0(x)y_n + h[\psi_0(x)f_n + \psi_1(x)f_{n+1}] + h^2[\gamma_0(x)g_n + \gamma_1(x)g_{n+1}] + h^3[\omega_0(x)l_n + \omega_1(x)l_{n+1}] \quad (3.2)$$

where

$$\begin{aligned} \phi_0(x) &= 1, \\ \psi_0(x) &= \left[\frac{-2(x-x_n)^6 + 6h(x-x_n)^5 - 5h^2(x-x_n)^4 + 2h^5(x-x_n)}{2h^5} \right], \\ \psi_1(x) &= \left[\frac{2(x-x_n)^6 - 6h(x-x_n)^5 + 5h^2(x-x_n)^4}{2h^5} \right], \\ \gamma_0(x) &= \left[\frac{-5(x-x_n)^6 + 16h(x-x_n)^5 - 15h^2(x-x_n)^4 + 5h^4(x-x_n)^2}{10h^4} \right], \\ \gamma_1(x) &= \left[\frac{-5(x-x_n)^6 + 14h(x-x_n)^5 - 10h^2(x-x_n)^4}{10h^4} \right], \\ \omega_0(x) &= \left[\frac{-10(x-x_n)^6 + 36h(x-x_n)^5 - 45h^2(x-x_n)^4 + 20h^3(x-x_n)^3}{120h^3} \right], \\ \omega_1(x) &= \left[\frac{10(x-x_n)^6 - 24h(x-x_n)^5 + 15h^2(x-x_n)^4}{120h^3} \right]. \end{aligned}$$

Evaluating the continuous scheme $y(x)$ in (3.2) at the points $x = x_{n+1}$ and x_{n+u} (where $u = \frac{1}{2}$) we obtain the coefficients of the first accurate multistep multi-derivative collocation method as follows,

$$y_{n+1} = y_n + \frac{h}{120}[60f_n + 60f_{n+1}] + \frac{h^2}{120}[12g_n - 12g_{n+1}] + \frac{h^3}{120}[l_{n+1} + l_n] \quad (3.3)$$

$$y_{n+u} = y_n + \frac{h}{3840}[1620f_n + 300f_{n+1}] + \frac{h^2}{3840}[282g_n - 102g_{n+1}] + \frac{h^3}{3840}[21l_n + 11l_{n+1}].$$

3.2 A tenth-order accurate multistep multi-derivative collocation method

Here we derive the second multistep multi-derivative collocation method of higher-order of accuracy for the direct numerical integration of chaotic systems. To obtain the coefficients of the second method we introduce and define the following $\xi = (x - x_n)$ which we shall use in the construction of the continuous scheme of the method. In a similar manner we obtain the continuous scheme from the multistep collocation formula (2.9) as follows,

$$y(x) = \phi_0(x)y_n + h \sum_{j=0}^2 \psi_j(x)f_{n+j} + h^2 \sum_{j=0}^2 \gamma_j(x)g_{n+j} + h^3 \sum_{j=0}^2 \omega_j(x)l_{n+j}. \quad (3.4)$$

Similarly, on expansion of (3.4) using the method of Taylor series we obtain the continuous scheme as,

$$\begin{aligned} y(x) = & \phi_0(x)y_n + h[\psi_0(x)f_n + \psi_1(x)f_{n+1} + \psi_2(x)f_{n+2}] \\ & + h^2[\gamma_0(x)g_n + \gamma_1(x)g_{n+1} + \gamma_2(x)g_{n+2}] \\ & + h^3[\omega_0(x)l_n + \omega_1(x)l_{n+1} + \omega_2(x)l_{n+2}] \end{aligned} \quad (3.5)$$

where

$$\begin{aligned} \phi_0(x) &= 1, \\ \psi_0(x) &= \left[\frac{2240\xi^9 - 21735h\xi^8 + 85560h^2\xi^7 - 172620h^3\xi^6 + 181944h^4\xi^5 - 83160h^5\xi^4 + 13440h^8\xi}{13440h^8} \right], \\ \psi_1(x) &= \left[\frac{-35\xi^9 + 315h\xi^8 - 1140h^2\xi^7 + 2100h^3\xi^6 - 2016h^4\xi^5 + 840h^5\xi^4}{105h^8} \right], \\ \psi_2(x) &= \left[\frac{2240\xi^9 - 18585h\xi^8 + 60360h^2\xi^7 - 96180h^3\xi^6 + 76104h^4\xi^5 - 24360h^5\xi^4}{13440h^8} \right], \\ \gamma_0(x) &= \left[\frac{280\xi^9 - 2765h\xi^8 + 11160h^2\xi^7 - 23380h^3\xi^6 + 26208h^4\xi^5 - 13440h^5\xi^4 + 2240h^7\xi^2}{4480h^7} \right], \\ \gamma_1(x) &= \left[\frac{-\xi^8 + 8h\xi^7 - 24h^2\xi^6 + 32h^3\xi^5 - 16h^4\xi^4}{8h^6} \right], \\ \gamma_2(x) &= \left[\frac{-280\xi^9 + 2275h\xi^8 - 7240h^2\xi^7 + 11340h^3\xi^6 - 8848h^4\xi^5 + 2800h^5\xi^4}{4480h^7} \right], \\ \omega_0(x) &= \left[\frac{280\xi^9 - 2835h\xi^8 + 11880h^2\xi^7 - 26460h^3\xi^6 + 33264h^4\xi^5 - 22680h^5\xi^4 + 6720h^6\xi^3}{40320h^6} \right], \\ \omega_1(x) &= \left[\frac{-35\xi^9 + 315h\xi^8 - 1125h^2\xi^7 + 1995h^3\xi^6 - 1764h^4\xi^5 + 630h^5\xi^4}{630h^6} \right], \\ \omega_2(x) &= \left[\frac{280\xi^9 - 2205h\xi^8 + 6840h^2\xi^7 - 10500h^3\xi^6 + 8064h^4\xi^5 - 2520h^5\xi^4}{40320h^6} \right]. \end{aligned}$$

Evaluating the continuous scheme $y(x)$ in (3.5) at the points $x = x_{n+1}$ and x_{n+2} we obtain the second multistep multi-derivative collocation method of higher-order accuracy as follows,

$$y_{n+2} = y_n + \frac{h}{315}[123f_n + 384f_{n+1} + 123f_{n+2}] + \frac{h^2}{315}[18g_n - 18g_{n+2}] + \frac{h^3}{315}[l_n + 16l_{n+1} + l_{n+2}] \quad (3.6)$$

$$y_{n+1} = y_n + \frac{h}{40320}[17007f_n + 24576f_{n+1} - 1263f_{n+2}] + \frac{h^2}{40320}[2727g_n - 5040g_{n+1} + 4230g_{n+2}] + \frac{h^3}{40320}[169l_n + 1024l_{n+1} - 41l_{n+2}]$$

4. Analysis of the accurate multistep multi-derivative collocation methods

4.1 Order, consistency, zero-stability and convergence of the MMDCMs

With the multistep collocation formula (2.9) we associate the linear difference operator ℓ defined by

$$\ell[y(x); h] = \sum_{j=0}^q \phi_j(x)y(x+jh) + h \sum_{j=0}^r \psi_j(x)y'(x+jh) + h^2 \sum_{j=0}^s \gamma_j(x)y''(x+jh) + h^3 \sum_{j=0}^t \omega_j(x)y'''(x+jh) \quad (4.1)$$

where $y(x)$ is an arbitrary function, continuously differentiable on $[x_0, T]$. Following [15], we can write the terms in (4.1) as a Taylor series expansion about the point x to obtain the expression,

$$\ell[y(x); h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_p h^p y^{(p)}(x) + \cdots \quad (4.2)$$

where the constant coefficients $C_p, p = 0, 1, 2, \dots$ are given as follows:

$$C_0 = \sum_{j=0}^q \phi_j$$

$$C_1 = \sum_{j=1}^q j \phi_j$$

$$C_2 = \frac{1}{2!} \left[\sum_{j=1}^q j^2 \phi_j - \sum_{j=0}^r \psi_j \right]$$

$$C_3 = \frac{1}{3!} \left[\sum_{j=1}^q j^3 \phi_j - 2 \sum_{j=1}^r j^2 \psi_j - \sum_{j=0}^s \gamma_j \right]$$

$$\vdots \quad \vdots$$

$$C_p = \frac{1}{p!} \left[\sum_{j=1}^q j^p \phi_j - \frac{1}{(p-1)!} \sum_{j=1}^r j^{p-1} \psi_j - \frac{1}{(p-2)!} \sum_{j=1}^s j^{p-2} \gamma_j - \frac{1}{(p-3)!} \sum_{j=0}^t j^{p-3} \omega_j \right], \quad p = 4, 5, \dots$$

According to [15], the multistep collocation formula (2.9) has order p if

$$\ell[y(x); h] = O(h^{p+1}), \quad C_0 = C_1 = \cdots = C_p = 0, \quad C_{p+1} \neq 0.$$

Therefore C_{p+1} is the error constant and $C_{p+1} h^{p+1} y^{(p+1)}(x)$ is the principal local truncation error. Hence from our calculation the order and error constants for the constructed methods are presented in Table 1. It is clear from the Table that the accurate multistep multi-derivative collocation methods are of high order. They have smaller error constants and hence more accurate than the conventional linear multistep methods of the Adams' family.

Table 1: Order and error constants of the multistep multi-derivative collocation methods

Method	Order	Error constant
Method (3.3)	$p = 6$	$C_7 = -1.1904 \times 10^{-3}$
	$p = 6$	$C_7 = -1.9047 \times 10^{-2}$
Method (3.6)	$p = 10$	$C_{11} = -2.4050 \times 10^{-6}$
	$p = 9$	$C_{10} = 2.7777 \times 10^{-3}$

Definition 4.1. [15] A linear multistep method is said to be consistent if the order of the individual method is greater than or equal to one, that is, if $p \geq 1$.

$$(i) \rho(1) = 0$$

$$(ii) \rho'(1) = \sigma(1), \text{ where } \rho(z) \text{ and } \sigma(z) \text{ are respectively the 1st and 2nd characteristic polynomials.}$$

From Table 1 and definition 4.1 we can attest that the accurate multistep multiderivative methods are consistent.

Definition 4.2. [15] A linear multistep method is said to be zero-stable if the root condition of the method is satisfied, that is, if the root condition is greater than or equal to one, that is, if $p \geq 1$.

$$\rho(\lambda) = \det \left[\sum_{i=0}^k A^i \lambda^{k-i} \right] = 0$$

satisfies $|\lambda_j| \leq 1$, $j = 1, 2, \dots, k$ and for those roots with $|\lambda_j| = 1$, the multiplicity does not exceed 2.

Based on definition 4.2 the newly constructed accurate multistep multi-derivative collocation methods are zero-stable.

Definition 4.3. [6] The necessary and sufficient conditions for a linear multistep method to be convergent are that it must be consistent and zero-stable. Also see [15], theorem 2.1 page 33.

From definitions 4.1 and 4.2 the accurate multistep multi-derivative collocation methods are convergent.

4.2 Regions of absolute stability of the accurate multistep multi-derivative collocation methods

Linear stability is always necessary in the design of a new algorithm to solve ordinary differential equations. For this reason, we consider for the "multistep multi-derivative" collocation methods the test equation of the form

$$\frac{dy}{dx} = \lambda y, \quad \lambda \in \mathbb{C} \text{ and } \Re \lambda < 0, \quad (4.3)$$

with a fixed positive step size $h > 0$. Since, the multistep multi-derivative collocation methods contain second derivative $g(x, y)$ as well as third derivative $l(x, y)$, it is natural to suppose that $g(x, y) = \lambda^2 y$ and $l(x, y) = \lambda^3 y$. Reformulating (3.3) and (3.6) as general linear methods, see Burrage and Butcher [2] and substituting the elements of the matrices A, C, B, D, D_1, D_2 into the recurrent relation (where D_1 and D_2 are the coefficients of the second and third derivatives respectively),

$$y^{[n-1]} = M(z)y^{[n]}, \quad n = 1, 2, 3, \dots, N-1, \quad z = \lambda h,$$

we obtain the stability matrix

$$M(z) = D + zB(I - zA)^{-1}C,$$

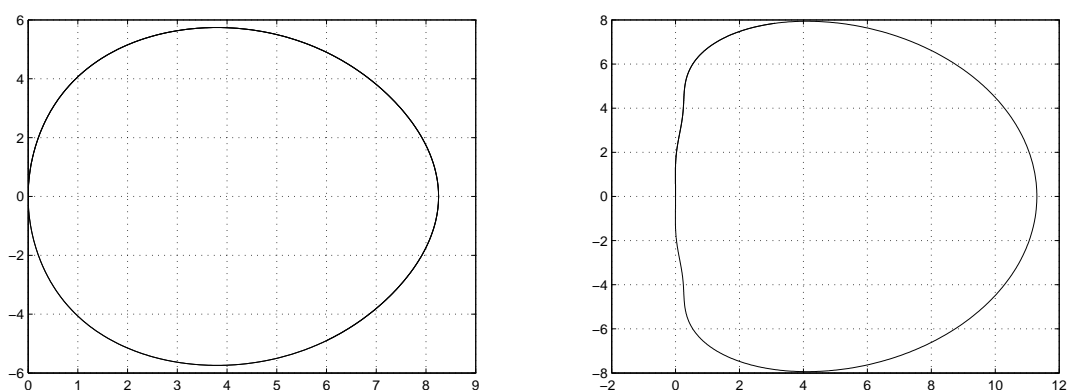
and the stability polynomial(stability function) of each method as follows,

$$\rho(\eta, z) = \det(r(A - Cz - D_1z^2 - D_2z^3) - B).$$

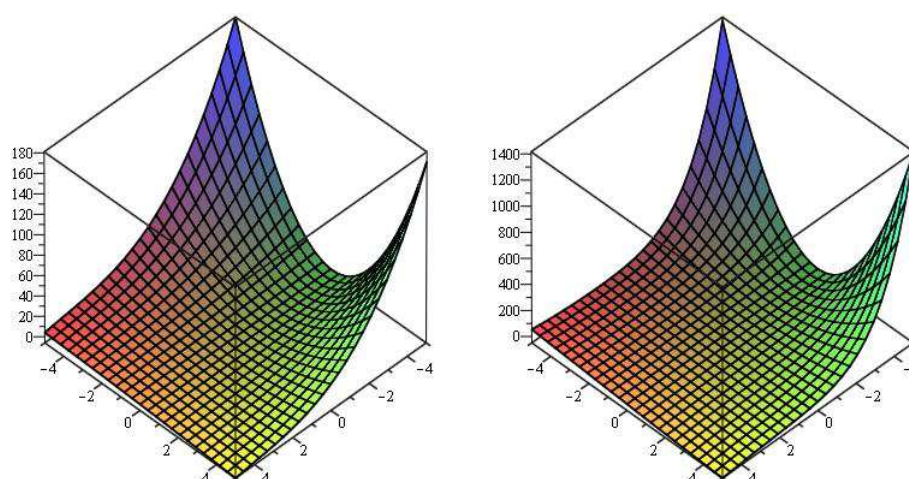
The region of absolute stability of the method is defined as

$$\mathcal{R} = \{x \in \mathbb{C} : \rho(\eta, z) = 1 \implies |\eta| \leq 1\}.$$

Computing the stability function gives the stability polynomial of each method, which is plotted to produce the required graph of the region of absolute stability of each method as shown in Figure 1.



Regions of Absolute Stability of Methods (3.3) and (3.6) respectively



Regions of Absolute Stability of Methods (3.3) and (3.6) respectively

Figure 1: Regions of Absolute Stability of the multistep multi-derivative collocation methods

Remark 4.4. In the accurate multistep multi-derivative collocation methods we added the matrices D1 and D2 obtained from the coefficients of h^2 and h^3 respectively to the stability matrices A, C, B and D which enabled us to plot the regions of absolute stability of the new methods in 2D and 3D (Dimension). The regions of absolute stability of the accurate multistep multi-derivative collocation methods are A-stable, since the regions are enclosed completely on the right-hand half-plane of the complex planes as shown in the Figures.

5. Numerical Examples

Our objective in this section is to see how well our methods compare with other recent derived methods which are also efficient for the numerical solution of system in ordinary differential equations. We report the numerical results obtained by applying the new multistep multi-derivative collocation methods to chaotic system as well as to oscillatory system in ordinary differential equations found in the literature. We present the computed results side by side in Tables. In the computation, we use, *nfe* to denote the number of function evaluations, and *Ext*, to indicate the exact solutions in the Figures. We used a MATLAB code for the computational purposes.

Example 1: The “Unplugged” Van der Pol’s Equation

The following nonlinear equation can be refer to as the unplugged well known Vander Pol’s equation,

$$\frac{d^2 y}{dx^2} + y = -\mu y^2 \frac{dy}{dx},$$

with initial conditions given by

$$y(0) = 1, \quad y'(0) = 0.$$

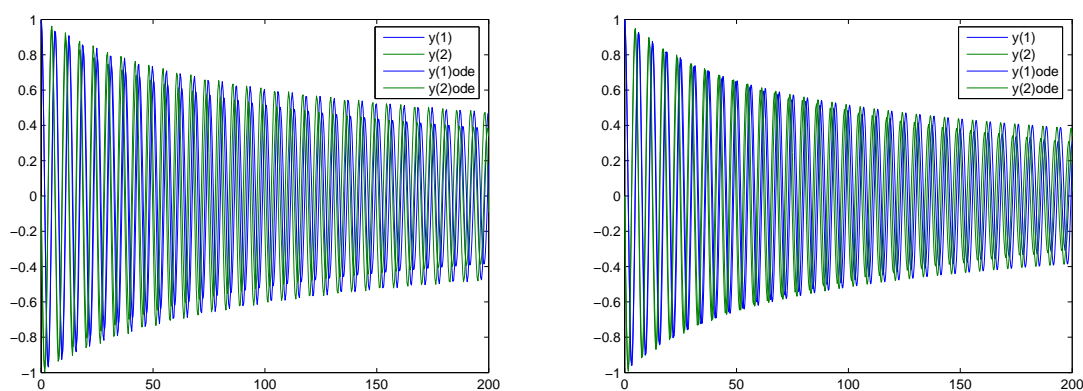
This equation describes the behaviour of oscillator circuits, where the parameter μ is set to be either large or small depending on whether the equation is stiff or non-stiff. In this case we consider $\mu = 0.1$. Therefore the equation can be equivalently written as a coupled system of two first order ordinary differential equations,

$$\begin{aligned} y_1'(x) &= y_2(x), \\ y_2'(x) &= -y_1(x) - \mu y_1(x)^2 y_2(x), \end{aligned}$$

and the initial conditions $y(0) = 1$ and $y'(0) = 0$ become

$$y_1(0) = 1, \quad y_2(0) = 0.$$

The reference solutions at the end point of the integration interval of $[0, 200]$ with $h = 0.2$ are plotted in Figure 2. Here we observe that the graphs of the approximated solutions and the graphs of the solutions from the ode solver coincide with each other.



Solution curves using method (3.3) and method (3.6) respectively, with nfe = 500

Figure 2: Graphical plots of example 1 using the multistep multi-derivative collocation methods

Example 2: In this example we consider the stiff system with eigenvalues lying close to the imaginary axis, $-1 \pm 15i$. This system is known to be very troublesome stiff problem, where some stiff integrators were known to perform inefficiently.

$$\begin{aligned} y_1'(x) &= -y_1(x) - 15y_2(x) + 15\exp(-x), & y_1(0) &= 1, \\ y_2'(x) &= 15y_1(x) - y_2(x) - 15\exp(-x), & y_2(0) &= 1. \end{aligned}$$

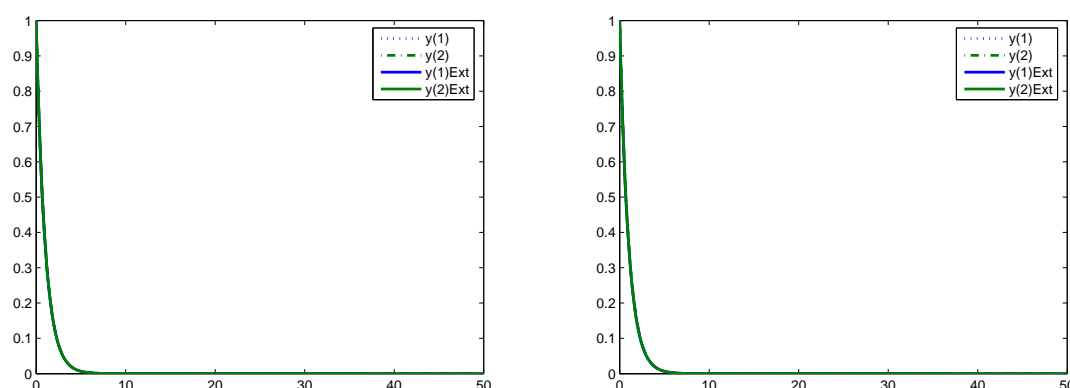
The exact solution is,

$$\begin{aligned} y_1(x) &= \exp(-x), \\ y_2(x) &= \exp(-x). \end{aligned}$$

We solve the system in the range of $[0, 50]$ with $h = 0.1$ and the results obtained are presented in Table 2. In this example we compare the results from one of the new methods of order six with another method from the literature side by side in the Table of values. The solution curves obtained are compared with the exact solutions in Figure 3 for both the two methods.

Table 2: Absolute errors in the numerical integration of example 2

x	y_i	Yakubu& Markus[26]	Method(3.3)
5	y_1	$6.229806545292090 \times 10^{-2}$	$6.436458351566370 \times 10^{-4}$
	y_2	$3.022126582723930 \times 10^{-2}$	$2.589041250120610 \times 10^{-4}$
50	y_1	$7.12661124351296 \times 10^{-4}$	$4.070066912710120 \times 10^{-5}$
	y_2	$2.59508488308931 \times 10^{-4}$	$1.53326923405380 \times 10^{-5}$
150	y_1	$3.171282663773070 \times 10^{-8}$	$1.867461844067540 \times 10^{-9}$
	y_2	$1.15479116504980 \times 10^{-8}$	$1.317373591156270 \times 10^{-9}$
250	y_1	$1.411194379754250 \times 10^{-12}$	$7.140447337033790 \times 10^{-14}$
	y_2	$5.13872453100512 \times 10^{-13}$	$8.46892843451680 \times 10^{-14}$
500	y_1	$1.864088597899370 \times 10^{-23}$	$7.079804720586480 \times 10^{-26}$
	y_2	$6.787893959484740 \times 10^{-24}$	$1.345691794641220 \times 10^{-24}$



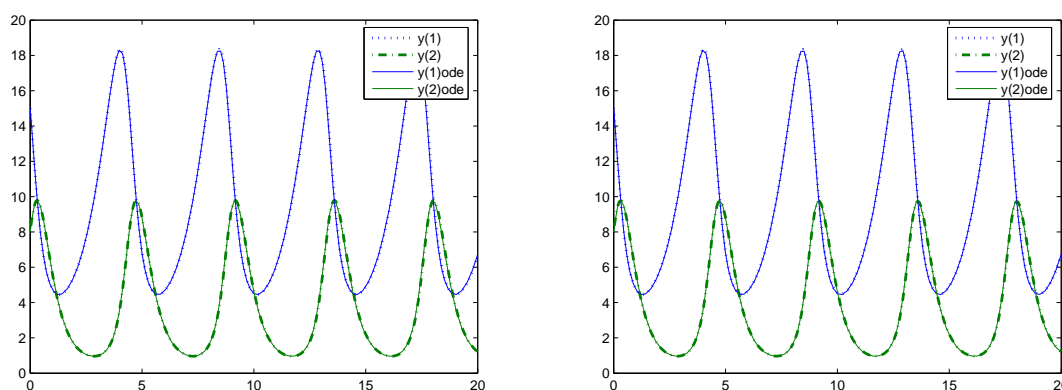
Solution curves using method (3.3) and method (3.6) respectively, with nfe =500

Figure 3: Graphical plots of example 2 using the multistep multi-derivative collocation methods**Example 3: Lotka Volterra system**

In this example we consider a real life problem of mathematical model for predicting the population dynamics of biological system in which two competing species interact, one of which is a predator, feeding on the other called a prey, see [22, 9]. If the numbers of the species alive at time t are denoted by $y_1(t)$ and $y_2(t)$, it is often assumed that, the birth rate of each of the species is simply proportional to the number of the species alive at that time and the death rate of each species depends upon the population of both species. The population of the pair of species is described by the system,

$$\begin{aligned} y_1'(t) &= 0.95y_1(t) - 0.25y_1(t)y_2(t), & y_1(0) &= 15, \\ y_2'(t) &= 0.25y_1(t)y_2(t) - 2.45y_2(t), & y_2(0) &= 8. \end{aligned}$$

We apply the newly derived multistep multi-derivative collocation methods to the system of Lotka-Volterra equations, subject to the given initial conditions and display the graphical plots obtained in Figure 4.



Solution curves using method(3.3) and method(3.6) respectively, with nfe =500

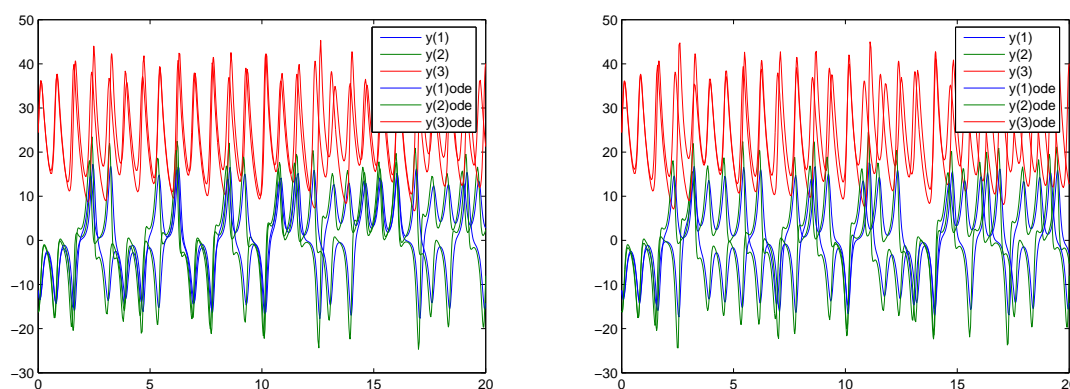
Figure 4: Graphical plots of example 3 using the multistep multi-derivative collocation methods

Example 4: Lorenz Model "Chaotic Attractor" [22]

This example is the well-known Lorenz system. The differential equations that describe the Lorenz system are given by,

$$\begin{cases} \frac{dy_1}{dx} = a(y_2 - y_1), \\ \frac{dy_2}{dx} = cy_1 - y_1y_3 - y_2, \\ \frac{dy_3}{dx} = y_1y_2 - by_3, \end{cases}$$

where a , b , and c are positive real numbers. For the parameters $(a,b,c)=(10,8/3, 28)$ the Lorenz system can display attractors. We apply our new methods to the Lorenz system and the solution curves obtained are shown in Figure 5. We compare the solution curves obtained with the ode solver of Matlab graphically,



Solution curves using method(3.3) and method(3.6) respectively, with nfe =500

Figure 5: Graphical plots of example 4 using the multistep multi-derivative collocation methods

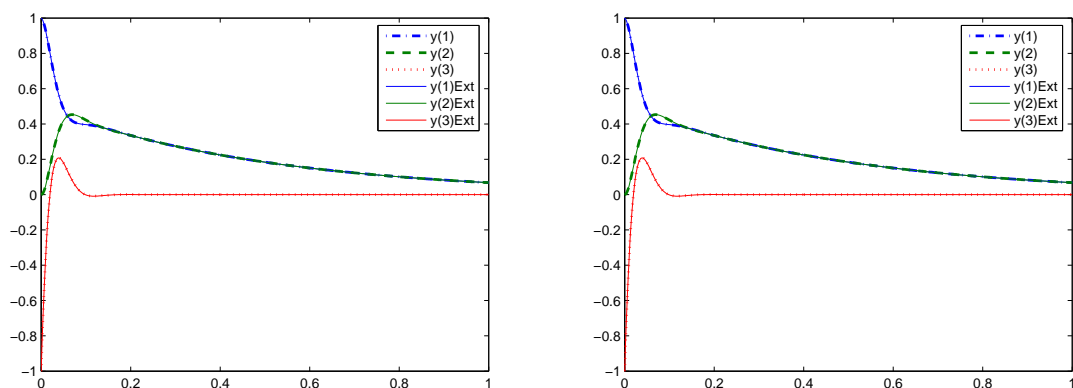
Example 5: The fifth example is a highly stiff homogeneous system with complex eigenvalues taken from Lambert [16],

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}, \quad \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

The eigenvalues of the Jacobian are $\lambda_1 = -2$, $\lambda_{2,3} = -40 \pm 40i$. This problem was posed by Lambert[16] to illustrate the difficulties experienced in solving stiff system. We solve the problem in the range of $[0,1]$ and the computed results are presented in Table 3, while the solution curves obtained are displayed in Figure 6. We compare the numerical results obtained with the results from the conventional Radau-Runge-Kutta method, which is widely recognized to be among the most efficient methods for stiff system, see, [3, 12].

Table 3: Absolute errors in the numerical integration of example 5

x	y_i	Radau IIA [3, 12],	Method(3.3)
5	y_1	$2.60451549216612 \times 10^{-10}$	$4.650613227852318 \times 10^{-12}$
	y_2	$2.60451687994490 \times 10^{-10}$	$4.650606288958414 \times 10^{-12}$
	y_3	$1.07396436188623 \times 10^{-9}$	$3.083699962047604 \times 10^{-12}$
50	y_1	$2.78099765438355 \times 10^{-12}$	$1.150857187326437 \times 10^{-12}$
	y_2	$2.78099765438355 \times 10^{-12}$	$1.150857187326437 \times 10^{-12}$
	y_3	$3.96613819020217 \times 10^{-10}$	$2.303427004121455 \times 10^{-12}$
250	y_1	$8.32667268468867 \times 10^{-17}$	$1.942890293094024 \times 10^{-16}$
	y_2	$1.38777878078145 \times 10^{-16}$	$1.665334536937735 \times 10^{-16}$
	y_3	$1.99011769660684 \times 10^{-16}$	$7.896102966055376 \times 10^{-18}$
500	y_1	$1.11022302462516 \times 10^{-16}$	$9.714451465470120 \times 10^{-17}$
	y_2	$1.11022302462516 \times 10^{-16}$	$9.714451465470120 \times 10^{-17}$
	y_3	$6.23470886821874 \times 10^{-18}$	$5.606066530391557 \times 10^{-18}$



Solution curves using Radau IIA [3, 12] and method(3.3) respectively, with nfe =500

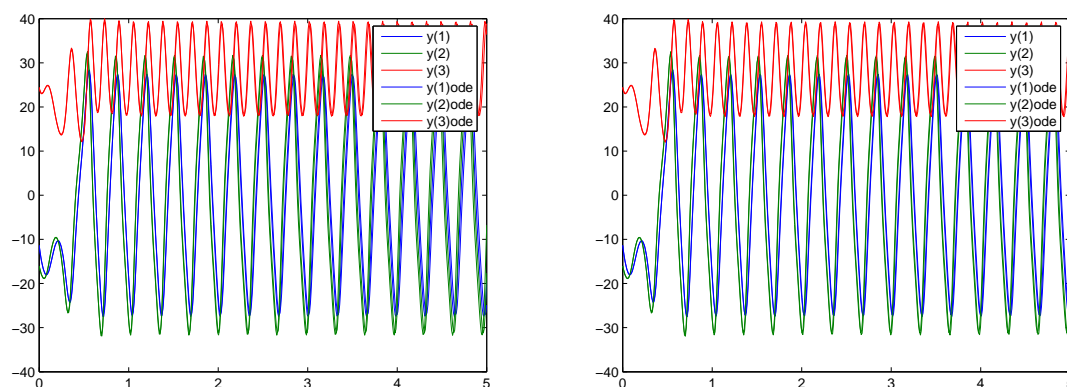
Figure 6: Graphical plots of example 5 using Radau IIA [3, 12] and method(3.3)

Example 6: Chen system “Nonlinear Chaotic Attractors” [5]

The nonlinear differential equations that describe the Chen system are given by,

$$\begin{cases} \frac{dy_1}{dx} = a(y_2 - y_1), \\ \frac{dy_2}{dx} = (c - a)y_1 - y_1y_3 + cy_2, \\ \frac{dy_3}{dx} = y_1y_2 - by_3, \end{cases}$$

where a, b and c are positive real numbers. Various studies [22, 24, 18] show that the Chen system has chaotic behavior for the parameters $(a, b, c) = (35, 12, 28)$. Figure 7 shows the solution curves obtained from the chaotic attractor of the Chen system, which is in good agreement with the graphical plots from the ode code of Matlab.



Solution curves using method(3.3) and method(3.6) respectively, with nfe =500

Figure 7: Graphical plots of example 6 using the multistep multi-derivative collocation methods

6. Concluding Remarks

The methods derived in this paper, based on the numerical experiments are accurate with better stability characteristics, suitable for the approximate numerical integration of system in ordinary differential equations. The methods proved to be accurate for the numerical integration of chaotic system of initial value problems when the high derivative terms are cheap to evaluate. We present two methods of orders six and ten that are intended for accurate integration of chaotic system in ordinary differential equations. We have also compared the numerical solutions obtained with the second derivative methods which are also accurate methods for system of equations. The solution curves are compared with the exact solutions as well as with the solutions obtained using the ode solver of Matlab graphically in Figures. From the numerical results obtained using the high-order multistep multi-derivatives collocation methods, we can conclude that a modest increase in the cost of function evaluations per step can lead to a significant improvement in accuracy and reliability of methods.

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