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IDENTITIES OF THE MODEL ALGEBRA OF MULTIPLICITY 2

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Abstract: We construct an additive basis of the free algebra of the variety generated by the model algebra of multiplicity 2 over an infinite field of characteristic not 2 and 3. Using the basis we remove a restriction on the characteristic in the theorem on identities of the model algebra (previously the same was proved in the case of characteristic 0). In particular, we prove that the kernel of the relatively free Lie-nilpotent algebra of index 5 coincides with the ideal of identities of the model algebra of multiplicity 2.

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Introduction

In this article, we consider associative algebras over an infinite field K of characteristic not 2 and 3. We will use the notation: $F = F_{\mathrm{Ass}}[X]$ is the free associative K-algebra over a countable set $X = \{x_1, x_2, \dots\}$ of free generators; $X_n = \{x_1, \dots, x_n\}$; while $[x_1, \dots, x_n]$ is the right-normed commutator of degree $n \geq 2$, i.e. $[x_1, x_2] = x_1x_2 - x_2x_1$, and $[x_1, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n]$. Also, $\mathrm{LN}(n)$: $[x_1, \dots, x_n] = 0$ is the identity of left-nilpotency of degree n; while $T^{(n)}$ and $V^{(n)}$ are the T-ideal and T-space of the algebra F which are generated by $[x_1, \dots, x_n]$. If $S \subset F$ then $(S)^T$ and $(S)^V$ denote the T-ideal and T-space generated by S; while $F^{(n)} = F/T^{(n)}$ is the relatively free algebra of countable rank with the identity $\mathrm{LN}(n)$; and $Z^*(A)$ is the kernel of A (the greatest ideal of A lying in $Z(A)$).

In the 1960s Latyshev [1, 2] began to study $F^{(3)}$ and $F^{(4)}$. In particular, he constructed an additive basis for $F^{(3)}$ and proved that the variety of associative algebras with $\mathrm{LN}(4)$ is Specht over an arbitrary field of characteristic 0.

Note also that the identities of the Grassmann algebra were studied by Krakowski and Regev [3]. In the article [4] of 1978 Volichenko constructed an additive basis for $F^{(4)}$ over a field of characteristic 0. To construct the additive basis from [1–4] it suffices to assume that the characteristic differs from 2 and 3.

Many papers are devoted to the study of $F^{(3)}$ and $F^{(4)}$. Note especially that $F^{(3)}$ was used to the negative answer to the Specht problem in the case of the finite characteristic of the main field [5–7].

In [8] was introduced the *model algebra* $E^{(n)}$ *of multiplicity* n. It was proved that $E^{(n)}$ satisfies $\mathrm{LN}(2n+1)$, and the conjecture was formulated that the variety of algebras with the identity $\mathrm{LN}(2n+1)$ is generated by $E^{(n)}$. It was assumed that $E^{(n)}$ generated the variety of algebras with $\mathrm{LN}(2n+1)$, whereupon $E^{(n)}$ was called a *model algebra*. The connection of model algebras with Clifford algebras, the correctness justification of an additive basis for $E^{(n)}$, and the refutation of the above conjecture were given in [9].

In [9, 10], under study were the central polynomials of $F^{(5)}$ and $F^{(6)}$ over a field of characteristic 0. Moreover, the important role was played by the *Hall elements*: $h(x, y, z) = [[x, y]^2, z]$ and $h'(x, y) = [[x, y]^2, y]$. Some description of the kernel $Z^*(F^{(5)}) = T(E^{(2)}) = (\mathrm{LN}(5), h')^T$ as a T-ideal was obtained, and it was proved that $Z(F^{(5)}) \subseteq T^{(4)}$. Furthermore, every proper central polynomial of $F^{(5)}$ was proved to lie in the T-space $\{h', [x_1, \dots, x_4]\}^T + (h)^V$. Analogous results were obtained for $F^{(6)}$.

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In [10], some algebras were pointed out, as well as superalgebras that possess the same identities as $E^{(2)}$ over a field of characteristic 0. The restriction on the characteristic is connected with the application of superalgebras to study the skew-symmetric elements.

Notice also that [11] presents an additive basis for the free algebra $F_3^{(n)}$ generated by three free generators. This was obtained from the Poincaré–Birkhoff–Witt basis (in abbreviated form: PBW) for F_3 by $T^{(n)}$.

The aim of this paper is to construct an additive basis for the free algebra A of countable rank in the variety $\operatorname{var} E^{(2)}$ generated by the model algebra of multiplicity 2 over an infinite field K of characteristic not 2 and 3. This result is an important intermediate stage to construct an additive basis for $F^{(5)}$. This paper is a continuation of [9, 10] and consists of four sections.

Notice the main difficulties that arise in construction of bases for A and $F^{(5)}$. If we start from the PBW-basis for a free associative algebra; then, factorizing by $T^{(5)}$, some identities of different types appear that are connected with the "concatenation" of variables from the commutators factors of basis words. The simplest type of concatenation is pointed out in Latyshev's Lemma: $[xyza][at] = 0$. The following type of concatenation is connected with Volichenko's Lemma implying that $[xyzt][ab][bc] = 0$. One more type of concatenation is connected with the centrality of a Hall element h, and it is of the shape $[abxy][ab] = 0$. It is proved that these types of concatenation "exhaust" all concatenations on the commutator words $v^{(4)}v_1^{(2)}v_2^{(2)}\dots\ \big(v_i^{(m)} \in V^{(m)}\big)$ of type $(4, 2, 2, \dots)$.

The concatenations on commutator words of type $(3, 2, 2, \dots)$ are more cumbersome, and we will not avoid their description here.

By [4], in order to construct some additive bases for A and $F^{(5)}$ it suffices to construct additive bases for $T^{(4)}$. To construct a basis for $F^{(5)}$ we evolve a descending sequence of T-ideals $T^{(4)} \supset H \supset H'$, where H and H' are the T-ideals generated by the Hall elements h and h', respectively. In § 1 we collect some available results on $F^{(5)}$. In § 2 we construct an additive basis for $T^{(4)}$ modulo the Hall ideal H, and then in § 3 we find an additive basis for H modulo the "weak Hall ideal" H' (i.e., an additive basis for A, in fact). Note that § 3 is most laborious and takes nearly 2/3 of the workload.

Notice also that in the article we additionally constructed an additive basis for the space of proper central polynomials of A.

In the last § 4 we prove the theorem on identities of $E^{(2)}$ over an infinite field K of finite characteristic $p \geq 5$. This theorem is an application of the constructed basis. We plan to devote a special article on other applications of the basis where we will present an additive basis for H', a description of the center of $F^{(5)}$, and the values of the sequence of codimensions of $T^{(5)}$.

§ 1. The Main Notions and Available Results

1.1. Proper identities and unitarily closed varieties. Let $A^{\#}$ be an algebra obtained from an algebra A by adjoining a unity externally. A variety $\mathfrak{M}$ is *unitarily closed* provided that $(\forall A)\, A \in \mathfrak{M} \Rightarrow A^{\#} \in \mathfrak{M}$.

Let $F = F_{\mathfrak{M}}[X]$ be a relatively free algebra of countable rank of a unitarily closed variety $\mathfrak{M}$. A set of free generators $X = \{x_1, \dots, x_n, \dots\}$ is assumed ordered by ascending indices.

A multilinear polynomial $f \in F$ is *proper* if $\frac{\partial f}{\partial x} = 0$ for every $x \in X$. The notion of proper polynomial can be generalized to the case of nonmultilinear polynomials as follows: A subalgebra of F generated by the Lie monomials (the commutators in generators) of degree ≥ 2 is the *subalgebra of proper polynomials*. Denote by $\Gamma_n(F)$ the space of multilinear proper polynomials of F which depend on the variables in X_n. Use the notation $\Gamma_n(\mathfrak{M})$ instead of $\Gamma_n(F)$ as well. Some new results on proper polynomials were obtained in [12].

Unless otherwise stated, when considering an identity of the relatively free algebra F, we will imply that the identity depends precisely on n variables in X_n. Notions, that are connected with the identities and varieties, may be found in [13].

1.2. Available results.

Latyshev's Lemma [2]. $[x, y, z, a][a, t] = 0$ *in* $F^{(5)}$.

Volichenko's Lemma [4]. $(T^{(3)})^2 \subseteq T^{(5)}$.

Denote the Jordan product of x and y by $x \circ y = xy + yx$. Recall that $(a, b, c)^+ = (a \circ b) \circ c - a \circ (b \circ c)$ denotes the Jordan associator of a, b, and c. In what follows, the well-known identity $(a, b, c)^+ = [b, [a, c]]$ is applied without further explanation.

Lemma 1.1 [9]. *The Hall polynomials in* $F^{(5)}$ *are such that* $h \in Z(F^{(5)})$ *and* $h' \in Z^*(F^{(5)})$.

Lemma 1.2 [10]. *In* $F^{(5)}$, *we have*
(a) $[x, a, a, y][a, z] \neq 0$ *is skew-symmetric in* x, y, *and* z;
(b) $(u, v, t)^+ = 0$ *if two elements among* u, v, *and* t *are commutators.*

These lemmas will be used frequently without further explanation.

1.3. The model algebra $E^{(m)}$. Following [9], recall the construction of $E^{(m)}$ introduced firstly in [8] under the names "model algebra" and "extended Grassmann algebra" of multiplicity m. Let E be an associative algebra with unity 1 over a field K which is given by the generators e_m ($m \in N$), θ_{ij} ($i, j \in N$, $1 \leq i \leq j$) and the defining relations $e_i \circ e_j = \theta_{ij}$ and $[\theta_{ij}, e_m] = 0$. The subalgebra $\Omega = K[\dots, \theta_{ij}, \dots]$ of E generated by 1 and θ_{ij} ($i, j \in N$), is a polynomial algebra in commuting variables, and E is a free module over Ω with the basis of *standard words* $e_{i_1} e_{i_2} \dots e_{i_n}$, where $1 \leq i_1 \leq i_2 \leq \dots \leq i_n$.

Let Θ be the ideal of E generated by θ_{ij}. Put $E^{(m)} = E/\Theta^m$. It was proved in [8] that the identity $\mathrm{LN}(2m+1)$ of Lie nilpotence of degree $2m+1$ holds in $E^{(m)}$. The initial hypothesis was that this identity is a defining identity of $E^{(m)}$. It was proved in [10] that each identity of $E^{(2)}$ over a field of characteristic 0 is a corollary of $\mathrm{LN}(5)$ and the weak Hall identity h'; in § 4 of this article we remove the restriction on characteristic.

§ 2. The T-Ideal $T^{(4)}$ of the Hall Algebra $F_{\mathfrak{H}}[X]$

2.1. Auxiliary lemmas. In this section we use the notations: $\mathfrak{H} = \mathrm{var}\langle h, \mathrm{LN}(5) \rangle$ is the Hall variety; and $A = F_{\mathfrak{H}}[X]$ is the free algebra of countable rank in $\mathfrak{H}$. For brevity, we omit commas in the right-normed commutators; for example, $[abcd] = [a, b, c, d]$. We agree also that the variables standing under one or two lines are under symmetrization; i.e., if f is a linear polynomial in the indicated variables then we put

$$f(\bar{a}, \bar{b}) = f(a, b) + f(b, a), \quad f(\overline{x_1}, \overline{x_2}, \overline{x_3}) = \sum_{\sigma \in S(3)} f(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}),$$

where $S(3)$ is the symmetric group of degree 3. Moreover, let us introduce the polynomial that we call *f-Jacobian*:

$$\sum_{a,b,c} f(a, b, c) := f(a, b, c) + f(b, c, a) + f(c, a, b).$$

Lemma 2.1. *We have*

$$[ab]^2[xy] = 0, \tag{1}$$

$$[abab][xy] = 0 \tag{2}$$

in A. *Consequently,* $V^{(4)} V^{(2)} = 0$, *and* $[ab][xy][zt]$ *is skew-symmetric in all variables.*

Proof. Notice firstly that by Lemma 1.2 the arrangement of parentheses is immaterial in the products of the shape $a \circ u \circ v$, $u \circ a \circ v$, where $a \in A$ and $u, v \in V^{(2)}$. Now,

$$[a \circ [ab] \circ [ab], y] = [[a^2, b] \circ [ab], y] = 0,$$

whence

$$[[ab]^2 \circ b, y] = 0,$$
$$2[ab]^2[xy] = [[ab]^2 \circ x, y] = -[[ab] \circ [ax] \circ b, y]$$
$$= -[[ax] \circ [ab] \circ b, y] = -[[ax] \circ [a, b^2], y] = 0,$$

which proves (1).

To verify (2) we note that if $v = [x, b]$ then by $h = 0$

$$2[abab][xy] = -2[abay][xb] = -2[[aba][xb], y] = 2[[xba][ab], y] = [[va] \circ [ab], y] = 0,$$

which proves (2).

Now, if $v \in V^{(2)}$ then by the Hall identity $h = 0$ and the Volichenko Lemma we have

$$[vbz] \circ [by] = [[vb] \circ [by], z] - [vb] \circ [[by]z] = 0.$$

Then $[vbz][by] = 0$, whence by the Latyshev Lemma and (2)

$$[b\bar{x}\bar{y}b]v = [bub]v = 0, \quad \text{where } u, v \in V^{(2)}.$$

Then by the Jacobi identity

$$0 = [b[xy]b]v = \{[bxyb] - [byxb]\}v = 2[bxyb]v.$$

Hence, $[abxy][zt]$ is skew-symmetric in a, b, y, z, and t. Whence by the Jacobi identity it is skew-symmetric in all variables, and $V^{(4)}V^{(2)} = 0$. By (1) $[ab][xy][zt]$ is skew-symmetric in all variables.

Lemma 2.2. *$V^{(3)}V^{(2)}V^{(2)} = 0$ in A.*

Proof. Note firstly that if $v \in V^{(2)}$ then

$$[abb][ax]v = -[abx][ab]v = 0 \tag{3}$$

by Lemma 2.1 and the identity $h = 0$.

Now, using Lemma 2.1, the Jacobi identity, and the linearization of (3), we see that

$$0 = [a[bc]][ax]v = \{[abc] - [acb]\}[ax]v = 2[abc][ax]v.$$

Therefore, $[abc][xy][zt]$ is skew-symmetric in all variables; in particular, $[abb][xy][zt] = 0$, and hence $V^{(3)}V^{(2)}V^{(2)} = 0$ by the Jacobi identity and the restriction on the characteristic $\mathrm{char}(K) \neq 3$.

Lemma 2.3. *Each proper polynomial of A contained in the T-ideal $T^{(4)}$ is a linear combination of the elements of the form*

(1) $[xyzt]$, where $x, y, z, t \in X$;

(2) $[xyz][ab]$, where $x, y, z, a, b \in X$.

Proof. By Lemmas 2.1 and 2.2 we have

$$[x^2, y, z, t] = [[x, y] \circ x, z, t] = [[x, y, z] \circ x, t] + [[x, y] \circ [x, z], t]$$
$$= [[x, y, z], t] \circ x + [x, y, z] \circ [x, t],$$

and by Lemma 2.1

$$[x^2, y, z] \circ [ab] = [[xy] \circ x, z] \circ [ab] = [xyz] \circ [ab] \circ x.$$

Hence, $T^{(4)}$ is generated as an ideal by the elements mentioned in the lemma, whence we get the required assertion by [12].

2.2. A basis for $\Gamma_4(\mathfrak{H}) \cap T^{(4)}$. We show firstly that the commutators $[x_4 x_{1\sigma} x_{2\sigma} x_{3\sigma}]$, $\sigma \in S(3)$, are linearly independent in A. By the PBW-Theorem [14] it suffices to understand that these elements are linearly independent in F. This is obvious, since the associative words $x_4 x_{1\sigma} x_{2\sigma} x_{3\sigma}$ are linearly independent.

Under construction of an additive basis for $F^{(5)}$, we will especially note the central basis polynomials having in mind their further application. Moreover, as some remarks we will give the dimensions of proper components of some T-ideals.

REMARK 1. $\dim_K(\Gamma_4(\mathfrak{H}) \cap T^{(4)}) = 6$.

2.3. A basis for $\Gamma_5(\mathfrak{H}) \cap T^{(4)}$. Construct a basis for $\Gamma_5(\mathfrak{H}) \cap T^{(4)}$ and prove that $\dim_K(\Gamma_5(\mathfrak{H}) \cap T^{(4)}) = 10$. Consider the space $X^{(3,2)} = [XXX][XX]$ spanned by the commutator words of type $(3,2)$. We verify firstly that $X^{(3,2)} = \operatorname{span}\langle [xay][az] \mid x, y, z, a \in K \cdot X\rangle$ is the linear span of the elements of the form $[xay][az]$. Indeed, if $[xay][az] = 0$ then $[xay][bz]$ is skew-symmetric in a, b, x, and z. Then, using the Jacobi identity we have

$$[xaa][yz] = -[xya][az] = \{[yax] + [axy]\}[az] = 0,$$

whence the required result follows. Consider the element $[x\bar{y}z][\bar{p}q] = [xyz][pq] + [xpz][yq]$ in variables from X_5. We can assume that $y < p$ and $x < q$, since $[a\bar{b}z][\bar{c}a] = 0$. Then we may suppose that either $x = x_1$, or $y = x_1$, or $z = x_1$ in $g = [x\bar{y}z][\bar{p}q]$.

It is easy to verify that

$$[a\bar{b}p][\bar{c}q] + [b\bar{c}p][\bar{a}q] + [c\bar{a}p][\bar{b}q] = 0. \tag{4}$$

Therefore, if $x = x_1$ in g then by (4) we may assume that x_1 is on the second position; i.e., $y = x_1$. Thus, we can suppose that either $y = x_1$ or $z = x_1$.

Linearize the Hall identity $[abp][ab] = 0$: $[xpp][xz] + [xzp][xp] = 0$. Applying the Jacobi identity to the second summand we have

$$[xpp][xz] + [xpz][xp] - [zpx][xp] = 0.$$

Hence,

$$[x\bar{p}p][\bar{x}z] + [x\bar{p}z][\bar{x}p] - [z\bar{p}x][\bar{x}p] = 0. \tag{5}$$

The element $[x\bar{y}z][\bar{p}q]$ in the variables from X_5 is *right* if $y = x_1$ and $x < q$.

Lemma 2.4. *Each element $[x\bar{y}z][\bar{p}q]$ over X_5 is a linear combination of right elements.*

PROOF. We will argue modulo the linear span of right elements, and denote by $\equiv$ the equivalence by this modulo. Assume that $z = x_1$ in $g = [x\bar{y}z][\bar{p}q]$. By the linearized identity (5), we have $[z\bar{p}x_1][\bar{x}p] \equiv 0$ and $[zpx_1][xp] \equiv 0$ which means by the above the required equivalence $[z\bar{p}x_1][\bar{y}t] \equiv 0$.

The number of right elements is $\mathrm{A}_4^2 = 12$. Put

$$\begin{aligned}
a_{12} &= [x_3\overline{x_1}x_4][\overline{x_2}x_5], & a_{13} &= [x_2\overline{x_1}x_4][\overline{x_3}x_5],\\
a_{14} &= [x_2\overline{x_1}x_3][\overline{x_4}x_5], & a_{15} &= [x_2\overline{x_1}x_3][\overline{x_5}x_4],\\
b_{12} &= [x_3\overline{x_1}x_5][\overline{x_2}x_4], & b_{13} &= [x_2\overline{x_1}x_5][\overline{x_3}x_4],\\
b_{14} &= [x_2\overline{x_1}x_5][\overline{x_4}x_3], & b_{15} &= [x_2\overline{x_1}x_4][\overline{x_5}x_3],\\
c_{12} &= [x_4\overline{x_1}x_3][\overline{x_2}x_5], & c_{13} &= [x_4\overline{x_1}x_2][\overline{x_3}x_5],\\
c_{14} &= [x_3\overline{x_1}x_2][\overline{x_4}x_5], & c_{15} &= [x_3\overline{x_1}x_2][\overline{x_5}x_4].
\end{aligned}$$

However, these 12 elements are linearly dependent, since the following are valid in A:

$$f_1 = a_{12} + a_{13} + b_{12} + b_{13} - a_{14} - a_{15} - c_{14} - c_{15} = 0,$$

$$f_2 = a_{12} - a_{15} + b_{13} + b_{14} - b_{15} + c_{12} - c_{13} - c_{14} = 0,$$

whose validity follows from the lemma.

Lemma 2.5. *The identities hold in A:*

(a) $f_1 = [y\bar{a}z][\bar{x}q] - [y\bar{a}x][\bar{z}q] - [x\bar{a}y][\bar{z}q] + [x\bar{a}z][\bar{y}q] + [x\bar{a}q][\bar{y}z] + [y\bar{a}q][\bar{x}z] - [x\bar{a}y][\bar{q}z] - [y\bar{a}x][\bar{q}z] = 0$;

(b) $f_2 = [y\bar{a}z][\bar{x}q] - [x\bar{a}y][\bar{q}z] + [x\bar{a}q][\bar{y}z] + [x\bar{a}q][\bar{z}y] - [x\bar{a}z][\bar{q}y] + [z\bar{a}y][\bar{x}q] - [z\bar{a}x][\bar{y}q] - [y\bar{a}x][\bar{z}q] = 0$.

Proof. (a) The element f_1 may be written without the symmetrization operator which is denoted by the line symbol:

$$\begin{aligned} f_1 = {} & [yaz][xq] + [yxz][aq] - [yax][zq] - [yzx][aq] - [xay][zq] - [xzy][aq] \\ & + [xaz][yq] + [xyz][aq] + [xaq][yz] + [xyq][az] + [yaq][xz] + [yxq][az] \\ & - [xay][qz] - [xqy][az] - [yax][qz] - [yqx][az]. \end{aligned}$$

Rearrange similar terms:

$$\begin{aligned} f_1 = {} & [yaz][xq] + [xaz][yq] - [xzy][aq] - [yzx][aq] \\ & + [xaq][yz] + [yaq][xz] - [xqy][az] - [yqx][az]. \end{aligned}$$

Now, redistribute the summands:

$$\begin{aligned} f_1 = {} & [yaz][xq] + [xaz][yq] + [zxy][aq] + [zyx][aq] \\ & + [xaq][yz] + [yaq][xz] + [qxy][az] + [qyx][az]. \end{aligned}$$

We see that f_1 is symmetric in x and y. Therefore, it suffices to show that

$$\begin{aligned} f_{11} & = [xaz][xq] + [zxx][aq] + [xaq][xz] + [qxx][az] \\ & = [xaz][xq] + [xaq][xz] + [zxx][aq] + [qxx][az] \end{aligned}$$

is equal to 0. Note that f_{11} is symmetric in z and q. Hence, it suffices to verify that $f_{12} = 0$; we have

$$\begin{aligned} f_{12} & = [x,a,z][x,z] + [z,x,x][a,z] = [x,a,z][x,z] - [z,a,x][x,z] \\ & = \{[x,a,z] - [z,a,x]\}[x,z] = \{-[a,x,z] + [a,z,x]\}[x,z] = [a,[x,z]][x,z] = 0. \end{aligned}$$

Item (a) is proved. Item (b) can be proved analogously. The lemma is proved.

Furthermore, using the computer program "Malcev" [15], it is easy to verify that f_1, f_2, and h are equivalent in $F^{(5)}$.

Lemma 2.6. *a_{1i}, b_{1i}, and c_{1j}, $i = 2,3,4,5$, $j = 2,3$, are linearly independent in A.*

Proof. Assume that in A we have

$$\begin{aligned} f = {} & \gamma_{12}[x_3\overline{x_1}x_4][\overline{x_2}x_5]_2 + \gamma_{13}[x_2\overline{x_1}x_4][\overline{x_3}x_5]_2 + \gamma_{14}[x_2\overline{x_1}x_3][\overline{x_4}x_5]_2 \\ & + \gamma_{15}[x_2\overline{x_1}x_3][\overline{x_5}x_4]_1 + \delta_{12}[x_3\overline{x_1}x_5][\overline{x_2}x_4]_1 + \delta_{13}[x_2\overline{x_1}x_5][\overline{x_3}x_4]_1 \\ & + \delta_{14}[x_2\overline{x_1}x_5][\overline{x_4}x_3]_1 + \delta_{15}[x_2\overline{x_1}x_4][\overline{x_5}x_3]_1 + \lambda_{12}[x_4\overline{x_1}x_3][\overline{x_2}x_5]_3 \\ & + \lambda_{13}[x_4\overline{x_1}x_2][\overline{x_3}x_5]_3 = 0. \end{aligned}$$

Since the linearization of the Hall element $[[a,x]^2,y]$ in $F^{(5)}$ is proportional to $[\bar{a}\bar{\bar{x}}y][\bar{b}\bar{\bar{z}}]$; therefore,

$$f = \sum_{a,b,x,y,z \in X} \alpha[\bar{a}\bar{\bar{x}}y][\bar{b}\bar{\bar{z}}], \quad \alpha \in K,$$

in $E^{(2)}$.

Substitute e_i ($i = \overline{1,5}$) for x_i. We have $[e_i\overline{e_j}e_k][\overline{e_p}e_q] = 4e_i(e_p\theta_{jk} + e_j\theta_{pk})e_q$ in $E^{(2)}$. By the skew-symmetry of $[e_i\overline{e_j}e_k][\overline{e_p}e_q]$ in e_i and e_q the elements of the form $[\bar{a}\bar{\bar{x}}y][\bar{b}\bar{\bar{z}}]$ are transformed into zero under the above substitutions.

Write the polynomial $\frac{1}{4}f$ as a sum of the three polynomials f_1, f_2, and f_3. The polynomial f_1 consists of summands, in which x_5 is either under the line or on the third position (these summands with index 1). The polynomial f_2 is the sum of elements with index 2. The polynomials f_3 are defined analogously.

1^0. Decompose $f_1(e_1, \dots, e_5)$ with respect to the basis words of $E^{(2)}$:

$$(-\delta_{14} - \delta_{15})\theta_{45}e_1e_2e_3 + (-\delta_{13} - \gamma_{15})\theta_{35}e_1e_2e_4 - \delta_{12}\theta_{25}e_1e_3e_4 \\ +(-\delta_{12} + \delta_{13} - \delta_{14})\theta_{15}e_2e_3e_4 - \delta_{15}\theta_{41}e_2e_3e_5 - \gamma_{15}\theta_{13}e_2e_4e_5.$$

2^0. Arrange $f_2(e_1, \dots, e_5)$ by analogy:

$$\gamma_{14}\theta_{13}e_2e_4e_5 + (-\gamma_{12} + \gamma_{13})\theta_{14}e_2e_3e_5 - \gamma_{12}\theta_{24}e_1e_3e_5 + (-\gamma_{13} - \gamma_{14})\theta_{34}e_1e_2e_5.$$

3^0. Do the same with $f_3(e_1, \dots, e_5)$:

$$-\lambda_{13}\theta_{12}e_3e_4e_5 - \lambda_{12}\theta_{13}e_2e_4e_5 + (-\lambda_{12} - \lambda_{13})\theta_{23}e_1e_4e_5.$$

The basis element $\theta_{12}e_3e_4e_5$ will be written as $\overline{\theta_{12}}$. Comparing the mentioned decompositions we get

$$-\lambda_{13}\overline{\theta_{12}} + (-\gamma_{15} + \gamma_{14} - \lambda_{12})\overline{\theta_{13}} + (-\delta_{15} - \gamma_{12} + \gamma_{13})\overline{\theta_{14}} \\ +(-\delta_{12} + \delta_{13} - \delta_{14})\overline{\theta_{15}} + (-\lambda_{12} - \lambda_{13})\overline{\theta_{23}} - \gamma_{12}\overline{\theta_{24}} \\ -\delta_{12}\overline{\theta_{25}} + (-\gamma_{13} - \gamma_{14})\overline{\theta_{34}} + (-\delta_{13} - \gamma_{15})\overline{\theta_{35}} + (-\delta_{14} - \delta_{15})\overline{\theta_{45}} = 0.$$

Since $\overline{\theta_{ij}}$ are linearly independent; therefore, we obtain a homogeneous system of ten equations in the ten variables γ_{1i}, δ_{1i}, and λ_{1j}. It is easy to understand that the system has only the trivial solution. Thus, we have proved that the system of ten right elements is linearly independent. The lemma is proved.

REMARK 2. $\dim_K(\Gamma_5(\mathfrak{H}) \cap T^{(4)}) = 10$. Furthermore, since $\theta_{12}e_3e_4e_5$ are not central in $E^{(2)}$, the ten "right" elements a_{1i}, b_{1i} ($i = 2, 3, 4, 5$), c_{12}, and c_{13} are linearly independent in $F^{(5)}$ modulo the center.

§ 3. The Hall T-Ideal H of $F_{\mathfrak{C}}[X]$

3.1. Linear generators of the Hall ideal. In what follows, $\mathfrak{C} = \operatorname{var}\langle \mathrm{LN}(5), h' \rangle$ is the variety of algebras satisfying the weak Hall identity; $A = F_{\mathfrak{C}}[X]$. Put $\varphi(a, x, y, b) = [a, \bar{x}] \circ [\bar{y}, b]$. Let $\Phi = \varphi(X, X, X, X)$ be the set of values of $\varphi(a, x, y, b)$ on X^4 of quadruples over X. Note that $\varphi(a, x, y, b)$ is symmetric in a, b and x, y, and in the sets (a, b) and (x, y) as well. Moreover,

$$\varphi(x, a, b, y) = \varphi(b, x, y, a) = \varphi(a, x, y, b). \tag{6}$$

Since $\varphi(a, a, a, z) = 0$; therefore, by the symmetry of $\varphi(a, x, y, b)$ in x and y we have the "Jacobi identity" with respect to a, b, and c:

$$\sum_{a,b,c} \varphi(a, b, c, z) = 0. \tag{7}$$

DEFINITION. The *proper* H-*polynomials* are the elements of the sets:

(1) $[\Phi, X][XX]^m$ ($m \ge 0$), (2) $\Phi[XX]^m$ ($m \ge 1$),

(3) $[XXX][XX]^m$ ($m \ge 2$), (4) $[XXXX][XX]^m$ ($m \ge 1$).

Note that the elements of types (1) and (4) are central. Later we prove that the remaining elements "are not" central. More precisely, we will find a basis for the space of proper central polynomials consisting of the elements of types (1) and (4) and extend the basis by the elements of types (2) and (3) to a basis of the space of proper polynomials of A.

Lemma 3.1. *The Hall* T*-ideal* H *is spanned by the elements of the shape* $h_i y_1 y_2 \dots y_j$*, where* h_i *are proper H-polynomials,* $y_i \in X$*, and* $y_1 \le y_2 \le \dots \le y_j$*.*

PROOF. In linearized form the Hall element h is of the shape $[\varphi(a,x,y,b),t]$. Putting $a = v \in [XX]$ by the Volichenko Lemma we get

$$[\varphi(v,x,y,b),t] = [[v\bar{x}] \circ [\bar{y}b], t] = [v\bar{x}t] \circ [\bar{y}b].$$

Now, by Lemma 1.2(b)

$$\varphi(a^2,x,y,b) = [a^2,\bar{x}] \circ [\bar{y},b] = a \circ [a,\bar{x}] \circ [\bar{y},b] = a \circ \varphi(a,x,y,b).$$

Similarly,

$$[\varphi(a^2,x,y,b),t] = [a \circ \varphi(a,x,y,b),t] = [\varphi(a,x,y,b),t] \circ a + \varphi(a,x,y,b) \circ [at],$$
$$[a^2,x,y] = a \circ [axy] - \tfrac{1}{2}\varphi(x,a,a,y),$$
$$[a^2,x,y] \circ v = a \circ [axy] \circ v - \varphi(x,a,a,y)v,$$
$$[a^2,x,y,z] = a \circ [axyz] + [axy] \circ [az] - \tfrac{1}{2}[\varphi(x,a,a,y),z],$$

whence we obtain the required result by (6) and induction on the polynomial degree.

3.2. An additive basis for $\Gamma_5(\mathfrak{E}) \cap H_1$. Denote by H_i the liner spaces that are spanned by the elements of type i; for example,

$$H_1 = \sum_{m \ge 0} K \cdot [\Phi, X][XX]^m.$$

Firstly, consider $\Gamma_5(\mathfrak{E}) \cap H_1$. Put

$$\psi(a,x,y,b,q) = [\varphi(a,x,y,b),q].$$

By the centrality of the Hall element h, $\psi(a,x,y,b,q) \in Z(F^{(5)})$. Note also that $\psi(a,x,y,b,q)$ is symmetric in a and b, in x and y, and in (a,b) and (x,y) as well. Furthermore, by the identity h' if $\psi(a,x,y,b,q)$ is of degree 3 with respect to one of the variables then it is zero. Therefore,

$$\sum_{a,b,q} \psi(a,x,y,b,q) = 0,$$

whence $\psi(a,x,y,b,q)$ is linearly expressed over X_5 via the elements of the shape $\psi(x_1,x,y,b,q)$, and, analogously, every such element is linearly expressed via $\psi(x_1,x_2,y,b,q)$. Now, x_3 occupies one of the three remaining positions. Since

$$\sum_{a,b,c} \psi(x_1,x_2,\bar{a},\bar{b},c) = 0,$$

$\psi(a,b,x,y,q)$ is linearly expressed over X_5 via the following five elements of the form

$$\psi_{12}(x_3,y,q), \quad \psi_{12}(y,x_3,q), \quad \psi_{12}(x_4,x_5,x_3), \tag{8}$$

where $\psi_{12}(x,y,z) = \psi(x_1,x_2,x,y,z)$, $y,q \in X_5$. Prove their linear independence. Assume that

$$\alpha\psi_{12}(x_3,x_4,x_5) + \beta\psi_{12}(x_3,x_5,x_4) + \gamma\psi_{12}(x_4,x_3,x_5)$$
$$+\delta\psi_{12}(x_5,x_3,x_4) + \lambda\psi_{12}(x_4,x_5,x_3) = 0$$

for some scalars $\alpha,\beta,\gamma,\delta,\lambda \in K$. Note that $\psi(\dots,V^{(2)}) = \psi(\dots,a)[at] = 0$ by the Latyshev Lemma and the centrality of the Hall element $[\varphi,t]$. Hence, putting $x_5 = v \in V^{(2)}$ and multiplying the sides of the equality by $[x_4 t]$, we get

$$0 = \lambda\psi_{12}(x_4,v,x_3)[x_4 t] = \lambda[\varphi(x_1,x_2,x_4,v),x_3][x_4 t],$$

whence

$$0 = [\varphi(x_1, x_2, x_4, v), x_4] = [[x_1, x_2] \circ [x_4, v] + [x_1, x_4] \circ [x_2, v], x_4]$$
$$= [[x_1, x_2] \circ [x_4, v], x_4] = 2[x_1, x_2][x_4, v, x_4]$$

in $E^{(2)}$ if $\lambda \neq 0$.

Recall that the property holds in $E^{(2)}$: if $f \neq 0$, while x and y do not enter into f; then $f[x, y] \neq 0$. Hence, $[v, a, a] = 0$; a contradiction. Thus, $\lambda = 0$, and

$$\alpha\psi_{12}(x_3, x_4, x_5) + \beta\psi_{12}(x_3, x_5, x_4) + \gamma\psi_{12}(x_4, x_3, x_5) + \delta\psi_{12}(x_5, x_3, x_4) = 0.$$

Letting again $x_5 = v \in V^{(2)}$, we obtain

$$0 = \beta\psi_{12}(x_3, v, x_4) + \delta\psi_{12}(v, x_3, x_4) = \beta[\varphi(x_1, x_2, x_3, v), x_4] + \delta[\varphi(x_1, x_2, v, x_3), x_4].$$

Put $x_3 = x_4 = a$. Then

$$0 = \beta[\varphi(x_1, x_2, a, v), a] + \delta[\varphi(x_1, x_2, v, a), a]$$
$$= \beta[[x_1, \overline{x_2}] \circ [\bar{a}, v], a] + \delta[[x_1, \overline{x_2}] \circ [\bar{v}, a], a]$$
$$= 2\beta[x_1, \overline{x_2}][[\bar{a}, v], a] + 2\delta[x_1, \overline{x_2}][[\bar{v}, a], a]$$
$$= 2\beta[x_1, x_2][[a, v], a] + 2\delta[x_1, x_2][[v, a], a],$$

whence $\beta = \delta$. Analogously, $\alpha = \gamma$, and so

$$\alpha\{\psi_{12}(x_3, x_4, x_5) + \psi_{12}(x_4, x_3, x_5)\} + \beta\{\psi_{12}(x_3, x_5, x_4) + \psi_{12}(x_5, x_3, x_4)\} = 0.$$

Putting $x_3 = x_5 = a$, we have

$$\alpha\{\psi_{12}(a, x_4, a) + \psi_{12}(x_4, a, a)\} + 2\beta\psi_{12}(a, a, x_4) = 0.$$

By the linearization of $\psi_{12}(a, a, a) = 0$ we get

$$-\alpha\psi_{12}(a, a, x_4) + 2\beta\psi_{12}(a, a, x_4) = 0.$$

Since $\varphi(x_1, x_2, a, a) \notin Z(E^{(2)})$, we get $\alpha = 2\beta$. If $\beta \neq 0$ then

$$2\{\psi_{12}(x_3, x_4, x_5) + \psi_{12}(x_4, x_3, x_5)\} + \{\psi_{12}(x_3, x_5, x_4) + \psi_{12}(x_5, x_3, x_4)\} = 0.$$

Multiplying the sides of this equality by $[x_4, t]$, by $\psi_{12}(a, a, x)[a, y] = 0$ we get

$$0 = \{\psi_{12}(x_3, x_4, x_5) + \psi_{12}(x_4, x_3, x_5)\}[x_4, t] = -\psi_{12}(x_4, x_4, x_5)[x_3, t].$$

Hence, $\varphi(x_1, x_2, a, a) \in Z(E^{(2)})$; a contradiction. Thus, the linear independence of the elements of the shape (8) is proved.

Remark 3. $\dim_K(\Gamma_5(\mathfrak{C}) \cap H_1) = 5$.

3.3. An additive basis for H_2. By (6) and (7) the linear span $K \cdot \Phi$ of the elements of the shape $\varphi(a, b, c, d)$ over X_4 possesses a basis $\varphi(x_1, x_2, x_i, x_j)$, where $\{i, j\} = \{3, 4\}$. So we have

Remark 4. $\dim_K(\Gamma_4(\mathfrak{C}) \cap K \cdot \Phi) = 2$.

A product of commutators $v' = [y_1 z_1] \dots [y_m z_m]$ is *right* provided that $y_1 < z_1 < \dots < y_m < z_m$; we write $y_0 < v'$ if $y_0 < y_1$.

Put

$$\varphi^+(a, b, x, y) = \varphi(a, b, x, y) + \varphi(a, b, y, x),$$
$$\varphi^-(a, b, x, y) = \varphi(a, b, x, y) - \varphi(a, b, y, x).$$

Lemma 3.2. *The* H*-elements of type* (2) *are linearly expressed via the elements that we call right φ-words:*

(2a) $\varphi(x_1, x_2, x_3, x_i)v'$ *and* $\varphi(x_1, x_2, x_i, x_3)v'$,

(2b) $\varphi^-(x_1, x_2, x_k, x_l)v''$,

where $i \geq 4$; $4 \leq k < l$, and v', v'' are some right products of commutators.

Proof. By [10, Lemma 5]

$$\varphi(a, a, a, b) = \varphi(b, a, a, a) = 0, \quad \varphi(a, x, y, a)[az] = \varphi(x, a, a, y)[az] = 0.$$

Then we can assume that the Hall element of type (2) is linearly expressed via the elements of the shape $\varphi(x_1, x_2, x_i, x_j)v$. Since

$$2\varphi(a, b, x, y) = \varphi^+(a, b, x, y) + \varphi^-(a, b, x, y);$$

every such element is linearly expressed via the elements of types (2a), (2b), and $\varphi^+(x_1, x_2, x_k, x_l)v''$ ($4 \leq k < l$). Finally, since $\varphi(x, y, a, a)[az] = 0$; thought $\varphi^+(x_1, x_2, x_k, x_l)v''$ ($4 \leq k < l$) is linearly expressed via the elements of type (2a). The lemma is proved.

Lemma 3.3. $\varphi^-(a, b, x, y)[x, y]$ *is not a central element in* $E^{(2)}$.

Proof. This is immediate from [10, Lemma 10].

Lemma 3.4. *The elements of types* (2a), (2b), *and* (4) *are linearly independent. Moreover, the elements of types* (2a) *and* (2b) *are linearly independent modulo the center Z.*

Proof. Assume that

$$\begin{aligned}&\sum_{i\geq 4}\{\alpha_i\varphi(x_1, x_2, x_3, x_i) + \beta_i\varphi(x_1, x_2, x_i, x_3)\}v' \\ &+ \sum_{4\leq k<l}\gamma_{kl}\varphi^-(x_1, x_2, x_k, x_l)v'' \equiv 0 \pmod{Z}.\end{aligned}$$

Let (k, l) be some fixed pair of indices. Then x_k and x_l stand under the function symbol φ only for one summand. Hence, multiplying the sides of the equivalence by $[x_k x_l]$, we get

$$\gamma_{kl}\varphi^-(x_1, x_2, x_k, x_l)[x_k x_l] \equiv 0 \pmod{Z},$$

whence $\gamma_{kl} = 0$ by Lemma 3.3. Therefore,

$$\sum_{i\geq 4}(\alpha_i\varphi(x_1, x_2, x_3, x_i) + \beta_i\varphi(x_1, x_2, x_i, x_3))v' \equiv 0 \pmod{Z}.$$

Assume that i is fixed. Put $x_2 = x_3 = a, x_i = u \in [XX]$. Then

$$\begin{aligned}0 \equiv \{\alpha_i\varphi(x_1, a, a, u) + \beta_i\varphi(x_1, a, u, a)\}v' &= \{2\alpha_i[x_1a] \circ [au] + \beta_i[x_1a] \circ [ua]\}v' \\ &= (2\alpha_i - \beta_i)([x_1a] \circ [au])v',\end{aligned}$$

whence $\beta_i = 2\alpha_i$, since $[xaa][ay] \notin Z(E^2)$. Thus,

$$\sum_{i\geq 4}\alpha_i(\varphi(x_1, x_2, x_3, x_i) + 2\varphi(x_1, x_2, x_i, x_3))v' \equiv 0 \pmod{Z}.$$

Note that if $x_1 = x_3 = x$, $x_2 = y$, and $x_i = a$, then

$$\begin{aligned}\varphi(x_1, x_2, x_3, x_i) + 2\varphi(x_1, x_2, x_i, x_3) &= \varphi(x, y, x, a) + 2\varphi(x, y, a, x) \\ &= [x\bar{y}] \circ [\bar{x}a] + 2[x\bar{y}] \circ [\bar{a}x] \\ &= [xy] \circ [xa] + 2[xy] \circ [ax] + 2[xa] \circ [yx] = -3[xy] \circ [xa].\end{aligned}$$

Hence,

$$\sum_{i\geq 4}\alpha_i([xy]\circ[xx_i])v'\equiv 0 \pmod{Z}.$$

Putting $x_i=u$ in this equivalence, we have $\alpha_i([xy]\circ[xu])v'=0$, whence $\alpha_i=0$ by $[xaa][ay]\notin Z(E^2)$. Thus, the lemma is proved.

Noting that the Hall elements of types (1) and (3) are of odd degree and the elements of types (2) and (4) are of even degree, we get

Proposition 3.1 (on the additive structure of H). *If $\overline{H_i}$ is the span of the elements of the shape $h_iy_1\dots y_s$, where $y_i\in X$, $y_1<\dots<y_s$, $h_i\in H_i$, then $H=(\overline{H_1}+\overline{H_3})\oplus(\overline{H_2}+\overline{H_4})$.*

Since the number of elements of types (2a) and (2b) in the variables of X_{2m} is equal to $2\,\mathrm{C}^1_{2m-3}+\mathrm{C}^2_{2m-3}=(2m-3)m$, we have

REMARK 5. $\dim_K(\Gamma_{2m}\cap H_2)=(2m-3)m$ with $m\geq 3$.

3.4. A basis for Ψ. In what follows, $\Psi=H_1$, and the degrees of all proper polynomials in Ψ are at least 7. Note that Ψ is spanned by the elements of the shape

$$[\varphi(a,x,y,b),q]v',$$

where $a,b,x,y,q,y_i,z_i\in X$ and $v'=[y_1z_1]\dots[y_mz_m]$.

DEFINITION. The *right ψ-words* are the elements of the shape (1) $[\varphi_i,y_0]v'$, (2) $[\varphi'_i,y_0]v'$, and (3) $[\varphi(x_1,x_2,x_k,x_l),x_3]v''$, where $\varphi_i=\varphi(x_1,x_2,x_3,x_i)$, $\varphi'_i=\varphi(x_1,x_2,x_i,x_3)$, $x_i,y_i\in X$, $k<l$, $v'=[y_1z_1]\dots[y_mz_m]$, and v'' are the right products of commutators and $y_0<v'$.

Lemma 3.5. *The right ψ-words form a basis for Ψ.*

PROOF. Verify firstly that the right ψ-words span Ψ. Note that $[abx]v'$ is skew-symmetric in $y_1,z_1,\dots,y_m,z_m$. If $w=[ab]$ then

$$[w^2,x]\circ[x,y]=[w^2,x\circ[x,y]]=[w^2,[x^2,y]]=0$$

by the centrality of the Hall element. So, $\psi=[\varphi(a,x,y,b),q]v'$ is skew-symmetric in $q,y_1,z_1,\dots,y_m,z_m$. By the weak Hall identity $[[a\bar{x}]\circ[\bar{y}a],a]=0$, whence

$$\sum_{a,b,c}[\varphi(a,x,y,b),c]=0. \tag{9}$$

By (6), (7), and (9) we may assume that $a=x_1$ in ψ. The same argument gives $x=x_2$. If x_3 coincides with one of the variables y or b then $\varphi(a,x,y,b)$ coincides either with φ_i or φ'_i. If $c=x_3$ then in view of $[\varphi(x_1,x_2,a,a),a]=0$ we can assume that $k<l$ in $[\varphi(x_1,x_2,x_k,x_l),x_3]$, since otherwise $[\varphi(x_1,x_2,\overline{x_k},\overline{x_l}),x_3]$ is a linear combination of the right ψ-words of types (1) and (2). Hence, Ψ is spanned by the right ψ-words.

Prove that the *right ψ-words are linearly independent.* Assume that

$$\sum_i\lambda_i[\varphi_i,y_0]v'+\sum_i\lambda'_i[\varphi'_i,y_0]v'+\sum_{4\leq k<l}\mu_{kl}[\varphi(x_1,x_2,x_k,x_l),x_3]v''=0. \tag{10}$$

Show firstly that $\mu_{kl}=0$. Fix $k<l$. Since $k\geq 4$; therefore, all remaining summands, which are different from $\mu_{kl}[\varphi(x_1,x_2,x_k,x_l),x_3]v''$, contain either x_k or x_l in $y_0\cup v'$ or v''. Hence, after multiplication of the sides of (10) on $[x_kx_l]$ we get $\mu_{kl}[\varphi(x_1,x_2,x_k,x_l),x_3]v''[x_kx_l]=0$, whence $\mu_{kl}=0$ by the Volichenko Lemma and Lemma 3.3. Then (10) turns into

$$\sum_i\lambda_i[\varphi_i,y_0]v'+\sum_i\lambda'_i[\varphi'_i,y_0]v'=0. \tag{11}$$

Prove that $\lambda_i = \lambda_i' = 0$ when $i \geq 5$. Assuming i fixed and multiplying the sides of (11) by $[x_i t]$, we obtain

$$0 = [\lambda_i \varphi_i + \lambda_i' \varphi_i', y_0][x_i t]v' = -[\lambda_i \varphi_i + \lambda_i' \varphi_i', x_i][y_0 t]v'.$$

Hence, without loss of generality we can assume that

$$[\lambda_i \varphi(x_1, x_2, x_3, x_i) + \lambda_i' \varphi(x_1, x_2, x_i, x_3), x_i] = 0.$$

Changing the variables, we see that $[\lambda_i \varphi(x, y, z, a) + \lambda_i' \varphi(x, y, a, z), a] = 0$, whence

$$[\lambda_i [x\bar{y}] \circ [\bar{z}a] + \lambda_i' [x\bar{y}] \circ [\bar{a}z], a] = 0.$$

With $z = y$ we have

$$0 = [2\lambda_i [xy] \circ [ya] + \lambda_i' [xy] \circ [ay], a] = (2\lambda_i - \lambda_i')[[xy] \circ [ya], a],$$

whence $\lambda_i' = 2\lambda_i$.

If $\lambda_i \neq 0$ then $[\varphi(x, y, z, a) + 2\varphi(x, y, a, z), a] = 0$. Take $v \in V^{(2)}$. Then

$$\begin{aligned} 0 &= [\varphi(x, y, z, v) + 2\varphi(x, y, v, z), a] = [[x\bar{y}] \circ [\bar{z}v] + 2[x\bar{y}] \circ [\bar{v}z], a] \\ &= [xy] \circ [zva] + [xz] \circ [yva] + 2[xy] \circ [vza] + 2[xva] \circ [yz] \\ &= [xz] \circ [yva] + [xy] \circ [vza] + 2[xva] \circ [yz], \end{aligned}$$

and with $z = a$ we get $0 = [xy] \circ [vaa] = 2[vaa][xy] \neq 0$ in $E^{(2)}$; a contradiction.

The lemma is proved. This lemma immediately implies

Remark 6. $\dim_K(\Gamma_{2m+3} \cap \Psi) = 2\,\mathrm{C}_{2m}^1 + \mathrm{C}_{2m}^2 = 2m^2 + 3m$ with $m \geq 2$.

3.5. Auxiliary relations for $\boldsymbol{\eta}$ and $\boldsymbol{\psi}$. Introduce the elements that describe the "concatenations" of a certain type:

$$\eta(a, x, q, y, b) = [a\bar{x}q][\bar{y}b],$$

$$\eta(a, x, b, b, b) = [a\bar{x}b][\bar{b}b] = [abb][xb] \notin Z,$$

$$\eta^-(a, b, x, y, c) = \eta(a, b, x, y, c) - \eta(a, b, y, x, c),$$

$$\eta^+(a, b, x, y, c) = \eta(a, b, x, y, c) + \eta(a, b, y, x, c).$$

The element $\eta(a, x, q, y, b)$ is skew-symmetric in a and b modulo Ψ. Notice also that H_3 is spanned by $\eta[a_1 b_1] \dots [a_m b_m]$ by Subsection 2.3.

We need some relations between η and ψ that will be established in the following four lemmas:

Lemma 3.6. *If $p \in K \cdot X$ then $\eta(x_3, x_2, x_1, p, p) \equiv \eta^-(x_2, x_1, x_3, p, p)$ modulo Ψ.*

Proof. Assuming that $a, b, c \in X$ we have

$$\eta(c, b, a, p, p) \equiv \eta^-(b, a, c, p, p),$$

$$\eta(c, b, a, p, p) + \eta(b, a, p, c, p) - \eta(b, a, c, p, p) \equiv 0,$$

$$[cpa][bp] + [bap][cp] + [bcp][ap] - [bpc][ap] \equiv 0,$$

$$[cpa][bp] + [cpb][ap] - [cpa][bp] - [cpb][ap] \equiv 0.$$

Since the last equivalence is an obvious equality, the lemma is proved.

Lemma 3.7. *If $w=[ab]$ then $2\eta^-(x,y,a,b,t)w=[w^2,x][yt]\neq 0$.*

PROOF. We have

$$\begin{aligned}\eta^-(x,y,a,b,t)w&=\{[x\bar{y}a][\bar{b}t]-[x\bar{y}b][\bar{a}t]\}w=\{[xba][yt]-[xab][yt]\}w\\&=\{[xba]-[xab]\}[yt]w=-[xw]w[yt]=[wx]w[yt],\end{aligned}$$

whence the required equality follows. The relation $[w^2,x][yt]\neq 0$ was proved in [9, Lemma 4].

Lemma 3.8. *$\eta(a,x,b,y,q)+\eta^-(\bar{x},b,\bar{y},a,q)$ belongs to Ψ.*

PROOF. Transform the inspected element

$$\begin{aligned}&[a\bar{x}b][\bar{y}q]+[x\bar{b}y][\bar{a}q]-[x\bar{b}a][\bar{y}q]+[y\bar{b}x][\bar{a}q]-[y\bar{b}a][\bar{x}q]\\&=[axb][yq]_1+[ayb][xq]_2+[xby][aq]_3+[xay][bq]_4-[xba][yq]_5\\&\underline{-[xya][bq]}+[ybx][aq]_6+[yax][bq]_7-[yba][xq]_8\underline{-[yxa][bq]}\end{aligned}$$

(the similar terms are underlined), and regrouping the summands we get

$$\begin{aligned}&=[y[ax]][bq]_4+[x[ay]][bq]_7-[yab][xq]_2-[yba][xq]_8\\&-[xab][yq]_1-[xba][yq]_5-[byx][aq]_6-[bxy][aq]_3\\&=-[axy][bq]-[ayx][bq]-[yab][xq]-[yba][xq]\\&-[xab][yq]-[xba][yq]-[byx][aq]-[bxy][aq].\end{aligned}$$

Put

$$\begin{aligned}C&=[axy][bq]+[ayx][bq]+[yab][xq]+[yba][xq]\\&+[xab][yq]+[xba][yq]+[byx][aq]+[bxy][aq]\\&=\{[axy][bq]+[ayx][bq]+[byx][aq]+[bxy][aq]\}\\&+\{[xab][yq]+[xba][yq]+[yab][xq]+[yba][xq]\}.\end{aligned}$$

Note that this expression is a linearization of $D=[axx][aq]+[xaa][xq]$. It remains to notice that modulo Ψ we have

$$D\equiv-[[aq],x][ax]-[[xq],a][xa]=\{-[[q,a],x]+[[q,x],a]\}[xa]=[q,[xa]][xa]\equiv 0.$$

3.6. Linear generators of $H_{1,3}=H_1+H_3$.

DEFINITION. The *right η-words of the first type* are the elements of type

$$\eta_1=\eta(x_3,x_2,x_1,x_4,x_5)v,\qquad \eta_2=\eta(x_4,x_1,x_2,x_3,x_5)v,$$

$$\eta_3=\eta(x_4,x_2,x_1,x_3,x_5)v,\qquad \eta_4=\eta(x_4,x_1,x_3,x_2,x_5)v,$$

where $v=[x_6x_7][x_8x_9]\dots$ is a right commutator word.

Lemma 3.9. *Let $\mathrm{H}^{(1)}$ be spanned by the right η-words of the first type. Then $H_{1,3}=\mathrm{H}^{(1)}+\widetilde{\mathrm{H}}+\Psi$, where $\widetilde{\mathrm{H}}$ is spanned by the elements having the shape*

$$\eta(x_3,x_1,x_2,x_i,b)v',\quad \eta(x_3,x_1,x_i,x_2,b)v',$$

$$\eta^-(x_2,x_1,x_i,x_j,b)v',\quad \eta^+(x_2,x_1,x_i,x_j,b)v',$$

with $3\le i<j$, $b<v'=[y_1z_1]\dots[y_mz_m]$ (i.e., b is less than every variable in v').

PROOF. We will proceed modulo Ψ.

Take $a, b, x, y, q \in X$. Firstly, we show that $\eta = \eta(a, x, q, y, b)v' = [a\bar{x}q][\bar{y}b]v'$ is linearly expressed (modulo Ψ) via the elements of the five types:

(1) $\eta(a, x_2, x_1, x_i, b)v'$,

(2) $\eta(a, x_1, x_3, x_2, b)v'$,

(3) $\eta(a, x_1, x_2, x_i, b)v'$, where $a, b, x_i \in X$ and $a < b < v'$,

(4) $\eta(x_3, x_1, x_i, x_2, b)v'$,

(5) $\eta(x_2, x_1, x_i, x_j, b)v'$, with $b, x_i, x_j \in X$ and $b < v'$.

The elements of types (1)–(4) are *prebasis words.*

The element $\eta(a, x, q, y, b)v'$ (modulo Ψ) is *skew-symmetric in the variables* $a, b, y_1, z_1, \dots, y_m, z_m$; we say that it is skew-symmetric in the set $a, b, \{v'\}$.

Notice also that $[a\bar{b}p][\bar{c}q] + [b\bar{c}p][\bar{a}q] + [c\bar{a}p][\bar{b}q] = 0$; i.e.,

$$\sum_{a,x,y} \eta(a, x, q, y, b) = 0. \tag{12}$$

1^0. If $q = x_1$ then $\eta = \eta(a, x, x_1, y, b)v'$ and $x_2 \in \{a, x, y\}$ by the skew-symmetry in a, b, and $\{v'\}$. Then we may assume that $x = x_2$ by (12); i.e., $\eta = \eta(a, x_2, x_1, x_i, b)v'$ is a prebasis word of type (1).

2^0. If $q \neq x_1$ then we can assume that $x_1 \in \{a, x, y\}$, and by (12) we can suppose that $\eta = \eta(a, x_1, q, y, b)v'$. Then by the skew-symmetry of η in a, b, and $\{v'\}$ one of the possibilities for x_2 takes place: (a) $q = x_2$, (b) $y = x_2$, and (c) $a = x_2$.

(a) If $q = x_2$ (i.e., $\eta = \eta(a, x_1, x_2, y, b)v'$); then we may assume that $\eta = \eta(a, x_1, x_2, x_i, b)v'$ and $a < b < v'$, i.e., η is a prebasis word of type (3).

(b) If $y = x_2$, i.e., $\eta = \eta(a, x_1, q, x_2, b)v'$, then we may assume that η coincides with one of the elements $\eta(a, x_1, x_3, x_2, b)v'$, where $a < b < v'$ or $\eta(x_3, x_1, x_i, x_2, b)v'$, with $b < v'$, and, consequently, it is a prebasis word of type (2) or (4).

(c) If $a = x_2$ then $\eta = \eta(x_2, x_1, x_i, x_j, b)v'$ is a prebasis word of type (5).

3^0. Prove that a prebasis word lies in $\mathrm{H}^{(1)} + \widetilde{\mathrm{H}} + \Psi$.

1. Let $\eta = \eta(a, x_2, x_1, x_i, b)v'$. If $a = x_3$ then by Lemma 3.6 η is linearly expressed via $\eta_1 = \eta(x_3, x_2, x_1, x_4, x_5)v'$ and the words of type (5). If $x_i = x_3$ then η is proportional to $\eta_3 = \eta(x_4, x_2, x_1, x_3, x_5)v'$.

2. $\eta(a, x_1, x_3, x_2, b)v'$ is proportional to $\eta_4 = \eta(x_4, x_1, x_3, x_2, x_5)v'$.

3. Let $\eta = \eta(a, x_1, x_2, x_i, b)v'$. If $a = x_3$ then $\eta = \eta(x_3, x_1, x_2, x_i, b)v' \in \widetilde{\mathrm{H}}$. If $x_i = x_3$ then η is proportional to $\eta_2 = \eta(x_4, x_1, x_2, x_3, x_5)v'$.

4. $\eta(x_3, x_1, x_i, x_2, b)v'$ belongs to $\widetilde{\mathrm{H}}$.

5. Given a prebasis word $\eta(x_2, x_1, x_i, x_j, b)v'$ of type (5), we have

$$2\eta(x_2, x_1, x_i, x_j, b)v' = \eta^-(x_2, x_1, x_i, x_j, b)v' + \eta^+(x_2, x_1, x_i, x_j, b)v'.$$

Hence, η belongs to $\mathrm{H}^{(1)} + \widetilde{\mathrm{H}} + \Psi$.

Lemma 3.10. *$H_{1,3} = \mathrm{H}^{(1)} + \mathrm{H}^{(2)} + \Psi$, where $\mathrm{H}^{(2)}$ is spanned by the regular η-words of type*

$$\eta^-(x_2, x_1, x_4, x_5, x_3)v, \quad \eta^-(x_3, x_1, x_2, x_i, b)v', \quad \eta^-(x_2, x_1, x_3, x_i, b)v',$$

$$\eta^+(x_3, x_1, x_2, x_i, b)v', \quad \eta^+(x_2, x_1, x_3, x_i, b)v', \quad \eta^+(x_2, x_1, x_j, x_k, x_3)v'';$$

here $b < v'$, $i \geq 4$, $4 \leq j < k$, and v, v', v'' are the right commutator words.

PROOF. Since

$$2\eta(x_3, x_1, x_2, x_i, b) = \eta^-(x_3, x_1, x_2, x_i, b) + \eta^+(x_3, x_1, x_2, x_i, b),$$

by Lemma 3.9 $\mathrm{H}^{(2)}$ is spanned by

$$\eta^-(x_3, x_1, x_2, x_i, b)v', \ \eta^+(x_3, x_1, x_2, x_i, b)v',$$

$$\eta^-(x_2, x_1, x_i, x_j, b)v', \ \eta^+(x_2, x_1, x_i, x_j, b)v'.$$

Consider $\eta^-(x_2, x_1, x_i, x_j, b)v'$. Assume that $i, j \geq 4$. Without loss of generality, we may assume that one of the indices i and j is 4 or 5. Let $i = 4$ and $j = 6$. Then $\eta^-(a, b, x, y, c)[x, y] \in \Psi$ for $a, b, c, x, y \in K \cdot X$ by Lemma 3.7. Hence,

$$\eta^-(x_2, x_1, x_4, x_6, c)[x_3, x_5] = -\eta^-(x_2, x_1, x_4, x_5, c)[x_3, x_6]$$
$$-\eta^-(x_2, x_1, x_3, x_6, c)[x_4, x_5] - \eta^-(x_2, x_1, x_3, x_5, c)[x_4, x_6].$$

This means that $\eta^-(x_2, x_1, x_4, x_6, c)[x_3, x_5]$ is a linear combination of $\eta^-(x_2, x_1, x_3, x_i, b)v'$ and $\eta^-(x_2, x_1, x_4, x_5, x_3)v$ modulo Ψ.

Lemma 3.11. $\eta^-(a, x, a, b, b) \equiv 0$.

Proof. We have

$$\eta^-(a, x, a, b, b) = \eta(a, x, a, b, b) - \eta(a, x, b, a, b) = [a\bar{x}a][\bar{b}b] - [a\bar{x}b][\bar{a}b]$$
$$= [aba][xb] - [axb][ab] \equiv -[xba][ab] + [xab][ab] = [x[ab]][ab] \equiv 0,$$

which was required.

Since $\eta^-(x_3, x_1, x_2, \overline{x_i}, \bar{b}) + \eta^-(x_2, x_1, x_3, \overline{x_i}, \bar{b}) \equiv 0$, Lemma 3.10 can be clarified.

Lemma 3.12. *$H_{1,3} = \mathrm{H}^{(1)} + \mathrm{H}^{(2)} + \Psi$, where $\mathrm{H}^{(2)}$ is spanned by the elements of the shape*

$$\eta^-(x_2, x_1, x_4, x_5, x_3)v, \quad \eta^-(x_3, x_1, x_2, x_4, x_5)v, \quad \eta^-(x_2, x_1, x_3, x_i, b)v',$$
$$\eta^+(x_3, x_1, x_2, x_i, b)v', \quad \eta^+(x_2, x_1, x_3, x_i, b)v', \quad \eta^+(x_2, x_1, x_j, x_k, x_3)v'';$$

here $b < v'$, $i \geq 4$, $4 \leq j < k$, and v', v'' are some right commutator words.

3.7. An additive basis for $H_{1,3}$.

Lemma 3.13. *In $F^{(5)}$ we have*
(a) $[a[yp]y][qr]$ *is skew-symmetric in p, q, and r;*
(b) $[a[xy]y] \cdot [bc] + [a[yb]x] \cdot [yc] = 0$.

Proof. By the Latyshev Lemma and Lemma 1.1

$$[a[yx]y][xc] = -[a[yx]c][xy] = -[[xy]ac][xy] = 0,$$

whence $[a[yp]y][qr]$ is skew-symmetric in p, q, and r. Then

$$[a[xy]y][bc] = -[a[yx]y][bc] = [a[yb]y][xc] = -[a[yb]x][yc],$$

which finishes the proof.

Lemma 3.14. *$\eta(a, x, y, a, z) \equiv \eta^-(x, y, a, z, a)$. In particular,*

$$\eta(x_3, x_2, x_1, x_4, x_5) + \eta(x_4, x_2, x_1, x_3, x_5)$$
$$-\eta^-(x_2, x_1, x_3, x_5, x_4) - \eta^-(x_2, x_1, x_4, x_5, x_3) \equiv 0.$$

Proof. We start with noting that the second equivalence is a linearization of the first. Demonstrate the first equivalence. Notice that

$$\eta(a, x, y, a, q, z) = [a\bar{x}y][\bar{a}z] = [axy][az],$$
$$\eta^-(x, y, a, z, a) = \eta(x, y, a, z, a) - \eta(x, y, z, a, a)$$
$$= [x\bar{y}a][\bar{z}a] - [x\bar{y}z][\bar{a}a] = [xya][za] + [xza][ya] - [xaz][ya].$$

Hence, by Lemma 3.13 we get

$$[\eta(a,x,y,a,z)-\eta^-(x,y,a,z,a),b]=[axyb][az]-[xyab][za]$$
$$-[xzab][ya]+[xazb][ya]=\{-[xayb]+[xyab]\}[az]$$
$$+\{[xazb]-[xzab]\}[ya]=-[x[ay]b][az]+[x[az]b][ya]$$
$$=[x[ay]a][bz]-[x[az]a][yb]=[x[ya]a][zb]+[x[za]a][yb]=0.$$

Definition. The *right η-words of type* (2) are the elements having the shape

$$\eta^-(x_2,x_1,x_3,x_i,b)v',\quad \eta^+(x_3,x_1,x_2,x_i,b)v',$$

$$\eta^+(x_2,x_1,x_3,x_i,b)v',\quad \eta^+(x_2,x_1,x_j,x_k,x_3)v'',$$

where $b<v'$, $i\ge 4$, $4\le j<k$, and v,v',v'' are some right commutator words.

Proposition 3.2. *The right η-words of type* (1) η_1,η_2,η_3, *the right η-words of type* (2), *and the right ψ-words form a basis for $H_{1,3}$.*

Furthermore, $H_{1,3}=\mathrm{H}^{(1)}\oplus\mathrm{H}^{(2)}\oplus\Psi$.

Proof. By Lemmas 3.14 and 3.8 modulo Ψ the equivalences hold

$$\eta^-(x_2,x_1,x_4,x_5,x_3)\equiv\eta_1+\eta_3-\eta^-(x_2,x_1,x_3,x_5,x_4),$$

$$\eta^-(x_3,x_1,x_2,x_4,x_5)\equiv-\eta_3-\eta^-(x_2,x_1,x_3,x_4,x_5).$$

Hence, by Lemma 3.12 $H_{1,3}$ is spanned modulo Ψ by the right η-words of type (1) or (2). Now, by Lemma 3.8 we have

$$\eta(x_3,x_4,x_1,x_2,x_5)+\eta^-(\overline{x_4},x_1,\overline{x_2},x_3,x_5)\equiv 0.$$

Since $\eta(x_3,x_2,x_1,x_4,x_5)=\eta(x_3,x_4,x_1,x_2,x_5)$; therefore,

$$\eta(x_3,x_2,x_1,x_4,x_5)+\eta^-(\overline{x_4},x_1,\overline{x_2},x_3,x_5)\equiv 0,$$

$$\eta(x_3,x_2,x_1,x_4,x_5)+\eta^-(x_4,x_1,x_2,x_3,x_5)+\eta^-(x_2,x_1,x_4,x_3,x_5)\equiv 0,$$

$$\eta_1(x_3,x_2,x_1,x_4,x_5)+\eta_2(x_4,x_1,x_2,x_3,x_5)$$
$$-\eta_4(x_4,x_1,x_3,x_2,x_5)-\eta^-(x_2,x_1,x_3,x_4,x_5)\equiv 0.$$

Consequently, η_4 is linearly expressed modulo Ψ via the right η-words η_1 and η_2 of type (1) and the right η-word $\eta^-(x_2,x_1,x_3,x_4,x_5)$ of type (2). Hence, $H_{1,3}$ is spanned modulo Ψ by the above-mentioned right η-words of types (1) and (2).

Prove that the right η-words specified in the lemma are linearly independent modulo Ψ. Assume that $z<v'$ and

$$\sum_{1\le i\le 3}\alpha_i\eta_i+\sum_{4\le i<j}\zeta_{ij}\eta^+(x_2,x_1,x_i,x_j,x_3)v''+\sum_{i\ge 4}\{\beta_i\eta^+(x_3,x_1,x_2,x_i,z)$$
$$+\gamma_i\eta^-(x_2,x_1,x_3,x_i,z)+\delta_i\eta^+(x_2,x_1,x_3,x_i,z)\}v'\equiv 0.$$

Prove that all scalars in this equivalence are zero. Present the argument as a sequence of cases. As before, we assume that the equivalences are considered modulo Ψ.

1^0. Prove firstly that $\zeta_{ij}=0$. Assuming that the pair of indices is fixed, multiply the sides of the last equivalence by $[x_ix_j]$; then

$$\zeta_{ij}\eta^+(x_2,x_1,a,b,x_3)[ab]v''\equiv 0$$

up to notation of the variables. If $\zeta_{ij} \neq 0$ then $\eta^+(x,y,a,a,z)[ab] \in Z(E^{(2)})$ and $[xaa][ab] \in Z(E^{(2)})$; a contradiction. Thus,

$$\sum_{1\le i\le 3} \alpha_i\eta_i + \sum_{i\ge 4}\{\beta_i\eta^+(x_3,x_1,x_2,x_i,z) + \gamma_i\eta^-(x_2,x_1,x_3,x_i,z)\}v' \\ + \sum_{i\ge 4}\delta_i\eta^+(x_2,x_1,x_3,x_i,z)v' \equiv 0.$$

2^0. Show that $\delta_i = 0$ and $\beta_i = 0$ if $i \ge 5$. Multiply the sides of the equivalence by $[x_3x_i]$, which yields $\delta_i\eta^+(x_2,x_1,x_3,x_i,z)v'[x_3x_i] \equiv 0$ by Lemma 3.7. Hence, $\delta_i = 0$, and

$$\sum_{1\le i\le 3} \alpha_i\eta_i + \sum_{i\ge 4}\{\beta_i\eta^+(x_3,x_1,x_2,x_i,z) + \gamma_i\eta^-(x_2,x_1,x_3,x_i,z)\}v' \\ +\delta_4\eta^+(x_2,x_1,x_3,x_4,x_5)v \equiv 0.$$

Analogously, multiplying by $[x_2x_i]$ we have $\beta_i\eta^+(x_3,x_1,x_2,x_i,z)v'[x_2x_i] \equiv 0$. Therefore, $\beta_i = 0$, and

$$\sum_{1\le i\le 3} \alpha_i\eta_i + \sum_{i\ge 4}\gamma_i\eta^-(x_2,x_1,x_3,x_i,z)v' \\ +\{\beta_4\eta^+(x_3,x_1,x_2,x_4,x_5) + \delta_4\eta^+(x_2,x_1,x_3,x_4,x_5)\}v \equiv 0.$$

3^0. Prove that $\delta_4 = 0$ and $\beta_4 = 0$. Multiply the sides of the equivalence by $[x_3x_4]$. Then $\delta_4\eta^+(x_2,x_1,x_3,x_4,x_5)v[x_3x_4] \equiv 0$, whence $\delta_4 = 0$ and

$$\sum_{1\le i\le 3} \alpha_i\eta_i + \sum_{i\ge 4}\gamma_i\eta^-(x_2,x_1,x_3,x_i,z)v' + \beta_4\eta^+(x_3,x_1,x_2,x_4,x_5)v \equiv 0.$$

Multiply the sides of the equivalence by $[x_2x_4]$. Then

$$\{\alpha_1\eta(x_3,x_2,x_1,x_4,x_5) + \beta_4\eta^+(x_3,x_1,x_2,x_4,x_5)\}[x_2x_4] \equiv 0.$$

Since $\eta(x_3,x_2,x_1,x_4,x_5)[x_2x_4] = 0$, we have $\beta_4 = 0$.

Thus,

$$\sum_{1\le i\le 3} \alpha_i\eta_i + \sum_{i\ge 4}\gamma_i\eta^-(x_2,x_1,x_3,x_i,z)v' \equiv 0.$$

4^0. Show that $\gamma_i = 0$ $(i \ge 5)$. Note that

$$\eta^-(e_m,e_n,e_i,e_j,e_l) = 4e_m(e_j\theta_{ni} - e_i\theta_{nj})e_l$$

in $E^{(2)}$. Indeed,

$$\begin{aligned}
[e_m\overline{e_n}e_i][\overline{e_j}e_l] &= 4e_m(e_j\theta_{ni} + e_n\theta_{ji})e_l,\\
\eta(e_m,e_n,e_i,e_j,e_l) &= 4e_m(e_j\theta_{ni} + e_n\theta_{ji})e_l,\\
\eta^-(e_m,e_n,e_i,e_j,e_l) &= 4e_m(e_j\theta_{ni} + e_n\theta_{ji})e_l - 4e_m(e_i\theta_{nj} + e_n\theta_{ji})e_l\\
&= 4e_m(e_j\theta_{ni} - e_i\theta_{nj})e_l.
\end{aligned}$$

In particular, substituting e_i for x_i, we infer that

(a) the summands from the first sum $\sum_{1\le i\le 3}\alpha_i\eta_i$ are of the shape $w\theta_{pq}$ $(p,q \le 4)$;

(b) the summands from the second sum $\sum_{i\ge 4}\gamma_i\eta^-(x_2,x_1,x_3,x_i,z)v'$ are of the form $w\theta_{13}$ and $w\theta_{1i}$ $(i \ge 4)$.

Hence, $\gamma_i = 0$ $(i \geq 6)$, and

$$\sum_{1 \leq i \leq 3} \alpha_i \eta_i + \gamma_4 \eta^-(x_2, x_1, x_3, x_4, x_5) + \gamma_5 \eta^-(x_2, x_1, x_3, x_5, x_4) \equiv 0.$$

Analogously, the last summand

$$\gamma_5 \eta^-(e_m, e_n, e_i, e_j, e_l) = 4e_2(e_5\theta_{13} - e_3\theta_{15})e_4$$

contains the element $-4\gamma_5 e_2 e_3 e_4 \theta_{15}$ which do not appear in other summands. Therefore, $\gamma_5 = 0$. Then

$$\alpha_1 \eta(x_3, x_2, x_1, x_4, x_5) + \alpha_2 \eta(x_4, x_1, x_2, x_3, x_5) + \alpha_3 \eta(x_4, x_2, x_1, x_3, x_5)$$
$$+ \gamma \eta^-(x_2, x_1, x_3, x_4, x_5) \equiv 0.$$

5^0. Finally, prove that all scalars in this equivalence are zero.

Substituting e_i for x_i and commuting the result with e_6, we have in $E^{(2)}$

$$(\alpha_1 - \alpha_2 - \alpha_3)e_3e_4e_5e_6\theta_{12} + (-\alpha_3 + \gamma)e_2e_4e_5e_6\theta_{13}$$
$$+ (-\alpha_1 - \gamma)e_2e_3e_5e_6\theta_{14} - \alpha_2 e_1e_4e_5e_6\theta_{23} = 0,$$

whence $\alpha_1 = \alpha_2 = \alpha_3 = \gamma = 0$. The lemma is proved.

A basis for $\Gamma_{2m+3} \cap H_{1,3}$ consists of the three right η-words of type (1), $3\mathrm{C}^1_{2m} + \mathrm{C}^2_{2m}$ right η-words of type (2), and $2\mathrm{C}^1_{2m} + \mathrm{C}^2_{2m}$ right ψ-words. Hence, $\dim(\Gamma_{2m+3} \cap H_{1,3}) = 4m^2 + 8m + 3$ with $m \geq 2$; i.e., we have

Remark 7. $\dim(\Gamma_{2m+1} \cap H_{1,3}) = 4m^2 - 1$ with $m \geq 3$.

3.8. An additive basis for H_4. Construct a basis for $\Gamma_{2m+4} \cap H_4$.

Let $M = 2m + 4$, $m \geq 1$. Introduce some proper polynomials that we call *right $V^{(4)}$-words of type* (1)–(4):

(1) $u_k = [x_1 x_M x_k y_0][y_1 z_1] \dots [y_m z_m]$,
(2) $v_k = [x_1 [x_M x_k] y_0][y_1 z_1] \dots [y_m z_m]$,
(3) $w_{ij} = [x_1 \overline{x_i} \overline{x_j} y_0][y_1 z_1] \dots [y_m z_m]$,
(4) $w_0 = [x_1[x_2 x_3]x_4][x_5 x_6] \dots [x_{M-1} x_M]$,

where $2 \leq k \leq x_{M-1}$, $2 \leq i < j < x_M$, and $y_0 < y_1 < z_1 < \dots < y_m < z_m$.

Lemma 3.15. *H_4 is spanned by the right $V^{(4)}$-words.*

Proof. Consider $p = [xyzt][a_1 b_1] \dots [a_m b_m] \in H_4$ over X_{2m+4}. By the Latyshev Lemma and the Jacobi identity this element is linearly expressed via the elements with the first letter x_1. Hence, we can assume that $x = x_1$ in p. We can suppose now that the greatest letter $q = x_M$ occupies one of the three positions y, z, and t.

(a) If $y = q$ then $p \in H_4^{(1)}$ which is a linear span of the right $V^{(4)}$-words of type (1).

(b) If $z = q$ then by the Jacobi identity modulo $H_4^{(1)}$ we can assume that p is of the shape $p' = [x[qy]t]v$, where $v = [a_1 b_1] \dots [a_m b_m]$. Clearly, $p' \in H_4^{(2)}$ which is the linear span of the right $V^{(4)}$-words of type (2).

(c) If $t = q$ then by Lemma 3.13(b) we have

$$[x[yb]q][zc] + [x[zb]q][yc] = -[x[qy]z][bc] - [x[qz]y][bc].$$

Hence, $p'' = [x[yz]q]v$ is skew-symmetric modulo $H_4^{(2)}$ with respect to all variables but x and q. Thus, $p'' \in K w_0 + H_4^{(2)}$. Since $2p = [x\bar{y}\bar{z}q]v + p''$, we finally obtain the claim.

Lemma 3.16. *The right $V^{(4)}$-words of types* (1), …, (4) *are linearly independent.*

PROOF. Assume that

$$\sum_k \alpha_k u_k + \sum_k \beta_k v_k + \sum_{i<j} \gamma_{ij} w_{ij} + \delta w_0 = 0 \tag{13}$$

for some scalars α_k, β_k, γ_{ij}, and δ.

Let (i, j) be a fixed pair of indices, and $i < j$. Note that w_{ij} is the unique right $V^{(4)}$-word in which the set of variables $y_0, y_1, z_1, \dots, y_m, z_m$ does not contain x_i and x_j. Hence, if we multiply the sides of (13) by $[x_i x_j]$ then $\gamma_{ij} w_{ij} = 0$; whence $\gamma_{ij}[x\bar{a}\bar{b}y][a, b] = 0$ in $E^{(2)}$ after renaming the variables. Then either $\gamma_{ij} = 0$ or $[x\bar{a}\bar{b}y][ab] = 0$. If the last identity holds then $[xaay][ab] = 0$ and $0 = [xaaa][yb] \neq 0$; a contradiction.

Hence, $\gamma_{ij} = 0$ and

$$\sum_k \alpha_k u_k + \sum_k \beta_k v_k + \delta w_0 = 0. \tag{14}$$

Let k be a fixed number in the segment $[2; M-1]$. Multiplying the sides of (14) by $[x_k x_M]$, we get $\alpha_k = 0$, and

$$\sum_k \beta_k v_k + \delta w_0 = 0. \tag{15}$$

If k is a fixed number in the segment $[4; M-1]$ then multiplying the sides of (15) by $[x_1 x_k]$ we obtain $\beta_k = 0$, whence $\beta_2 v_2 + \beta_3 v_3 + \delta w_0 = 0$.

Then the identities hold in $E^{(2)}$:

$$\beta_2[x_1[x_6x_2]x_3][x_4x_5] + \beta_3[x_1[x_6x_3]x_2][x_4x_5] + \delta[x_1[x_2x_3]x_4][x_5x_6] = 0,$$
$$\beta_2[x_1[x_6x_2]x_3] + \beta_3[x_1[x_6x_3]x_2] + \delta[x_1[x_2x_3]x_6] = 0,$$
$$\beta_2[x[ab]c] + \beta_3[x[ac]b] + \delta[x[bc]a] = 0.$$

Subsequently putting $a = b$ and $a = c$, we get $\beta_3 = -\delta$ and $\beta_2 = \delta$. Hence, either $\beta_2 = \beta_3 = \delta = 0$ or $[x[ab]c] - [x[ac]b] + [x[bc]a] = 0$. If the last identity holds; then, putting $a = x$, we obtain $2[[xb][xc]] = 0$. We arrive at a contradiction, since there are no identities of degree 4 in $E^{(2)}$ (see [9, Lemma 10]). The lemma is proved.

Finally, calculate the number of right $V^{(4)}$-words. There are $2(M-2)$ words of types (1) and (2), C^2_{M-2}-words of type (3), and one word of type (4). Hence, the number of right $V^{(4)}$-words of degree $M = 2m+4$ is $2(M-2) + \mathrm{C}^2_{M-2} + 1 = 2m^2 + 7m + 6$; i.e., $\dim(\Gamma_{2m+4} \cap H_4) = 2m^2 + 7m + 6$ with $m \geq 1$; i.e., $\dim(\Gamma_{2m} \cap H_4) = 2m^2 - m$ when $m \geq 3$, whence by Remark 5 we have

REMARK 8. $\dim(\Gamma_{2m} \cap H_{2,4}) = 4m^2 - 4m$ with $m \geq 3$.

§ 4. The Identities of $E^{(2)}$

Theorem. *The ideal of identities of the model algebra $E^{(2)}$ of multiplicity 2 over an infinite field K of characteristic not 2 and 3 is generated as a T-ideal by LN(5) and h'.*

PROOF. Let A be the free algebra of countable rank of the variety given by the identities LN(5) and h'. Since $E^{(2)}$ possesses the unity, by the infiniteness of K we can assume that $E^{(2)}$ is defined by a system $S \subseteq A$ of homogeneous proper identities.

Prove that if $f \in S$ then $f = 0$ in A. A product of commutators is a *commutator word*. An element f is a linear combination of commutator words. Since $(T^{(3)})^2 \subseteq T^{(5)}$ by the Volichenko Lemma; therefore, a commutator word contains at most one long commutator (of degree 3 or 4), and if a long commutator enters into a commutator word then the order of commutators is unimportant.

Now, if $\deg_a f \geq 4$ then $f = 0$ (see [10]). By the restriction $\mathrm{char}(K) \neq 2, 3$ we may assume that f is multilinear. Furthermore, it is easy to understand that the proper polynomial f belongs to $T^{(4)}$. Decomposing f by the constructed additive basis we note that f should belong to H'. Thus, the theorem is proved.

Corollary. *The kernel of $F^{(5)}$ coincides with the ideal of identities of $E^{(2)}$.*

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