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GENERALIZED RIGID METABELIAN GROUPS

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Abstract: We study the generalized rigid groups (r-groups), in the metabelian case in more detail. The periodic r-groups are described. We prove that each divisible metabelian r-group decomposes as a semidirect product of two abelian subgroups, each metabelian r-group independently embeds into a divisible metabelian r-group, and the intersection of each collection of divisible subgroups of a metabelian r-group is divisible too.

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1. Introduction

The author has studied algebraic and model-theoretic properties of rigid groups rather thoroughly in the papers and with joint papers with Myasnikov [1–9]. Recall the definition of rigid (soluble) group.

Assume that a group G has a normal series

$$G = G_1 > G_2 > \cdots > G_m > G_{m+1} = 1, \tag{1}$$

with the abelian quotients G_i/G_{i+1}. The action of G on G_i by conjugation $x \to x^g = g^{-1}xg$ enables us to define on G_i/G_{i+1} the structure of a right module over the group ring $\mathbb{Z}[G/G_i]$. The group G is called *rigid* whenever for all i this module is $\mathbb{Z}[G/G_i]$-torsion-free, meaning that every nonzero element of the ring $\mathbb{Z}[G/G_i]$ acts nontrivially on each nonzero element of G_i/G_{i+1}.

A generalization of that concept appeared in [10]. Essentially, the condition that the $\mathbb{Z}[G/G_i]$-module is G_i/G_{i+1}-torsion-free is replaced with the following: Firstly, denote by R_i the quotient ring of $\mathbb{Z}[G/G_i]$ by the annihilator of G_i/G_{i+1}, so that we can regard G_i/G_{i+1} as the right R_i-module. The new condition is that G_i/G_{i+1} is R_i-torsion-free and that the canonical mapping $\mathbb{Z}[G/G_i] \to R_i$ is injective on G/G_i. For brevity, the generalized rigid groups are called *r-groups*. Some basic properties of these groups were obtained in [10]. In particular, it was shown that series (1) of existent is uniquely determined by G, and the terms of this *rigid series* are denoted by $G_i = \rho_i(G)$. Each subgroup of G is also an r-group, and we obtain a rigid series of the latter by intersecting it with (1) and omitting repetitions. The ring R_i defined above is called the ring *associated* with the quotient G_i/G_{i+1} of the series (1). It was shown to be a (left and right) Ore domain; therefore, R_i embeds into its skew field of fractions. The concept of a *divisible* r-group was introduced: For that each module G_i/G_{i+1} must be a divisible R_i-module and then we may regard it as a (right) vector space over the skew field of fractions of R_i.

The results of this article apply mainly to metabelian r-groups, and we understand by metabelian groups those of solubility length ≤ 2. Note right away that by definition abelian r-groups can be of the two types: (1) abelian groups of prime period p, and they are all divisible in the sense of r-groups; (2) torsion-free abelian groups, and here the concept of a divisible object in the sense of r-groups coincides with the similar concept in the sense of abelian groups. We obtain the following main results. The periodic r-groups are described; in particular, they all turn out metabelian. We prove that each divisible metabelian r-group decomposes as a semidirect product of two abelian subgroups. We prove that each metabelian r-group independently embeds into a divisible metabelian r-group. We established that the intersection of every collection of divisible subgroups of a metabelian r-group is divisible too.

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2. Periodic r-Groups

Given a pair of primes (p, q) such that p divides $q^n - 1$ for some positive integer n and a cardinal $\alpha \geq 1$, construct a length 2 soluble periodic r-group denoted by $E(p, q, \alpha)$. As usual, denote by F_q the field with q elements and by $\overline{F}_q$, the algebraic closure of F_q. In the multiplicative group $\overline{F}_q^*$ take the unique cyclic subgroup C of order p. The subring of $\overline{F}_q$ generated by C is actually some subfield F_{q^n}. Consider a vector space T over field F_{q^n} with basis of cardinality α. Put $E(p, q, \alpha)$ to be the group of matrices $\begin{pmatrix} C & 0 \\ T & 1 \end{pmatrix}$. This group decomposes as a semidirect product of C and the additive group of T. It is an r-group with the rigid series

$$\begin{pmatrix} C & 0 \\ T & 1 \end{pmatrix} > \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix} > 1.$$

For p dividing $q - 1$ the group $E(p, q, 1)$ is the nonabelian group of order pq.

The proof of the next statement was outlined by Mazurov, to whom the author is grateful.

Lemma. *For two primes p and q such that p divides $q - 1$ and $E = E(p, q, 1)$ is the nonabelian group of order pq, this group is not a subgroup of the multiplicative group of any skew field.*

Proof. Assume on the contrary that E lies in the multiplicative group of a skew field D. Were the characteristic of D prime, E would generate a finite subring of D, which must be a skew subfield, and therefore a field, in contradiction with the noncommutativity of E. Thus, the characteristic of D must be 0.

Take a nonzero element $v \in D$ and denote by V the $\mathbb{Q}$-subspace generated by $v \cdot E$. Note that V is finite-dimensional, and E acts on V by right multiplication fixed-point-freely. The last condition is equivalent to the property that the linear transformation of V, representing a nontrivial element of E, always lacks eigenvalue 1. Extend V to the space W over the field of complexes $\mathbb{C}$. The group E also acts on W fixed-point-freely. Take a subgroup $A = \langle a \rangle$ of order p and a subgroup $B = \langle b \rangle$ of order q in E. Then B is normal in E and $aba^{-1} = b^r$, where $1 < r < q$ and $r^p \equiv 1 \mod q$. Denote a primitive degree q root of unity by ξ. We may assert that $\{\xi, \xi^2, \dots, \xi^{q-1}\}$ is the set of all eigenvalues of b. The space W decomposes as the direct sum $W_1 \oplus \cdots \oplus W_{q-1}$ of the subspaces $W_i = \{x \in W \mid xa = \xi^i x\}$. Take $0 \neq w \in W_1$. Since $(wa)b = w(aba^{-1})a = (wb^r)a = \xi^r \cdot wa$, it follows that $W_1 a = W_r$. Furthermore, we find that $w, wa, \dots, wa^{p-1}$ lie in distinct W_i. Thus, the vector $w + wa + \cdots + wa^{p-1}$ is nonzero, but it is fixed under the transformation a. This contradiction completes the proof. $\square$

Proposition 1. *Let G be an r-group of solubility length m. Assume that some quotient $\rho_i(G)/\rho_{i+1}(G)$ of the rigid series of G has $\mathbb{Z}$-torsion. Then this quotient has prime period p and either $i = 1$ or $i = m$. In the case $i = 1$ and $m > 1$ the order of the quotient equals p.*

Proof. Since by assumption $\rho_i(G)/\rho_{i+1}(G)$ has $\mathbb{Z}$-torsion, it contains a nontrivial element u of prime order p. In the additive language, $up = 0$. But then $p\mathbb{Z}$ lies in the kernel of the action of the ring $\mathbb{Z}$, and the whole quotient $\rho_i(G)/\rho_{i+1}(G)$ has period p.

Assume that $i = 1$ and $m > 1$ and put $A = \rho_1(G)/\rho_2(G)$. As we observed, A is an abelian group of period p. Suppose that A is not cyclic. Then A includes a subgroup A' isomorphic to the direct product of two cyclic groups of order p. Denote by R_2 the ring associated with $\rho_2(G)/\rho_3(G)$. Note that R_2 is commutative, embeds into a field, and is a homomorphic image of the group ring $\mathbb{Z}A$. Furthermore, A and, therefore, A' embed into R_2^*. However, the multiplicative group of a field cannot include a subgroup isomorphic to A'. The resulting contradiction implies that A is the cyclic group of order p.

Suppose that $1 < i < m$. In this case $\rho_i(G)$ is a nonabelian r-group; furthermore, the first quotient of the rigid series of G has prime period, for instance, q. Basing on the above, this quotient is the cyclic group of order q. Replacing G with $\rho_{i-1}(G)$, we may assume that $i = 2$, while $B = \rho_2(G)/\rho_3(G)$ is the cyclic group of order q, and $m \geq 3$. Then the ring associated with $\rho_2(G)/\rho_3(G)$ is F_q. The group $A = \rho_1(G)/\rho_2(G)$ must embed into the multiplicative group of F_q, and the period of A must be a prime p dividing $q-1$. We infer that $G/\rho_3(G)$ is isomorphic to $E(p, q, 1)$, which is a nonabelian group of order pq.

This group, in turn, must embed into the multiplicative group of the ring associated with $\rho_3(G)/\rho_4(G)$, and so, into the multiplicative group of the skew field of fractions of this ring. The latter contradicts the lemma. $\square$

Theorem 1. *The periodic r-groups are exhausted by the abelian groups of prime period and the metabelian groups $E(p, q, \alpha)$.*

PROOF. Everything is clear for abelian groups. Assume that G is a periodic nonabelian r-group. Proposition 1 implies the following properties: G is length 2 soluble, the quotient $A = \rho_1(G)/\rho_2(G)$ is the cyclic group of prime order p, and the group $\rho_2(G)$ is of prime period, for instance q. Consider the ring associated with $\rho_2(G)$. It is a homomorphic image of F_qA, so it is finite, has no zero divisors, and so presents some field F_{q^n}. The group A embeds into the multiplicative group of F_{q^n} and generates F_{q^n} as a subring. Therefore, p divides $q^n - 1$ and n is minimal with this property. Now it is obvious that $G \cong E(p, q, \alpha)$, where α is the dimension of $\rho_2(G)$ over F_{q^n}. $\square$

3. Divisible Metabelian r-Groups

3.1. The important property of divisible rigid groups is that they decompose as iterated semidirect products of abelian subgroups [5]. Let us verify a similar property for divisible metabelian r-groups.

Theorem 2. *Suppose that G is a length 2 soluble r-group and $\rho_2(G)$ is a divisible R-module, where R is the ring associated with $\rho_2(G)$. Then the subgroup $\rho_2(G)$ splits in G, meaning that there is a subgroup $A \cong G/\rho_2(G)$ such that G is the semidirect product $A \cdot \rho_2(G)$. All other decompositions of G of this sort are of the form $A^f \cdot \rho_2(G)$, where $f \in \rho_2(G)$. Each element $g \in G \setminus \rho_2(G)$ is conjugate to an element of A by an element of $\rho_2(G)$.*

Corollary. *Every divisible metabelian r-group decomposes as the semidirect product of some divisible abelian subgroup and $\rho_2(G)$.*

PROOF OF THEOREM 2. By Theorem 2 of [11], there exists a strict R-decomposition of G over $\rho_2(G)$, i.e., a group of matrices $\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix}$, where $\overline{G} = G/\rho_2(G)$ and $D(G)$ is a torsion-free right R-module, while G embeds into this group of matrices as $g = \begin{pmatrix} \bar{g} & 0 \\ d(g) & 1 \end{pmatrix}$, with $\bar{g} = g \cdot \rho_2(G)$; furthermore, the elements $d(g)$ generate the module $D(G)$. Denote by K the field of fractions of R and by V, the vector space over K obtained as the natural extension of the R-module $\rho_2(G)$. It is clear that $\rho_2(G)$ is a subspace of V. Fixing some $g_0 \in G \setminus \rho_2(G)$, put $a = \bar{g}_0$ and $t = d(g_0)$; i.e., $g_0 = \begin{pmatrix} a & 0 \\ t & 1 \end{pmatrix}$ with $a \neq 1$. Given $g = \begin{pmatrix} \bar{g} & 0 \\ d(g) & 1 \end{pmatrix} \in G$, we have

$$[g_0, g] = \begin{pmatrix} 1 & 0 \\ t(\bar{g} - 1) - d(g)(a - 1) & 1 \end{pmatrix} \in \rho_2(G).$$

Since the module $\rho_2(G)$ is divisible, it follows that

$$u_g = \begin{pmatrix} 1 & 0 \\ t(\bar{g} - 1)(a - 1)^{-1} - d(g) & 1 \end{pmatrix}$$

lies in $\rho_2(G)$ too. We have

$$gu_g = \begin{pmatrix} \bar{g} & 0 \\ t(a - 1)^{-1}(\bar{g} - 1) & 1 \end{pmatrix} = \begin{pmatrix} \bar{g} & 0 \\ v_0(\bar{g} - 1) & 1 \end{pmatrix},$$

where $v_0 = t(a-1)^{-1} \in V$. It is not difficult to verify that the elements $\begin{pmatrix} \bar{g} & 0 \\ v_0(\bar{g} - 1) & 1 \end{pmatrix}$ constitute in G a subgroup A isomorphic to $\overline{G}$: the mapping $\begin{pmatrix} \bar{g} & 0 \\ v_0(\bar{g} - 1) & 1 \end{pmatrix} \to \bar{g}$ is an isomorphism. The group G is the semidirect product of A and $\rho_2(G)$.

Regarding $\rho_2(G)$ as a right R-module, denote it by T. We can identify G with the group of matrices $\begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$. Take $g = \begin{pmatrix} a & 0 \\ t & 1 \end{pmatrix} \notin \rho_2(G)$, then $a \neq 1$. With $f = \begin{pmatrix} 1 & 0 \\ t(a-1)^{-1} & 1 \end{pmatrix}$ we have $f^{-1}gf = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in A$, so that every element $g \in G \setminus \rho_2(G)$ is conjugate by the element $f \in \rho_2(G)$ to an element of A.

Suppose that $A' \cdot \rho_2(G)$ is another decomposition of G as a semidirect product and take $1 \neq a' \in A'$. Choose $f \in \rho_2(G)$ with $f^{-1}a'f \in A$. Since both A and A' coincide with the centralizers of their nontrivial elements, we can assert that $f^{-1}A'f = A$. $\square$

3.2. Suppose that G is an r-group of solubility length m and take the tuple $(R_1, \dots, R_m)$ of rings associated with the corresponding quotients of the rigid series of G. By definition, $G/\rho_i(G)$ embeds into the multiplicative group of R_i and generates the latter. Suppose that H is a subgroup of G of the same solubility length m. Then H is also an r-group; furthermore, $\rho_i(H) = H \cap \rho_i(G)$ and we may assume that the ring $R_i(H)$ associated with $\rho_i(H)/\rho_{i+1}(H)$ is the subring of R_i generated by $H/\rho_{i+1}(H) \leq G/\rho_{i+1}(G)$. By analogy with rigid groups, say that H is an *independent* subgroup of G whenever for all i every system of elements of $\rho_i(H)/\rho_{i+1}(H)$ linearly independent over $R_i(H)$ remains linearly independent over the larger ring $R_i = R_i(G)$. Observe that every subgroup of an abelian r-group is independent. In accordance with this definition, we can discuss the independent embeddings of one r-group into another.

Theorem 3. *Each metabelian r-group embeds independently into a divisible metabelian r-group.*

Proof. If the group G is abelian then either its period is a prime and then G is divisible according to our definition, or G is a torsion-free group and then G embeds into the divisible completion (divisible hull), which is a complete torsion-free abelian group uniquely defined up to a G-isomorphism.

Suppose that the group G is length 2 soluble, put $A = G/\rho_2(G)$, and denote by R the ring associated with $\rho_2(G)$. Since by Theorem 2 of [11] we can replace G by its strict R-decomposition, assume right away that G is the semidirect product of A and $\rho_2(G)$.

Suppose firstly that A is a group of prime period p. By Proposition 1 it must be the cyclic group of order p. If the period of $\rho_2(G)$ is finite then Theorem 1 shows that G is isomorphic to some group $E(p, q, \alpha)$ and is itself divisible. Suppose that the period of $\rho_2(G)$ is infinite. Denote by K the field of fractions of R which is of characteristic 0. In the R-module $\rho_2(G)$ choose a maximal system of elements $\{e_i \mid i \in I\}$ which is linearly independent over R and embed $\rho_2(G)$ into the vector space V over K with basis $\{e_i \mid i \in I\}$. The group of matrices $\begin{pmatrix} A & 0 \\ V & 1 \end{pmatrix}$ is a divisible r-group, and G embeds naturally into it as an independent subgroup.

Suppose that A is a torsion-free abelian group. It embeds into the multiplicative group of R. Denote by K the field of fractions of R and by $\overline{K}$ the algebraic closure of K. The multiplicative group of $\overline{K}$ is a complete abelian group. Denote by A' the divisible hull of A in this group, which is a divisible torsion-free abelian group. Denote by R' the subring in $\overline{K}$ generated by A' and by K' its field of fractions. Choose in $\rho_2(G)$ a maximal system of elements $\{e_i \mid i \in I\}$ which is linearly independent over R and embed $\rho_2(G)$ into the vector space V over K' with basis $\{e_i \mid i \in I\}$. The group G independently embeds into the divisible metabelian r-group $\begin{pmatrix} A' & 0 \\ V & 1 \end{pmatrix}$. $\square$

Proposition 2. *The intersection of each collection of divisible subgroups in a metabelian r-group is a divisible subgroup.*

Proof. Suppose that G is a metabelian r-group. Take some set $\{G_i \mid i \in I\}$ of the divisible subgroups of G and put $H = \bigcap G_i$. The claim is obvious if G is an abelian group or $H \leq \rho_2(G)$ and then $H = \bigcap \rho_2(G_i)$ is the intersection of abelian divisible r-groups.

Suppose that there exists $a \in H \setminus \rho_2(G)$. Denote by A the centralizer of a in G and put $A_i = A \cap G_i$. Theorem 2 implies that there are decompositions of G_i as semidirect products $A_i \cdot \rho_2(G_i)$. Then $H = B \cdot \rho_2(H)$, where $B = \bigcap A_i$ and $\rho_2(H) = \bigcap \rho_2(G_i)$. It is clear that B is a divisible abelian r-group. Let R

be the ring associated with $\rho_2(G)$ in G. The subring R_i generated by A_i is associated in G_i with $\rho_2(G_i)$, while the subring R_B generated by B is associated in H with $\rho_2(H)$. It is clear that $R_B \leq \bigcap R_i$. We have to prove that $\rho_2(H)$ is a divisible R_B-module. Take $t \in \rho_2(H)$ and $0 \neq u \in R_B$. Since $u \in R_i$ for every i, it follows that t is divisible by u in each $\rho_2(G_i)$. The quotient is uniquely determined and independent of i. Suppose that $su = t$. We have $s \in \bigcap \rho_2(G_i) = \rho_2(H)$. $\square$

Problem. *Is the intersection of each collection of divisible subgroups in an arbitrary r-group a divisible subgroup?*

Remark 1. Resting on Proposition 2, we can define the divisible closure of an arbitrary subgroup H of a metabelian divisible r-group G as the intersection of all divisible subgroups which include H.

Remark 2. For an arbitrary rigid group G it was established [5] that there exists a divisible closure (in a suitable divisible rigid group) in which G is an independent subgroup, and this closure is uniquely determined up to a G-isomorphism. The existence of a similar closure for every metabelian r-group follows from Theorem 3, but uniqueness does not.

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