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ON THE PRONORMALITY OF SUBGROUPS OF ODD INDEX IN SOME EXTENSIONS OF FINITE GROUPS

W. Guo, N. V. Maslova, and D. O. Revin UDC 512.542

Abstract: We study finite groups with the following property $(*)$: All subgroups of odd index are pronormal. Suppose that G has a normal subgroup A with property $(*)$, and the Sylow 2-subgroups of G/A are self-normalizing. We prove that G has property $(*)$ if and only if so does $N_G(T)/T$, where T is a Sylow 2-subgroup of A. This leads to a few results that can be used for the classification of finite simple groups with property $(*)$.

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1. Introduction

We will consider only finite groups, so the term a "group" will always mean a "finite group."

Since the celebrated Feit–Thompson Theorem [1] stating that the order of a nonabelian simple group is even, the subgroups of odd index have played an important role in the theory of finite nonsolvable and, in particular, simple groups (see [2–8]). This paper is concerned with pronormality of the subgroups of odd index in nonsolvable groups.

Following Hall, we say that a subgroup H of a group G is *pronormal* if the subgroups H and H^g are conjugate in $\langle H, H^g\rangle$ for every $g \in G$.

In this paper, we study the groups G that enjoy the property:

$$\textit{Every subgroup of } G \textit{ of odd index is pronormal in } G. \qquad (*)$$

It is clear that property $(*)$ is inherited by homomorphic images. We are interested in the conditions for a group G with a normal subgroup A to have property $(*)$ provided that so do A and G/A. This is motivated by the following

Problem A. *Which finite simple groups have property* $(*)$*?*

Problem A, studied in [9–12], stems from the article [13] describing the normalizers of Sylow 2-subgroups in finite simple groups as well as the observation: In a group G, a subgroup H of odd index is pronormal if and only if H and H^g are conjugate in $\langle H, H^g\rangle$ for all $g \in N_G(S)$, where S is a fixed Sylow 2-subgroup of G lying in H. Since most of the finite simple groups have a self-normalizing Sylow 2-subgroup (that is, coinciding with its normalizer), we see that the finite simple groups satisfy property $(*)$ in general. Also, property $(*)$ was established for many simple groups with non-self-normalizing Sylow 2-subgroups (for example, for all sporadic simple groups, the groups of Lie type over fields of characteristic 2, and

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the groups $\mathrm{PSL}_2(q)$ and ${}^2G_2(q)$; see [10]). At the same time there exist simple groups that do not have property $(*)$: If $q \equiv \pm 3 \pmod 8$ and $n \neq 2^k, 2^k(2^{2m}+1)$, then $\mathrm{PSp}_{2n}(q)$ has a nonpronormal subgroup of odd index [12]. Note that if $q \not\equiv \pm 3 \pmod 8$ or $n = 2^k$, then $\mathrm{PSp}_{2n}(q)$ has property $(*)$ (see [12]).

An attempt of induction in studying Problem A for the classical groups raises the question about the conditions under which property $(*)$ is enjoyed by a group such that property $(*)$ is enjoyed by the factors of some subnormal series. The following situation is typical in this connection. Given a group G, suppose that a subgroup H of odd index and an element g of the normalizer of $S \in \mathrm{Syl}_2(H)$ lie in a maximal subgroup M. To prove that H is pronormal, it suffices to show that M has property $(*)$. The description of maximal subgroups of odd index [4, 5, 14, 15] and the inductive hypothesis usually guarantee that all composition factors of M have property $(*)$. The group M itself is often a direct or central product of groups with property $(*)$ or a wreath product of a group with property $(*)$ and a symmetric group (which has property $(*)$ since all Sylow 2-subgroups of the latter are self-normalizing).

The difficulty here is that property $(*)$ is not transferred even to such a simple type of extension as a direct product[1)] and, furthermore, even to a direct product of simple groups. We say that a group L *belongs to the list* $\mathscr{L}_0$ if $L = J_1$ or $L = {}^2G_2(3^{2m+1})$ for some natural m. It is known [10] that the groups of $\mathscr{L}_0$ have property $(*)$ and the normalizer of a Sylow 2-subgroup in such a group is a 2-Frobenius group of order $2^3 \cdot 7 \cdot 3$ (see [13]).

Proposition 1. *A direct product $L \times K$ of finite groups has a nonpronormal subgroup of odd index if $L \in \mathscr{L}_0$ and the index of a Sylow 2-subgroup S of K in $N_K(S)$ is a multiple of 7. In particular, a direct product of two subgroups from $\mathscr{L}_0$ does not satisfy property $(*)$.*

Thus, property $(*)$ is not inherited by direct products. It is not inherited by normal subgroups either. Indeed, a Sylow 2-subgroup of $G = \mathrm{Aut}(\mathrm{PSp}_6(3))$ is self-normalizing (see, for instance, [17]), and so G has property $(*)$, while its normal subgroup $N = \mathrm{PSp}_6(3)$ has a nonpronormal subgroup of odd index [11, Theorem 2].

On the other hand, it is easy to show that the Sylow 2-subgroups of a group G are self-normalizing provided that G has a normal subgroup A such that the Sylow 2-subgroups of both A and G/A are self-normalizing. As we mentioned, in the study of Problem A, we often need to check property $(*)$ for a wreath product of a group satisfying property $(*)$ by some symmetric group. This raises the question of whether or not the subgroups of odd index are pronormal in an extension of a group with property $(*)$ by a group whose Sylow 2-subgroups are self-normalizing.

In general, the answer is in the negative (for example, in a wreath product $X \wr S_n$, where X is an abelian group of odd order and the degree n is not coprime to $|X|$, the wreathing group is not pronormal [11, Corollary to Theorem 1]). Nevertheless, in this paper we find a criterion for the affirmative answer. Also, we demonstrate the power of this criterion by proving that the direct products of some simple groups and some wreath products have property $(*)$.

Notation. Fix a prime p.

We write $H \leq_p G$ if $H \leq G$ and p does not divide $|G : H|$.

Let $\mathscr{X}_p$ be the class of all finite groups with self-normalizing Sylow p-subgroups, and let $\mathscr{Y}_p$ be the class of all finite groups G in which $H \leq_p G$ implies that H is pronormal in G.

In this notation, $\mathscr{Y}_2$ is exactly the class of groups with property $(*)$. Observe also that $\mathscr{X}_p \subseteq \mathscr{Y}_p$ (see Lemma 1).

Theorem 1. *Suppose that a group G has a normal subgroup A such that $A \in \mathscr{Y}_p$ and $G/A \in \mathscr{X}_p$. Let T be a Sylow p-subgroup of A. Then the following are equivalent:*

(1) $G \in \mathscr{Y}_p$;

(2) $N_G(T)/T \in \mathscr{Y}_p$.

This assertion turns out a useful tool for proving property $(*)$ in wreath products. As an example of how Theorem 1 can be used, we prove Theorem 2 which in turn will be used to prove $(*)$ in direct

[1)] For example, [16] was addressed the question of when a given subgroup of a direct product is pronormal.

products of simple groups (see Theorem 3), as well as in studying Problem A for some simple groups of classical series: symplectic, linear, and unitary groups.

Notation. We define the partial order $\preceq$ on the set of naturals as follows: Given numbers a and b with binary expansions

$$a = \sum_{i=0}^{\infty} \alpha_i \cdot 2^i, \quad b = \sum_{i=0}^{\infty} \beta_i \cdot 2^i,$$

where $\alpha_i, \beta_i \in \{0, 1\}$ and almost all a_i and b_i are zero, we write

$$a \preceq b \quad \text{if and only if } \alpha_i \leq \beta_i \text{ for all } i.$$

It is clear that $\preceq$ is a suborder of the natural linear order on the set of naturals.

Theorem 2. *Let A be an abelian group and $G = \prod_{i=1}^{t}(A \wr S_{n_i})$, where all wreath products are natural permutations. Then $G \in \mathscr{Y}_2$ if and only if for every natural m such that $m \preceq n_i$ for some i, the number $(|A|, m)$ is a power of 2.*

The main result of this paper is

Theorem 3. *Let $G = \prod_{i=1}^{t} G_i$, where $G_i \cong \mathrm{PSp}_{n_i}(q_i)$ for all $i \in \{1, \dots, t\}$, while all n_i are powers of 2 and all q_i are odd. Then every subgroup of odd index is pronormal in G.*

This theorem is an important step in studying property $(*)$ for the symplectic groups $\mathrm{PSp}_{2n}(q)$ in the only open case when $q \equiv \pm 3 \pmod 8$ and $n = 2^k(2^{2m} + 1)$.

Theorem 3 and Proposition 1 suggest the conjecture:

Conjecture 1. *Let $G = \prod_{i=1}^{t} G_i$, where for every $i \in \{1, \dots, t\}$ G_i is a nonabelian simple group such that G_i satisfies condition $(*)$ and $N_{G_i}(S_i)/S_i$ is abelian, with S_i a Sylow 2-subgroup of G_i. Then G satisfies $(*)$ too.*

2. Preliminaries

Our terminology and notation are mostly standard and can be found in [17, 18]. We write $H \operatorname{prn} G$ to abbreviate that "H is a pronormal subgroup of a group G."

As usual, given a set π of primes, π' stands for the set of all primes not in π. Also, if n is a natural, then n_π is the largest natural divisor of n such that all prime divisors of n_π are in π.

Given a subset π of the set of all primes, let $O_\pi(G)$ denote the π-radical (the largest normal π-subgroup) of G.

We write $\mathrm{Soc}(G)$ for the socle of G (the subgroup generated by all minimal nonidentity normal subgroups of G). Recall that a group is *almost simple* if its socle is a nonabelian simple group.

We write $\mathrm{Syl}_p(G)$ for the set of Sylow p-subgroups of G.

Lemma 1 [9, Lemma 5]. *Let G be a group and $H \leq G$. Suppose that H has a Sylow subgroup S of G. Then*

(1) *$H \operatorname{prn} G$;*

(2) *H and H^g are conjugate in $\langle H, H^g \rangle$ for every $g \in N_G(S)$.*

Lemma 2 [16, Propositions 2.1, 4.3, 4.4, and Corollary 4.7]. *Let $G = G_1 \times G_2$ and $H \leq G$. For $i = 1, 2$, let π_i be the coordinate projection $G \to G_i$ and let $C_i = \{x \in G_i \mid [x, \pi_i(H)] \leq G_i \cap H\}$. Then*

(1) *$C_i = N_G(H) \cap G_i$;*

(2) *if $\pi_i(H) \operatorname{prn} G_i$ for $i = 1, 2$ and*

$$N_G(H) = \langle N_{G_i}(\pi_i(H)) \mid i = 1, 2 \rangle = N_{G_1}(\pi_1(H)) \times N_{G_2}(\pi_2(H)),$$

then $H \operatorname{prn} G$;

(3) *if at least one of the groups* G_i, $i = 1, 2$, *is solvable, then* $H \operatorname{prn} G$ *if and only if* $\pi_i(H) \operatorname{prn} G_i$ *for* $i = 1, 2$ *and*

$$N_G(H) = \langle N_{G_i}(\pi_i(H)) \mid i = 1, 2\rangle = N_{G_1}(\pi_1(H)) \times N_{G_2}(\pi_2(H));$$

(4) *if at least one of the groups* G_i, $i = 1, 2$, *is solvable, then* $H \operatorname{prn} G$ *if and only if* $\pi_i(H) \operatorname{prn} G_i$ *and* $N_{G_i}(\pi_i(H)) \leq C_i$ *for* $i = 1, 2$.

Lemma 3 [10, Lemma 5]. *Let* H *and* M *be subgroups of* G, *with* $H \leq M$. *Then*

(1) *if* $H \operatorname{prn} G$, *then* $H \operatorname{prn} M$;

(2) *if* $S \leq H$ *for some Sylow subgroup* S *of* G, *with* $N_G(S) \leq M$, *and* $H \operatorname{prn} M$, *then* $H \operatorname{prn} G$.

Lemma 4. *Suppose that* G *is a group,* A *is a normal subgroup of* G *and* H *is a subgroup of* A. *Then the following conditions are equivalent:*

(1) $H \operatorname{prn} G$;

(2) $H \operatorname{prn} A$ *and* $G = N_G(H)A$;

(3) $H \operatorname{prn} A$ *and* $H^G = H^A$.

Proof. It suffices to establish the equivalence of (1) and (2), since it is well known that for G, A, and H as in the hypothesis of the lemma, $G = N_G(H)A$ if and only if $H^G = H^A$.

The fact that (1) implies (2) is well known, and we provide a proof only for completeness. Suppose that $H \operatorname{prn} G$ and $g \in G$. Then $H \operatorname{prn} A$ by Lemma 3 and there is t such that

$$t \in \langle H, H^g\rangle \leq A, \quad H^t = H^g.$$

Thus $gt^{-1} \in N_G(H)$, and so $g \in N_G(H)A$.

We prove now that (2) implies (1). Suppose that $H \operatorname{prn} A$ and $G = N_G(H)A$. Let $g \in G$. There are $n \in N_G(H)$ and $a \in A$ such that $g = na$. Therefore $H^g = H^{na} = H^a$ and since H is pronormal in A, we have that H and H^a are conjugate by $t \in \langle H, H^a\rangle = \langle H, H^g\rangle$. It follows that H and H^g are conjugate by $t \in \langle H, H^g\rangle$. $\square$

Lemma 5 [10, Lemma 3; 19, Chapter I, Proposition (6.4)]. *Suppose that* H *is a subgroup and* N *is a normal subgroup of* G *and let* $^- : G \to G/N$ *be the natural epimorphism. Then*

(1) *if* $H \operatorname{prn} G$, *then* $\overline{H} \operatorname{prn} \overline{G}$;

(2) $H \operatorname{prn} G$ *if and only if* $\overline{H} \operatorname{prn} \overline{G}$ *and* $H \operatorname{prn} N_G(HN)$;

(3) *if* $N \leq H$, *then* $H \operatorname{prn} G$ *if and only if* $\overline{H} \operatorname{prn} \overline{G}$.

In particular, a subgroup H *of odd index is pronormal in* G *if and only if* $H/O_2(G)$ *is pronormal in* $G/O_2(G)$.

Lemma 6. *Let* N *be a normal subgroup of* G *such that* $G/N \in \mathscr{X}_p$. *Then* $H \operatorname{prn} G$ *and* $H \operatorname{prn} HN$ *are equivalent for* $H \leq_p G$.

Proof. By Lemma 3, it suffices to show that the pronormality of H in HN implies the pronormality of H in G. Let $^- : G \to G/N$ be the natural epimorphism.

Suppose that $H \operatorname{prn} HN$. By Lemma 5(2), it is sufficient to prove that $\overline{H} \operatorname{prn} \overline{G}$ and $H \operatorname{prn} N_G(HN)$.

Let T be a Sylow p-subgroup of G such that $T \leq H$ and let $\bar{g} \in N_{\overline{G}}(\overline{T})$. Since $\overline{G} \in \mathscr{X}_p$, it follows that $\bar{g} \in N_{\overline{G}}(\overline{T}) = \overline{T} \leq \overline{H}$ and, by Lemma 1, $\overline{H} \operatorname{prn} \overline{G}$.

If $K = N_G(HN)$, then the Frattini argument yields

$$K = HNN_K(T) \leq HNN_K(NT) = HNNT = HN,$$

and so $H \operatorname{prn} HN = K = N_G(HN)$. $\square$

Lemma 7 [13, Corollary of Theorems 1–3]. *Let* G *be a nonabelian simple group and* $S \in \mathrm{Syl}_2(G)$.

(1) *If* G *is isomorphic to* ${}^2G_2(3^{2n+1})$ *or* J_1, *then* $N_G(S) \cong 2^3 \rtimes (7 \rtimes 3) < \mathrm{Hol}(2^3)$.

(2) *If* G *is isomorphic to* $\mathrm{PSp}_{2n}(q)$, *with* q *an odd prime power, then* $N_G(S) = S$ *for* $q \equiv \pm 1 \pmod 8$; *and if* $q \equiv \pm 3 \pmod 8$, *then* $N_G(S)/S$ *is an elementary abelian* 3*-group of order* 3^t, *where* t *is defined by the binary expansion*

$$n = 2^{s_1} + \dots + 2^{s_t}, \quad s_1 > \dots > s_t \geq 0.$$

Lemma 8 [9, Lemma 4]. *Let H be a subgroup of G and let $g \in G$. Suppose that for some $y \in \langle H, H^g \rangle$ the subgroups H^y and H^g are conjugate in $\langle H^y, H^g \rangle$. Then H and H^g are conjugate in $\langle H, H^g \rangle$.*

Lemma 9. *Let Q be a subgroup of odd index in a group $L = L_1 \times L_2 \times \dots \times L_n$, where L_i are finite groups, and let π_i be the projection from L to L_i. If there is i such that L_i is almost simple, $L_i/\operatorname{Soc}(L_i)$ is a 2-group and $\pi_i(Q) = L_i$, then $L_i \le Q$.*

Proof. Since $L_i \trianglelefteq L$, we have $Q \cap L_i \trianglelefteq Q$, and so $\pi_i(Q \cap L_i)$ is a normal subgroup of $\pi_i(Q) = L_i$.

Choose $S \in \operatorname{Syl}_2(L)$ such that $S \le Q$. Then $S \cap L_i \in \operatorname{Syl}_2(L_i)$ and $S \cap L_i = \pi_i(S \cap L_i) \le \pi_i(Q \cap L_i)$. Therefore, $\pi_i(Q \cap L_i)$ is a normal subgroup of odd index in L_i. The group L_i is almost simple and $L_i/\operatorname{Soc}(L_i)$ is a 2-group, so $\pi_i(Q \cap L_i) = L_i$, whence $L_i \le Q$. $\square$

Lemma 10 [11, Theorem 1]. *Let H and V be subgroups of a group G such that V is abelian and normal in G and $G = HV$. Then the following are equivalent:*

(1) *H is pronormal in G;*

(2) *$U = N_U(H)[H, U]$ for every H-invariant subgroup $U \le V$.*

Lemma 11. *Let $G = V \rtimes B$, where V is an abelian normal subgroup of a group G and $B \le G$, and let H be a subgroup of G. Define the mapping $\tau : G \to B$ that sends $g \in G$ to $b \in B$ such that $g = vb$ for some $v \in V$. Then τ is a homomorphism and $H \operatorname{prn} HV$ whenever $\tau(H) \operatorname{prn} \tau(H)V$.*

Proof. The assertion that τ is a homomorphism is trivial.

Let U be an H-invariant subgroup of V. As V is abelian, we have that U is $\tau(H)$-invariant. By Lemma 10, it follows from $\tau(H) \operatorname{prn} \tau(H)V$ that $U = N_U(\tau(H))[\tau(H), U]$. Since $U \cap \tau(H)$ is trivial, $N_U(\tau(H)) = C_U(\tau(H))$.

Again using the fact that V is abelian, we see that $C_U(H) = C_U(\tau(H))$ and $[\tau(H), U] = [H, U]$. This yields

$$U = C_U(H)[H, U] \le N_U(H)[H, U].$$

Thus $U = N_U(H)[H, U]$. By Lemma 10, $H \operatorname{prn} HV$. $\square$

Lemma 12. *Let C be an abelian group of odd order, while $L = C \wr S_{2^t}$ and $G = L \wr S_n$ be the natural permutation wreath products. Suppose also that A is a normal subgroup of G equal to the base of the corresponding wreath product. Let $S \in \operatorname{Syl}_2(L)$ and $T \in \operatorname{Syl}_2(A)$. Then*

(1) $N_L(S) \cong C \times S$;

(2) $T \cong \underbrace{S \times \dots \times S}_{n \text{ times}}$;

(3) $N_A(T) \cong \underbrace{N_L(S) \times \dots \times N_L(S)}_{n \text{ times}}$;

(4) $N_G(T) \cong N_L(S) \wr S_n$;

(5) $N_G(T)/T \cong C \wr S_n$.

Proof. Assertions (1)–(3) are obvious. Assertion (4) easily follows from (3) and the Frattini argument. Assertion (5) follows from (1) and (4). $\square$

Lemma 13. *Let $G = \mathrm{PSL}_2(q)$, where $q \equiv \pm 3 \pmod 8$, let H be a subgroup of G containing $S \in \operatorname{Syl}_2(G)$, and let $g \in N_G(S)$ be an element of odd order. Suppose that $\langle H, H^g \rangle < G$ and the following holds: If $\langle H, H^g \rangle \le M < G$, where M is a maximal subgroup of G, then M is isomorphic to a dihedral group. Then $g = 1$.*

Proof. Suppose that $g \ne 1$. Using the fact that $N_G(S) \cong A_4$, we have $N_G(S) = \langle S, g \rangle$. Fix a maximal subgroup M of G including $\langle H, H^g \rangle$. Since M has a normal cyclic 2-complement V, H has a normal cyclic 2-complement U too and $U \le V$. Also $M = SV$ and $H = SU$. It follows that U^g is a normal 2-complement in H^g and since $H^g \le M$, we have that $U^g \le V$. So U and U^g coincide being subgroups of the same order in the cyclic group V. As $g \in N_G(S)$, we see that $H^g = S^g U^g = SU = H$; therefore, $g \in N_G(H)$ and H is normalized by $N_G(S) = \langle S, g \rangle$. Now $\langle H, H^g \rangle = H$ is a subgroup of the solvable group $HN_G(S)$ which does not lie in any maximal subgroup of G isomorphic to a dihedral group because $HN_G(S)$ has the subgroup $N_G(S) \cong A_4$; a contradiction. $\square$

3. Proof of Proposition 1

In this section we prove Proposition 1, and thereby show that property $(*)$ is not inherited by direct products even if the factors are simple groups.

Lemma 14. *A direct product of a Frobenius group of order* 21 *and a cyclic group of order* 7 *has a nonpronormal subgroup.*

Proof. Let X be a direct product of a Frobenius group F of order 21 and a cyclic group of order 7. We can identify X with a subgroup of the form $F \times C$ in the Cartesian square $F \times F$ of F, where $C = \langle c \rangle$ is the kernel of F of order 7. We claim that $Y = \langle (c, c) \rangle$ is not pronormal in X. Indeed, choose a nonidentity element t in a Frobenius complement of F, and put $g = (t, 1) \in X$. Then $c^t \neq c$, and so Y and $Y^g = \langle (c^t, c) \rangle$ are distinct and not conjugate in the abelian subgroup $\langle Y, Y^g \rangle \leq C \times C$. $\square$

Proof of Proposition 1. Let $L \in \mathscr{L}_0$ and let K be a finite group such that the index of a Sylow 2-subgroup S_K of K in $N_K(S_K)$ is a multiple of 7. By Lemma 7, the normalizer in L of its Sylow 2-subgroup S_L has a subgroup that is a Frobenius group of order 21. Thus, the normalizer in $L \times K$ of its Sylow 2-subgroup $T = S_L \times S_K$ contains a direct product X of a Frobenius group of order 21 and a cyclic group of order 7, and X in turn contains a nonpronormal subgroup Y by Lemma 14. Now we have that YT is a nonpronormal subgroup of odd index in $L \times K$: it is not pronormal even in XT since its homomorphic image YT/T is not pronormal in XT/T. $\square$

4. Proof of Theorem 1

First, we prove the implication (1) $\Rightarrow$ (2) of Theorem 1. Let $G \in \mathscr{Y}_p$ and $H \leq_p N_G(T)$. By Lemma 5, it suffices to show that $N_G(T) \in \mathscr{Y}_p$. Applying the Frattini argument, we see that $N_G(T) \leq_p G$ and so $H \leq_p G$. Hence $H \operatorname{prn} G$ and, by Lemma 3, $H \operatorname{prn} N_G(T)$.

The implication (2) $\Rightarrow$ (1) of Theorem 1 is a consequence of the following

Lemma 15. *Suppose that $A \trianglelefteq G$, while T is a Sylow p-subgroup of A, $N_G(T)/T \in \mathscr{Y}_p$, and $H \leq_p G$. Then*

(1) *H is pronormal in $N_{HA}(H \cap A)$;*
(2) *if $A \in \mathscr{Y}_p$, then H is pronormal in HA;*
(3) *if $A \in \mathscr{Y}_p$ and $G/A \in \mathscr{X}_p$, then $G \in \mathscr{Y}_p$.*

Proof. Let $H \leq_p G$. Then H has a Sylow p-subgroup S of G. Without loss of generality, we may assume that $S \cap A = T$, and so $T \in \mathrm{Syl}_p(H \cap A) \subseteq \mathrm{Syl}_p(A)$.

We put $X = N_{HA}(H \cap A)$ and $Y = N_A(H \cap A)$, and let $\bar{\ } : X \to X/(H \cap A)$ be the natural homomorphism. It is clear that $H \leq N_{HA}(H \cap A) = X$; hence, $X = N_{HA}(H \cap A) = HN_A(H \cap A) = HY$.

Now we prove assertions (1)–(3) of the lemma.

(1) Suppose that $N_G(T)/T \in \mathscr{Y}_p$. We claim that $H \operatorname{prn} X$.

Applying the Frattini argument to the subgroups H and Y having the normal subgroup $H \cap A$, since $T = S \cap A \in \mathrm{Syl}_p(H \cap A)$, we conclude that

$$H = N_H(T)(H \cap A), \quad Y = N_Y(T)(H \cap A),$$
$$X = HY = N_H(T)N_Y(T)(H \cap A).$$

In particular,

$$\overline{H} = \overline{N_H(T)}, \quad \overline{X} = \overline{N_H(T)N_Y(T)} = \overline{\langle N_H(T), N_Y(T) \rangle}. \tag{1}$$

Observe that $S \leq N_H(T)$ and so the index of $N_H(T)$ in its every overgroup is coprime to p. In particular, $N_H(T) \leq_p N_G(T)$ and $N_H(T)/T \leq_p N_G(T)/T$. Since $N_G(T)/T \in \mathscr{Y}_p$, it follows that

$N_H(T)/T \operatorname{prn} N_G(T)/T$. By Lemma 5, $N_H(T) \operatorname{prn} N_G(T)$, and together with (1), Lemma 3, and the fact that

$$N_H(T) \le \langle N_H(T), N_Y(T)\rangle \le N_G(T),$$

this yields $N_H(T) \operatorname{prn} \langle N_H(T), N_Y(T)\rangle$. Applying Lemma 5 twice, we have

$$\overline{H} = \overline{N_H(T)} \operatorname{prn} \overline{\langle N_H(T), N_Y(T)\rangle} = \overline{X}, \quad H \operatorname{prn} X.$$

(2) Suppose that $N_G(T)/T \in \mathscr{Y}_p$ and $A \in \mathscr{Y}_p$. We show that $H \operatorname{prn} HA$.

Take $g \in HA$. Then $g = ha$, where $h \in H$ and $a \in A$. The index $|A : (H \cap A)|$ is coprime to p since $H \cap A$ contains $T \in \mathrm{Syl}_p(A)$, and so $(H \cap A) \le_p A$. Since $A \in \mathscr{Y}_p$, it follows that $(H \cap A) \operatorname{prn} A$. Thus, $H^y \cap A = H^a \cap A$ for some $y \in \langle H \cap A, H^a \cap A\rangle = \langle H \cap A, H^g \cap A\rangle \le \langle H, H^g\rangle$. By Lemma 8, without loss of generality, we can replace H by H^y and assume that $H \cap A = H^a \cap A = H^g \cap A$, which implies that $g \in N_{HA}(H \cap A) = X$. But $H \operatorname{prn} X$ by assertion (1). Hence H and H^g are conjugate in $\langle H, H^g\rangle$ if $g \in HA$ and so $H \operatorname{prn} HA$, proving assertion (2).

(3) Suppose that $N_G(T)/T \in \mathscr{Y}_p$, $A \in \mathscr{Y}_p$, and $G/A \in \mathscr{X}_p$. We prove that $H \operatorname{prn} G$ and so $G \in \mathscr{Y}_p$.

Take $g \in N_G(S)$. Since $SA/A \in \mathrm{Syl}_p(G/A)$ and $G/A \in \mathscr{X}_p$, it follows that $N_{G/A}(SA/A) = SA/A$ and

$$g \in N_G(S) \le N_G(SA) = SA \le HA.$$

But $H \operatorname{prn} HA$ by assertion (2). Thus, H and H^g are conjugate in $\langle H, H^g\rangle$ for every $g \in N_G(S)$, which shows that $H \operatorname{prn} G$ by Lemma 1. $\square$

We also note the following assertion that generalizes Theorem 1 and provides a criterion for an extension of a group in $\mathscr{Y}_p$ by another group in $\mathscr{Y}_p$ to be in $\mathscr{Y}_p$.

Theorem 4. *Let a group G have a normal subgroup A and let $\overline{} : G \to G/A$ be the natural epimorphism. Let T be a Sylow p-subgroup of A. Suppose that $A \in \mathscr{Y}_p$ and $\overline{G} \in \mathscr{Y}_p$. Then the following are equivalent:*

(1) $G \in \mathscr{Y}_p$;

(2) $N_G(T)/T \in \mathscr{Y}_p$, and $\overline{N_G(H)} = N_{\overline{G}}(\overline{H})$ for every subgroup $H \le_p G$.

Proof. If G and A are as in the hypothesis of the theorem and H is a subgroup of G, then $\overline{N_G(H)} = N_{\overline{G}}(\overline{H})$ is equivalent to the equality $H^A = H^{N_G(HA)}$.

To prove the implication (1) $\Rightarrow$ (2), let $G \in \mathscr{Y}_p$ and $H \le_p G$. The fact that $N_G(T)/T \in \mathscr{Y}_p$ can be proved in the same manner as in Theorem 1. Notice that $H \operatorname{prn} HA$ and $H \operatorname{prn} N_G(HA)$ by Lemma 5. Now Lemma 4 implies that $H^A = H^{N_G(HA)}$.

To prove the implication (2) $\Rightarrow$ (1), suppose that $H \le_p G$. By Lemma 15, $H \operatorname{prn} HA$. The equality $H^A = H^{N_G(HA)}$ and Lemma 4 yield $H \operatorname{prn} N_G(HA)$. By Lemma 5 and the fact that $G/A \in \mathscr{Y}_p$, it follows that $H \operatorname{prn} G$, and so $G \in \mathscr{Y}_p$ since H is an arbitrary subgroup. $\square$

5. Proof of Theorem 2

We begin with the following result:

Lemma 16. *Let A be an abelian group of odd order and let*

$$G = \prod_{i=1}^{t} (A \wr S_{n_i}) = \prod_{i=1}^{t} V_i H_i,$$

where all wreath products are natural permutations, while V_i denotes the base group of the corresponding wreath product and $H_i \cong S_{n_i}$. Suppose that $(|A|, n_i) = 1$ for every i. Then $H = \prod_{i=1}^{t} H_i$ is pronormal in G.

Proof. Observe that $G = V \rtimes H$, where $V = \langle V_i \mid i = 1, \dots, t \rangle$. We write the group V additively, and so $V = \bigoplus_{i=1}^{t} V_i$.

To prove the pronormality of H in G, we use Lemma 10. Let T_i be a subgroup generated by a cycle of length n_i in H_i and let $T = \prod_{i=1}^{t} T_i$. Note that $(|T|, |V|) = 1$ because of the condition imposed on n_i.

It is easy to see that the centralizer $C_{V_i}(T_i)$ is the whole diagonal subgroup of the base group V_i in $A \wr S_{n_i} = V_i H_i$ and $C_{V_i}(T_i)$ coincides with $C_{V_i}(H_i)$. This yields

$$C_V(T) = \bigoplus_{i=1}^{t} C_{V_i}(T_i) = \bigoplus_{i=1}^{t} C_{V_i}(H_i) = C_V(H).$$

Let U be an arbitrary H-invariant subgroup of V. Then

$$C_U(H) = U \cap C_V(H) = U \cap C_V(T) = C_U(T).$$

Since the orders of T and V are coprime, from [20, Chapter 4, Lemma 4.28] it follows that

$$U = C_U(T) + [T, U] = C_U(H) + [T, U] \le C_U(H) + [H, U] = N_U(H) + [H, U],$$

and so by Lemma 10 $H \operatorname{prn} HV = G$. $\square$

Proof of Theorem 2. Let A be an abelian group and let

$$G = \prod_{i=1}^{t} (A \wr S_{n_i}) = \prod_{i=1}^{t} V_i H_i, \tag{2}$$

where all wreath products are natural permutations, $V_i H_i = A \wr S_{n_i}$, V_i denotes the base group of the corresponding wreath product (written additively), and $H_i = S_{n_i}$. Put

$$V = \langle V_i \mid i = 1, \dots, t \rangle = \bigoplus_{i=1}^{t} V_i, \quad B = \langle B_i \mid i = 1, \dots, t \rangle = \prod_{i=1}^{t} H_i.$$

It is clear that $G = VB$.

By Lemma 5, we can assume that the order of A is odd. We need to check that G satisfies property $(*)$ if and only if the following holds:

$$\begin{gathered} \text{if for a natural } m, \text{ we have } m \preceq n_i \\ \text{for one of the naturals } n_i, \text{ then } (m, |A|) = 1. \end{gathered} \tag{$**$}$$

Suppose first that there are $m \in \mathbb{N}$ and $i \in \{1, \dots, t\}$ such that $m \preceq n_i$ and $(|A|, m) \ne 1$. We claim that G has a nonpronormal subgroup of odd index. Without loss of generality, we may assume that $i = t$. In this case the subgroup $T \le S_{n_t}$, which is isomorphic to $S_m \times S_{n_t - m}$, has odd index in S_{n_t} by the main result of [15]. Define the subgroup M of G as

$$M = \left(\prod_{i=1}^{t-1} (A \wr S_{n_i}) \right) \times (A \wr T) = \left(\prod_{i=1}^{t-1} (A \wr S_{n_i}) \times (A \wr S_{n_t - m}) \right) \times (A \wr S_m) = M_1 \times M_2,$$

where

$$M_1 \cong \prod_{i=1}^{t-1} (A \wr S_{n_i}) \times (A \wr S_{n_t - m}), \quad M_2 \cong A \wr S_m.$$

The factor group $G/M_1 \cong M_2$ has a nonpronormal subgroup $H \cong S_m$ of odd index by [11, Corollary 1]. Applying Lemma 5, we see that the full preimage of H in G is a nonpronormal subgroup of odd index in G.

Suppose now that G satisfies $(**)$ but $(*)$ does not hold.

Let G be a group of the form (2) that satisfies $(**)$, includes a nonpronormal subgroup H of odd index, and has the minimal possible order among the groups satisfying $(**)$ but not satisfying $(*)$.

It follows by Lemma 6 that H is not pronormal in HV. By Lemma 11, we can assume that $H \leq B$. If $H = B$, then $H \operatorname{prn} HV$ by Lemma 16, which is a contradiction. Thus $H < B$.

Let π_i be the projection from H to H_i. By Lemma 9, there is j such that the projection $\pi_j(H)$ of H to H_j is a proper subgroup in H_j. Without loss of generality, we may assume that $j = t$. It follows that there is a maximal subgroup M of H_t of odd index such that $\pi_t(H) \leq M < H_t$. Then

$$H \leq \left(\prod_{i=1}^{t-1} V_i H_i\right) \times (V_{n_t} M) = G_1.$$

Assume that H_t acts on the set $\Omega = \{1, \dots, n_t\}$ naturally and regard the base group V_t as a permutation module of H_t.

Suppose that M is intransitive on Ω. Then $M \cong S_m \times S_{n_t - m}$ and

$$G_1 = \left(\prod_{i=1}^{t-1} (A \wr S_{n_i})\right) \times (A \wr S_m) \times (A \wr S_{n_t - m}).$$

Using the fact that the index of M in H_t is odd and the main result of [15], we conclude that $n_t \succeq m$ and $n_t \succeq n_t - m$. Hence G_1 is a subgroup of the form (2) satisfying $(**)$. By the choice of G, it follows that H is pronormal in G_1. Then $H \operatorname{prn} HV$ since $V \leq G_1$; a contradiction.

Thus M is transitive on Ω. Suppose that M is imprimitive. Then M is the stabilizer of a partition of Ω into u subsets $\Gamma_1, \dots, \Gamma_u$ of size $r > 1$, with $n = ru$, and $M \cong S_r \wr S_u$. The main result of [15] implies that r is a power of 2.

We write $\rho : M \to S_u$ for the action of M on the set of blocks $\Sigma = \{\Gamma_1, \dots, \Gamma_u\}$.

Let $L \trianglelefteq M$ be the kernel of the homomorphism ρ. Then L is the direct product of u isomorphic copies of S_r corresponding to the blocks $\Gamma_1, \dots, \Gamma_u$ of M, while $W = LV_t$ is isomorphic to the direct product of u isomorphic copies of $A \wr S_r$.

Observe that

$$G_2 = \left(\prod_{i=1}^{t-1} A \wr S_{n_i}\right) \times W \trianglelefteq G_1$$

is a group of the form (2) satisfying $(**)$. Hence subgroups of odd index are pronormal in G_2. Also

$$G_1/G_2 = MV_t/W = MV_t/LV_t \cong M/L \cong S_u,$$

and so Sylow 2-subgroups of G_1/G_2 are self-normalizing by [21, Lemma 4]. Let $S \in \mathrm{Syl}_2(G_1)$ be such that $S \leq H$ and $T = S \cap G_2 \in \mathrm{Syl}_2(G_2)$. By Lemma 12, $N_{G_1}(T)/T \cong A^{t-1} \times A \wr S_u$, and by the minimality of G, the subgroups of odd index are pronormal in $N_{G_1}(T)/T$. Applying Theorem 1, we see that the subgroups of odd index are pronormal in G_1, and so $H \operatorname{prn} HV$ since $V \leq G_1$; a contradiction.

Thus M is primitive. Since M contains some Sylow 2-subgroup of H_t, it follows that M contains a transposition. Then by [22, Theorem 13.3] $M = H_t$, which is a contradiction because M is a proper subgroup of H_t. $\square$

6. Proof of Theorem 3

We begin with the following

Lemma 17. *If $G = \prod_{i=1}^{t} G_i$, where $G_i \cong \mathrm{PSL}_2(q_i) = \mathrm{PSp}_2(q_i)$ is a simple group and $q_i > 3$ is odd for all $i = 1, \dots, t$, then the subgroups of G of odd index are pronormal.*

Proof. Suppose that the lemma is false, and let G be a counterexample of minimal order, i.e., G is a group of the form as in the hypothesis, G has a nonpronormal subgroup H whose index in G is odd, and G has minimal possible order among such groups. Let $S \in \mathrm{Syl}_2(H) \subseteq \mathrm{Syl}_2(G)$. By Lemma 1, $N_G(S) > S$ and $N_G(S)$ contains an element g of odd order such that H and H^g are not conjugate in $K = \langle H, H^g \rangle$.

Observe that $q_i \equiv \pm 3$ for all $i = 1, \dots, t$. Otherwise, let $I \subseteq \{1, \dots, t\}$ be the set of those indices i for which $q_i \not\equiv \pm 3 \pmod 8$ and let J be the set of the other indices. Then

$$G = AB, \quad \text{ where } A = \langle G_i \mid i \in J \rangle \text{ and } B = \langle G_i \mid i \in I \rangle.$$

Since $B \in \mathscr{X}_2$ and $A \in \mathscr{Y}_2$ by the choice of G, by Theorem 1 it suffices to show that $N_G(T)/T \in \mathscr{Y}_2$, where $T \in \mathrm{Syl}_2(A)$. The group $N_G(T)/T$ is isomorphic to the direct product of an elementary abelian group V of order $3^{|J|}$ and B. By Lemma 6, to prove the pronormality of a group X of odd index in $V \times B$, it suffices to prove that X is pronormal in VX, which in turn follows from Lemma 11.

As usual, π_i denotes the coordinate projection $G \to G_i$.

Suppose that there is an index i such that $\pi_i(H) = G_i$. Then applying Lemma 9, we see that $G_i \leq H$. By the choice of G, the subgroups of odd index are pronormal in $G/G_i \cong \prod_{j \neq i} G_j$, and so $H \operatorname{prn} G$ by Lemma 5; a contradiction.

Thus, $\pi_i(H) < G_i$ for every i. Then for every i there is a maximal subgroup of G_i that includes $\pi_i(H)$. In general, such a subgroup is not uniquely determined. The following possibilities are given:

(I) There is an index j such that $\pi_j(H)$ lie in some nonsolvable maximal subgroup M_j of G_j. By the main result of [14], there are two further subcases here.

(I1) $M_j = C_{G_j}(\sigma)$ for a field automorphism σ of prime odd order r of $\mathrm{PSL}_2(q_j)$. By [18, Proposition 4.5.4], $M_j \cong \mathrm{PSL}_2(\tilde{q}_j)$, where $q_j = \tilde{q}_j^r$.

(I2) $M_j \cong A_5 \cong \mathrm{PSL}_2(5)$.

Thus in Case I

$$H \leq G^* = \Big(\prod_{i \neq j} G_i \Big) \times M_j,$$

and comparing the indices $|N_G(S) : S|$ and $|N_{G^*}(S) : S|$, it is easy to see that G^* includes $N_G(S)$. The choice of G yields $H \operatorname{prn} G^*$, and since G^* contains the normalizer of a Sylow 2-subgroup of G, it follows from Lemma 3 that H is pronormal in G; a contradiction.

(II) For every i, all maximal subgroups of G_i including $\pi_i(H)$ are solvable; and so by the main result of [14], each of them is isomorphic to either A_4, or the dihedral group $D_{q_i - \varepsilon_i}$, where $\varepsilon_i = \pm 1$ is chosen so that 4 divides $q_i - \varepsilon_i$.

Given i, we choose the maximal subgroup M_i containing $\pi_i(H)$ as follows.

If $\pi_i(H)$ lies in some maximal subgroup isomorphic to A_4 then we take it as M_i (for instance, this is always the case if the order of $\pi_i(H)$ is a power of 2). In this case, it is easy to see that every overgroup of a Sylow 2-subgroup is normal in M_i. Therefore, $N_{G_i}(H \cap G_i) = N_{G_i}(\pi_i(H)) = M_i$. Since the index of a Sylow 2-subgroup

$$S_i \in \mathrm{Syl}_2(H \cap G_i) \subseteq \mathrm{Syl}_2(M_i) \subseteq \mathrm{Syl}_2(G_i)$$

in M_i is equal to 3, it follows that $H \cap G_i, \pi_i(H) \in \{S_i, M_i\}$.

Otherwise, take M_i to be dihedral. Note that in this case $\pi_i(H)$ is dihedral too and $O(\pi_i(H)) \neq 1$. Furthermore, if M_i is dihedral, then by the choice of M_i, the group $N_{G_i}(\pi_i(H))$ is contained in some dihedral maximal subgroup too and we can assume that this subgroup is M_i. Moreover, we claim that

$$H \cap G_i = \pi_i(H) = N_{G_i}(\pi_i(H)) = N_{G_i}(H \cap G_i)$$

in this case, and so our assumption that $N_{G_i}(\pi_i(H)) \leq M_i$ in fact does not impose any additional restrictions on M_i. Indeed, as in the proof of Lemma 9, $H \cap G_i \trianglelefteq \pi_i(H)$. But a Sylow 2-subgroup of the dihedral group $\pi_i(H)$ is self-normalizing, and so is its every overgroup; in particular,

$$\pi_i(H) = N_{\pi_i(H)}(H \cap G_i) = H \cap G_i.$$

Similarly, since every maximal subgroup of G_i, including $\pi_i(H)$, is dihedral; it follows that

$$N_{M_i}(\pi_i(H)) = \pi_i(H) = N_{G_i}(\pi_i(H)).$$

Thus in Case II $N_{G_i}(H \cap G_i) = N_{G_i}(\pi_i(H))$ for all i.

We derive a contradiction by showing that in Case II the subgroups H and H^g are conjugate in $K = \langle H, H^g \rangle$.

By Lemma 9, it follows that there is j such that $\pi_j(\langle H, H^g \rangle) < G_j$. Without loss of generality, we can assume that $\pi_j(\langle H, H^g \rangle) \leq M_j$, where the maximal subgroup M_j of G_j is chosen as above. (The only hypothetical case when the type of the chosen subgroup M_j differs from the type of the maximal subgroup containing $\pi_j(\langle H, H^g \rangle)$ is the situation where $\pi_j(H)$ is equal to a Sylow 2-subgroup of G_j. It is clear that in this situation $\pi_j(g)$ normalizes $\pi_j(H)$, and so $\pi_j(\langle H, H^g \rangle)$ is a Sylow 2-subgroup in G_j too.)

Let $G^* = \langle M_j, G_i \mid i \neq j \rangle = G_0 \times M_j$, where $G_0 = \langle G_i \mid i \neq j \rangle = G_1 \times \cdots \times G_{j-1} \times G_{j+1} \times \cdots \times G_n$. The group G^* contains H. By the choice of M_j, it follows that whenever M_j is dihedral, every maximal subgroup of G_j including $\pi_j(\langle H, H^g \rangle) = \langle \pi_j(H), \pi_j(H)^{\pi_j(g)} \rangle$ is also dihedral and, by Lemma 13, $\pi_j(g) = 1$. Hence,

$$\pi_j(g) \in N_{G_j}(H \cap G_j) = N_{G_j}(\pi_j(H)) \quad \text{and} \quad \pi_j(g) \in M_j \leq G^*$$

regardless of the structure of M_j. Since $\pi_i(g) \leq G_i \leq G^*$ for $i \neq j$, it follows that

$$g \in \langle \pi_i(g) \mid i = 1, \dots, n \rangle \leq G^*.$$

To derive a contradiction, it suffices to show that $H \operatorname{prn} G^*$. To this end, we use Lemma 2.

Denote the coordinate projections from $G^* = G_0 \times M_j$ to G_0 and M_j by γ and μ. It is clear that

$$\pi_i(h) = \pi_i(\gamma(h)) \text{ for } i \neq j, \quad \pi_j(h) = \pi_j(\mu(h)) = \mu(h)$$

for every $h \in H$. The minimality of G yields $\gamma(H) \operatorname{prn} G_0$ and using the structure of M_j, we see that $\mu(H) \operatorname{prn} M_j$.

From the above, $H \cap M_j = H \cap G_j$ is a normal subgroup of $N_{M_j}(\mu(H)) = N_{G_j}(\pi_j(H)) = N_{G_j}(H \cap G_j)$ and the factor group $N_{M_j}(\mu(H))/(H \cap M_j)$ is abelian. Hence

$$N_{M_j}(\mu(H)) \leq \{x \in M_j \mid [x, \mu(H)] \leq H \cap M_j\}.$$

By Lemma 2, it suffices to prove that $N \leq C$, where

$$N = N_{G_0}(\gamma(H)), \quad C = \{x \in G_0 \mid [x, \gamma(H)] \leq H \cap G_0\}.$$

Given $i \neq j$, we have $\pi_i(H) = \pi_i(\gamma(H)) \trianglelefteq \pi_i(N)$; therefore,

$$N \leq N_0 = \langle N_{G_i}(\pi_i(H)) \mid i \neq j \rangle.$$

Since $H \cap G_i \trianglelefteq N_{G_i}(\pi_i(H))$, each of the groups $N_{G_i}(\pi_i(H))/(H \cap G_i)$ is abelian and $H \cap G_i \leq H \cap G_0$, we conclude that $H \cap G_0$ is a normal subgroup of N_0 and $N_0/(H \cap G_0)$ is abelian. Thus,

$$[y, \gamma(H)](H \cap G_0)/(H \cap G_0) \leq [N_0(H \cap G_0)/(H \cap G_0), N_0(H \cap G_0)/(H \cap G_0)] = 1$$

for all $y \in N$. Hence $[y, \gamma(H)] \leq H \cap G_0$ and $y \in \{x \in G_0 \mid [x, \gamma(H)] \leq H \cap G_0\} = C$.

This shows that $N \leq C$ and thereby proves Lemma 17. $\square$

Proof of Theorem 3. We proceed by induction on the order of G. Observe that the simple groups $\mathrm{PSL}_2(q) = \mathrm{PSp}_2(q)$, and $A_5 \cong \mathrm{PSL}_2(5) \cong \mathrm{PSp}_2(5)$ in particular, also can be factors in G.

Let H be a subgroup of odd index in G and let S be a Sylow 2-subgroup of G such that $S \leq H$. We claim that $H \operatorname{prn} G$. Suppose that this is false, and let G be a counterexample of minimum possible order.

Let π_i be the projection from G to G_i. Suppose that there is an index i such that $\pi_i(H) = G_i$. Then $G_i \leq H$ by Lemma 9. By the inductive hypothesis, the subgroups of odd index are pronormal in $\prod_{j \neq i} G_j$, and so $H \operatorname{prn} G$ by Lemma 5; a contradiction.

Thus $\pi_i(H) < G_i$ for all i. Then there is a maximal subgroup M_i of G_i such that $\pi_i(H) \leq M_i < G_i$ for all i.

It is easy to see that the subgroup H lies in

$$G_0^i = \prod_{j \neq i} G_j \times M_i$$

for all i. Assume that there is an index i such that $n_i \geq 4$. By the main result of [14], the following possibilities for M_i are open.

Case (1): $M_i = C_A(\sigma)$ with σ a field automorphism of $\mathrm{PSp}_{n_i}(q_i)$ of prime odd order r. Then [18, Proposition 4.5.4] yields $M_i \cong \mathrm{PSp}_{n_i}(\tilde{q}_i)$, where $q_t = \tilde{q}_t^r$. Since r is odd, it is easy to see that $q_i \equiv \pm 3 \pmod 8$ if and only if $\tilde{q}_i \equiv \pm 3 \pmod 8$. By the inductive hypothesis, H is pronormal in G_0^i. By Lemma 7, $|N_G(S) : S| = |N_{G_0^i}(S) : S|$ and so $N_G(S) \leq G_0^i$. Thus $H \operatorname{prn} G$ by Lemma 3; a contradiction.

Case (2): M_i is the stabilizer of an orthogonal decomposition $V^i = \bigoplus V_j^i$ of the natural projective module of G_i into isometric subspaces V_j^i of dimension s_i, where $s_i = 2^{w_i} \geq 2$.

By [18, Proposition 4.2.10],

$$M_i \cong 2^{u_i - 1}.(\mathrm{PSp}_{s_i}(q_i) \wr S_{u_i}), \quad \text{where } n_i = s_i u_i.$$

Also, if $q_i \equiv \pm 3 \pmod 8$, then M_i has an element of order 3 which normalizes $S \cap M_i$; i.e., $|N_{M_i}(S \cap M_i) : S \cap M_i| = 3$ by Lemma 7. Hence, $N_G(S) = N_{G_0^i}(S)$ and Lemma 3 implies that $H \operatorname{prn} G$ if and only if $H \operatorname{prn} G_0^i$.

The subgroup $O_2(M_i)$ is normal in G_0^i and, as a 2-group, it lies in H. Consider the natural homomorphism ${}^{-}\colon G_0^i \to G_0^i/O_2(M_i)$. By Lemma 5, $H \operatorname{prn} G_0^i$ if and only if $\overline{H} \operatorname{prn} \overline{G}_0^i$. Clearly,

$$\overline{G}_0^i \cong \Big(\prod_{j \neq i} G_j\Big) \times (\mathrm{PSp}_{s_i}(q_i) \wr S_{u_i}).$$

Now, we consider a normal subgroup A of $\overline{G}_0^i$ isomorphic to $\big(\prod_{j \neq i} G_j\big) \times (\mathrm{PSp}_{s_i}(q_i))^{u_i}$. Observe that $\overline{G}_0^i/A \cong S_{u_i}$, and so the Sylow 2-subgroups of $\overline{G}_0^i/A$ are self-normalizing by [21, Lemma 4]. By the inductive hypothesis, the subgroups of odd index are pronormal in A. Furthermore, if $S_0 \in \mathrm{Syl}_2(A)$, then

$$N_{\overline{G}_0^i}(S_0)/S_0 \cong \Big(\prod_{j \neq i} C_j\Big) \times C_i \wr S_{u_i},$$

where C_k is trivial if $q_k \equiv \pm 1 \pmod 8$ and isomorphic to C_3 if $q_k \equiv \pm 3 \pmod 8$. Applying Theorem 2, we conclude that subgroups of odd index are pronormal in $N_{\overline{G}_0^i}(S_0)/S_0$. By Theorem 1, the subgroups of odd index are pronormal in G_0^i, which is a contradiction.

Case (3): $q_i \equiv \pm 3 \pmod 8$, $n_i = 4$ and $M_i \cong 2^4.A_5$. In this situation $O_2(M_i)$ is normal in G_0^i and, being a 2-group, it lies in H. The inductive hypothesis yields $H/O_2(M_i) \operatorname{prn} G_0^i/O_2(M_i)$. By Lemma 5, $H \operatorname{prn} G_0^i$. Lemma 7 implies that $|N_G(S) : S| = |N_{G_0^i}(S) : S|$. Hence $N_G(S) \leq G_0^i$. It follows that $H \operatorname{prn} G$ by Lemma 3; a contradiction.

Thus all n_i are equal to 2. But then $H \operatorname{prn} G$ by Lemma 17, which contradicts the choice of H. $\square$

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W. Guo
University of Science and Technology of China, Hefei, P. R. China
E-mail address: `wbguo@ustc.edu.cn`

N. V. Maslova
Krasovskii Institute of Mathematics and Mechanics
Ural Federal University, Ekaterinburg, Russia
E-mail address: `butterson@mail.ru`

D. O. Revin
Sobolev Institute of Mathematics
Novosibirsk State University, Novosibirsk, Russia
University of Science and Technology of China, Hefei, P. R. China
E-mail address: `revin@math.nsc.ru`