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Existence and concentration of solution for a class of fractional Hamiltonian systems with subquadratic potential

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Abstract. In this article, we consider the following fractional Hamiltonian systems:

$$ {}_tD_{\infty}^{\alpha}({}_{-\infty}D_t^{\alpha}u) + \lambda L(t)u = \nabla W(t,u), \quad t \in \mathbb{R}, $$

where $\alpha \in (1/2, 1)$, $\lambda > 0$ is a parameter, $L \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ and $W \in C^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$. Unlike most other papers on this problem, we require that $L(t)$ is a positive semi-definite symmetric matrix for all $t \in \mathbb{R}$, that is, $L(t) \equiv 0$ is allowed to occur in some finite interval $\mathbb{I}$ of $\mathbb{R}$. Under some mild assumptions on W, we establish the existence of nontrivial weak solution, which vanish on $\mathbb{R}\setminus\mathbb{I}$ as $\lambda \to \infty$, and converge to $\tilde{u}$ in $H^{\alpha}(\mathbb{R})$; here $\tilde{u} \in E_0^{\alpha}$ is nontrivial weak solution of the Dirichlet BVP for fractional Hamiltonian systems on the finite interval $\mathbb{I}$. Furthermore, we give the multiplicity results for the above fractional Hamiltonian systems.

Keywords. Liouville–Weyl fractional derivative; fractional Sobolev space; critical point theory; variational method; positive semi-definite.

Mathematics Subject Classification. 26A33, 34C37, 35A15, 35B38.

1. Introduction

In this paper, we investigate the solvability of the following non homogeneous fractional Hamiltonian system:

$$ {}_tD_{\infty}^{\alpha}({}_{-\infty}D_t^{\alpha}u) + \lambda L(t)u = \nabla W(t,u), \quad t \in \mathbb{R}, \tag{1.1} $$

where $\alpha \in (1/2, 1)$, $W \in C(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$, the parameter $\lambda > 0$, ${}_{-\infty}D_t^{\beta}$ and ${}_tD_{\infty}^{\beta}$ denote the left- and right-Liouville–Weyl fractional derivative of order α respectively and are defined by

$$ {}_{-\infty}D_t^{\beta}u = \frac{\mathrm{d}}{\mathrm{d}t}{}_{-\infty}I_t^{\alpha}u, \quad {}_tD_{\infty}^{1-\alpha}u = -\frac{\mathrm{d}}{\mathrm{d}t}{}_tI_{\infty}^{1-\alpha}. $$

and the matrix L satisfies the following conditions:

(L_1) $L(t) \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ is a symmetric matrix for all $t \in \mathbb{R}$; there exists a nonnegative continuous function $l : \mathbb{R} \to \mathbb{R}$ and a constant $k > 0$ such that

$$ (L(t)u(t), u(t)) \geq l(t)|u(t)|^2, $$

and the set $\{l < k\} = \{t \in \mathbb{R} : \ l(t) < k\}$ is nonempty with $C_\alpha^2|\{l < k\}| < 1$, where $|.|$ is the Lebesgue measure and C_α is the Sobolev constant (see §2).
(L_2) $\mathbb{J} = \mathrm{int}(l^{-1}(0))$ is a nonempty finite interval and $\overline{\mathbb{J}} = l^{-1}(0)$.
(L_3) There exists an open interval $\mathbb{I} \subset \mathbb{J}$ such that $L(t) \equiv 0$ for all $t \in \overline{\mathbb{I}}$.

Fractional differential equations appear naturally in a number of fields such as physics, chemistry, biology, economics, control theory, signal and image processing, blood flow phenomena, etc. During the last few decades, the theory of fractional differential equations is an area intensively developed, mainly due to the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes (see [6,7,9,13,21] and the references therein).

Physical models containing left and right fractional differential operators have recently renewed attention from scientists which is mainly due to applications as models for physical phenomena exhibiting anomalous diffusion (see [2–5,10,11,17–20]). A strong motivation for investigating the fractional differential equation (1.1) comes from symmetry fractional advection–dispersion equation. A fractional advection–dispersion equation is a generalization of the classical ADE in which the second-order derivative is replaced with a fractional-order derivative. In contrast to the classical ADE, the fractional ADE has solutions that resemble the highly skewed and heavy-tailed breakthrough curves observed in field and laboratory studies [2,3], in particular in contaminant transport of ground-water flow [4]. In [4], the authors have stated that solutes moving through highly heterogeneous aquifer violations violates the basic assumptions of local second-order theories because of large deviations from the stochastic process of Brownian motion. Moreover, models involving a fractional differential oscillator equation, which contains a composition of left and right fractional derivatives, are proposed for the description of the processes of emptying the silo [10] and the heat flow through a bulkhead filled with granular material [17], respectively. Their studies show that the proposed models based on fractional calculus are efficient and describe well the processes.

Very recently, in [18] the author considered (1.1), where $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric matrix-valued function for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. Assuming that L and W satisfy the following hypotheses:

(L) $L(t)$ is a positive definite symmetric matrix for all $t \in \mathbb{R}$, and there exists an $l \in C(\mathbb{R}, (0, \infty))$ such that $l(t) \to +\infty$ as $t \to \infty$ and

$$(L(t)x, x) \geq l(t)|x|^2, \quad \text{for all } t \in \mathbb{R} \ \text{ and } \ x \in \mathbb{R}^n. \tag{1.2}$$

(W_1) $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, and there is a constant $\mu > 2$ such that

$$0 < \mu W(t, x) \leq (x, \nabla W(t, x)), \quad \text{for all } t \in \mathbb{R} \ \text{ and } x \in \mathbb{R}^n \setminus \{0\}.$$

(W_2) $|\nabla W(t, x)| = o(|x|)$ as $x \to 0$ uniformly with respect to $t \in \mathbb{R}$.
(W_3) There exists $\bar{W} \in C(\mathbb{R}^n, \mathbb{R})$ such that

$$|W(t, x)| + |\nabla W(t, x)| \leq |\overline{W(x)}| \ \text{ for every } x \in \mathbb{R}^n \text{ and } t \in \mathbb{R}.$$

The author showed that (1.1) has at least one nontrivial solution via Mountain pass theorem. By using the genus properties of critical point theory, Zhang and Yuang in [23], generalized the result of [18] and established some new criterion to guarantee the existence of infinitely many solutions of (1.1) for the case that $W(t, u)$ is sub-quadratic as $|u| \to +\infty$.

As is well-known, the condition (L) is the so-called coercive condition and is a little demanding. In fact, for a simple choice like $L(t) = sId_n$, the condition (L) is not satisfied, where $s > 0$ and Id_n is the $n \times n$ identity matrix. Considering this in [24], the recent results of [18,23] are generalized and significantly improved. More precisely, in [24], the authors considered the case that $L(t)$ is bounded in the sense that

(L)$'$ there are constants $0 < \tau_1 < \tau_2 < +\infty$ such that

$$\tau_1 |u|^2 \leq (L(t)u, u) \leq \tau_2 |u|^2 \quad \text{for all} \;\; (t, u) \in \mathbb{R} \times \mathbb{R}^n.$$

By using the genus properties of critical point theory, the authors proved that (1.1) has infinitely many nontrivial solutions. Xu *et al.* [22] by using the fountain theorem of critical point theory, have established the existence of infinitely many solutions of (1.1) for the case that $W(t, u)$ is subquadratic as $|u| \to 0$ and superqudratic as $|u| \to \infty$.

Motivated by these previous results, we consider problem (1.1) where the symmetric matrix L is positive semi-definite and we study the existence of nontrivial weak solutions when W is sub-quadratic. Furthermore, we shall explore the phenomenon of concentrations of weak solution as $\lambda \to \infty$, which seems to be rarely concerned in the previous studies of solutions for fractional Hamiltonian systems. To reduce our statements, we make the following assumptions on W:

(W$_1$) $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and there exist a constant $p \in (1, 2)$ and a function $\xi(t) \in L^{\frac{2}{2-p}}(\mathbb{R}, \mathbb{R}^+)$ such that

$$|\nabla W(t, u)| \leq \xi(t) |u|^{p-1}, \quad \text{for all} \;\; (t, u) \in \mathbb{R} \times \mathbb{R}^n. \tag{1.3}$$

(W$_2$) There exist three constants $\eta, \delta > 0$ and $\nu \in (1, 2)$ such that

$$|W(t, u)| \geq \eta |u|^{\nu}, \quad \forall \, t \in \mathbb{I} \text{ and } \; |u| \leq \delta. \tag{1.4}$$

On the existence of solutions we have the following result.

Theorem 1.1. *Assume that the conditions* (L_1), (L_2), (W_1) *and* (W_3) *hold. Then there exists* $\Lambda > 0$ *such that for every* $\lambda > \Lambda$, *problem* (1.1) *has at least one weak solution* u_λ.

For technical reasons, we consider that there exists $0 < \mathbb{T} < +\infty$ such that $\mathbb{I} = (0, \mathbb{T})$, where $\mathbb{I}$ is given by (L_3). On the concentration of solutions, we have the following result.

Theorem 1.2. *Assume that the conditions* (L_1), (L_3), (W_1) *and* (W_3) *hold. Let* u_λ *be a solution of problem* (1.1) *obtained in Theorem* 1.1, *then* $u_\lambda \to \tilde{u}$ *strongly in* $H^\alpha(\mathbb{R})$ *as* $\lambda \to \infty$, *where* $\tilde{u}$ *is a nontrivial weak solution of the equation*

$$\begin{aligned} & {}_tD^{\alpha}_{\mathbb{T}}{}_0D^{\alpha}_t u = \nabla W(t, u), \quad t \in (0, \mathbb{T}), \\ & u(0) = u(\mathbb{T}) = 0. \end{aligned} \tag{1.5}$$

We recall that, in particular, if $\alpha = 1$, (1.1) reduces to the standard second order Hamiltonian systems

$$u'' - L(t)u + \nabla W(t, u) = 0, \tag{1.6}$$

where $W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a given function and $\nabla W(t, u)$ is the gradient of W at u. The existence of homoclinic solution is one of the most important problems in the history of that kind of equations, and has been studied intensively by many mathematicians. By assuming that $L(t)$ and $W(t,u)$ are independent of t, or T-periodic in t or $L(t)$ and $W(t,u)$ are neither autonomous nor periodic in t, many authors have studied the existence of homoclinic solutions for (1.6) via critical point theory and variational methods (see [12,14]). Furthermore, recently a second-order Hamiltonian system like (1.6) with positive semi-definite matrix was considered in [16]. Assuming that $W \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ is an indefinite potential satisfying asymptotically quadratic condition at infinity on u, Sun and Wu [16] with a little mistake in their embedding results, have proved the existence of two homoclinic solutions of (1.6).

The rest of the paper is organized as follows: In §2, we describe the Liouville–Weyl fractional calculus and we introduce the fractional space that we use in our work and some propositions are proven which will aid in our analysis. In §3, we prove Theorem 1.1. In §4, we prove Theorem 1.2. Finally, in section §5, we comment about the multiplicity result for the fractional Hamiltonian systems (1.1).

2. Preliminary results

2.1 *Liouville–Weyl fractional calculus*

We first introduce some basic definitions of fractional calculus. The Liouville–Weyl fractional integrals of order $0<\alpha<1$ are defined as

$$ {}_{-\infty}I_x^{\alpha}u(x)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-\xi)^{\alpha-1}u(\xi)\mathrm{d}\xi, \tag{2.1}$$

and

$$ {}_{x}I_{\infty}^{\alpha}u(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(\xi-x)^{\alpha-1}u(\xi)\mathrm{d}\xi. \tag{2.2}$$

The Liouville–Weyl fractional derivative of order $0<\alpha<1$ are defined as the left-inverse operators of the corresponding Liouville–Weyl fractional integrals

$$ {}_{-\infty}D_x^{\alpha}u(x)=\frac{\mathrm{d}}{\mathrm{d}x}\,{}_{-\infty}I_x^{1-\alpha}u(x), \tag{2.3}$$

and

$$ {}_{x}D_{\infty}^{\alpha}u(x)=-\frac{\mathrm{d}}{\mathrm{d}x}\,{}_{x}I_{\infty}^{1-\alpha}u(x). \tag{2.4}$$

Furthermore, if $u(x)$ is defined on $(-\infty,\infty)$, then the Fourier transform of the Liouville–Weyl integral and differential operator satisfies

$$ \widehat{{}_{-\infty}I_x^{\alpha}u(x)}(w)=(iw)^{-\alpha}\hat{u}(w),\quad \widehat{{}_{x}I_{\infty}^{\alpha}u(x)}(w)=(-iw)^{-\alpha}\hat{u}(w), \tag{2.5}$$

$$ \widehat{{}_{-\infty}D_x^{\alpha}u(x)}(w)=(iw)^{\alpha}\hat{u}(w),\quad \widehat{{}_{x}D_{\infty}^{\alpha}u(x)}(w)=(-iw)^{\alpha}\hat{u}(w). \tag{2.6}$$

2.2 *Fractional derivative space*

Our aim is to establish a variational structure that enables us to reduce the existence of solutions of (1.1) to finding critical points of the corresponding functional, and it is necessary to construct appropriate function spaces. We denote by $L^p(\mathbb{R}, \mathbb{R}^n)$, $p \in [2, +\infty]$, the Banach space of functions endowed with the norm

$$\|u\|_{L^p} = \left(\int_{\mathbb{R}} |u(t)|^p \mathrm{d}t \right)^{1/p},$$

and $L^\infty(\mathbb{R}, \mathbb{R}^n)$ is the Banach space of essentially bounded functions equipped with the norm

$$\|u\|_\infty := \operatorname{ess\,sup}\{|u(t)| : t \in \mathbb{R}\}.$$

Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty([0,T], \mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\alpha,p} = \left(\int_0^T |u(t)|^p \mathrm{d}t + \int_0^T |{}_0D_t^\alpha u(t)|^p \mathrm{d}t \right)^{1/p}, \quad \forall\, u \in E_0^{\alpha,p}.$$

This space can be characterized by $E_0^{\alpha,p} = \{u \in L^p([0,T], \mathbb{R}^n)/{}_0D_t^\alpha u \in L^p([0,T], \mathbb{R}^n)$ and $u(0) = u(T) = 0\}$. Moreover $(E_0^{\alpha,p}, \|.\|_{\alpha,p})$ is a reflexive and separable Banach space (see [8]).

For $\alpha > 0$, define the semi-norm $|u|_{I_{-\infty}^\alpha} = \|{}_{-\infty}D_x^\alpha u\|_{L^2}$, and the norm

$$\|u\|_{I_{-\infty}^\alpha} = \left(\|u\|_{L^2}^2 + |u|_{I_{-\infty}^\alpha}^2 \right)^{1/2}, \tag{2.7}$$

and let

$$I_{-\infty}^\alpha(\mathbb{R}, \mathbb{R}^n) = \overline{C_0^\infty(\mathbb{R}, \mathbb{R}^n)}^{\|.\|_{I_{-\infty}^\alpha}},$$

where $C_0^\infty(\mathbb{R}, \mathbb{R}^n)$ denotes the space of infinitely differentiable functions from $\mathbb{R}$ into $\mathbb{R}^n$ with vanishing property at infinity.

On the other hand, we define the fractional Sobolev space $H^\alpha(\mathbb{R}, \mathbb{R}^n)$ in terms of the Fourier transform. Choose $0 < \alpha < 1$ and define the semi-norm

$$|u|_\alpha = \||w|^\alpha \hat{u}\|_{L^2} \tag{2.8}$$

and the norm

$$\|u\|_\alpha = \left(\|u\|_{L^2}^2 + |u|_\alpha^2 \right)^{1/2},$$

and let

$$H^\alpha(\mathbb{R}, \mathbb{R}^n) = \overline{C_0^\infty(\mathbb{R}, \mathbb{R}^n)}^{\|.\|_\alpha}.$$

Moreover, we note a function $u \in L^2(\mathbb{R}, \mathbb{R}^n)$ belongs to $I_{-\infty}^\alpha(\mathbb{R}, \mathbb{R}^n)$ if and only if $|w|^\alpha \hat{u} \in L^2(\mathbb{R}, \mathbb{R}^n)$ and we have

$$|u|_{I_{-\infty}^\alpha} = \||w|^\alpha \hat{u}\|_{L^2}. \tag{2.9}$$

Therefore $I_{-\infty}^\alpha(\mathbb{R}, \mathbb{R}^n)$ and $H^\alpha(\mathbb{R}, \mathbb{R}^n)$ are equivalent with equivalent semi-norm and norm.

Let $C(\mathbb{R}, \mathbb{R}^n)$ denote the space of continuous functions from $\mathbb{R}$ into $\mathbb{R}^n$. Then we obtain the following Sobolev theorem.

Theorem 2.1 **[18].** *If $\alpha > \frac{1}{2}$, then $H^{\alpha}(\mathbb{R}, \mathbb{R}^n) \subset C(\mathbb{R}, \mathbb{R}^n)$ and there is a positive constant C_α such that*

$$\|u\|_\infty \leq C_\alpha \|u\|_\alpha. \tag{2.10}$$

In what follows, we introduce a new fractional space in which we will construct the variational framework of (1.1). Let

$$X^\alpha = \left\{ u \in H^\alpha(\mathbb{R}, \mathbb{R}^n) \middle| \int_{\mathbb{R}} \left[|{}_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t), u(t)) \right] \mathrm{d}t < \infty \right\},$$

then X^α is a reflexive and separable Hilbert space with the inner product

$$\langle u, v\rangle_{X^\alpha} = \int_{\mathbb{R}} [{}_{-\infty}D_t^\alpha u(t) \cdot {}_{-\infty}D_t^\alpha v(t) + (L(t)u(t), v(t))] \mathrm{d}t$$

and the corresponding norm

$$\|u\|^2_{X^\alpha} = \langle u, u\rangle_{X^\alpha}.$$

For $\lambda > 0$, we also need the following inner product

$$\langle u, v\rangle_\lambda = \int_{\mathbb{R}} [{}_{-\infty}D_t^\alpha u(t) \cdot {}_{-\infty}D_t^\alpha v(t) + \lambda (L(t)u(t), v(t))] \mathrm{d}t,$$

and the corresponding norm $\|u\|^2_\lambda = \langle u, u\rangle_\lambda$. It is clear that $\|u\|_{X^\alpha} \leq \|u\|_\lambda$ for $\lambda \geq 1$. Set $X_\lambda = (X^\alpha, \|.\|_\lambda)$.

Lemma 2.1. *Suppose that* (L_1) *and* (L_2) *hold. Then, the embedding* $X^\alpha \hookrightarrow H^\alpha(\mathbb{R}, \mathbb{R}^n)$ *is continuous.*

Proof. By (L_1), (L_2) and (2.10), we have

$$\begin{aligned} \int_{\mathbb{R}} |u(t)|^2 \mathrm{d}t &= \int_{\{l<k\}} |u(t)|^2 \mathrm{d}t + \int_{\{l \geq k\}} |u(t)|^2 \mathrm{d}t \\ &\leq \|u\|^2_\infty |\{l < k\}| + \frac{1}{k} \int_{\mathbb{R}} l(t)|u(t)|^2 \mathrm{d}t \\ &\leq C_\alpha^2 |\{l < k\}| \left(\int_{\mathbb{R}} (|{}_{-\infty}D_t^\alpha u(t)|^2 + |u(t)|^2) \mathrm{d}t \right) \\ &\quad + \frac{1}{k} \int_{\mathbb{R}} (L(t)u(t), u(t)) \mathrm{d}t. \end{aligned}$$

Therefore

$$\|u\|^2_{L^2} \leq \frac{\max\left\{ C_\alpha^2 |\{l < k\}|, \frac{1}{k} \right\}}{1 - C_\alpha^2 |\{l < k\}|} \|u\|^2_{X^\alpha}. \tag{2.11}$$

Then, by (2.11) we get

$$\|u\|^2_\alpha \leq \left(1 + \frac{\max\{C_\alpha^2 |\{l < k\}|, \frac{1}{k}\}}{1 - C_\alpha^2 |\{l < k\}|} \right) \|u\|^2_{X^\alpha}, \tag{2.12}$$

which implies that the embedding $X^\alpha \hookrightarrow H^\alpha(\mathbb{R})$ is continuous . □

Lemma 2.2. *Suppose that* (L_1) *and* (L_2) *hold. Then, there exists* $\Lambda > 0$ *such that, for all* $\lambda \geq \Lambda$, *the embedding* $X_\lambda \hookrightarrow L^r(\mathbb{R}, \mathbb{R}^n)$ *is continuous for all* $2 \leq r < \infty$.

Proof. Let $\Lambda = \frac{1}{kC_\alpha^2|\{L<k\}|}$. Using the same ideas of the proof of Lemma 2.1, for all $\lambda \geq \Lambda$, we also obtain

$$\|u\|_{L^2}^2 \leq \frac{C_\alpha^2|\{L < k\}|}{1 - C_\alpha^2|\{L < k\}|}\|u\|_\lambda^2 = \frac{1}{\theta_0}\|u\|_\lambda^2, \tag{2.13}$$

where $\theta_0 = \frac{1-C_\alpha^2|\{L<k\}|}{C_\alpha^2|\{L<k\}|}$. Furthermore, using (2.13), for each $r \in (2, \infty)$ and $\lambda \geq \Lambda$ we have

$$\begin{aligned}\int_{\mathbb{R}} |u(t)|^r \mathrm{d}t &\leq \|u\|_\infty^{r-2} \int_{\mathbb{R}} |u(t)|^2 \mathrm{d}t \\ &\leq C_\alpha^{r-2}\left(\int_{\mathbb{R}} |{}_{-\infty}D_t^\infty u(t)|^2 + |u(t)|^2 \mathrm{d}t\right)^{\frac{r-2}{2}} \frac{C_\alpha^2|\{l < k\}|}{1 - C_\alpha^2|\{l < k\}|}\|u\|_\lambda^2 \\ &\leq \frac{1}{\theta_0^{r/2}|\{l < k\}|^{\frac{r-2}{2}}}\|u\|_\lambda^r.\end{aligned}$$

Therefore, for all $r \in (2, \infty)$,

$$\|u\|_{L^r}^r \leq \frac{1}{\theta_0^{r/2}|\{L < k\}|^{\frac{r-2}{2}}}\|u\|_\lambda^r. \tag{2.14}$$

□

In order to prove Theorem 1.1, we use the following result by Rabinowitz [15]

Lemma 2.3. *Let* $\mathfrak{E}$ *be a real Banach space and* $\Phi \in C^1(\mathfrak{E}, \mathbb{R})$ *satisfy the* (PS)*-condition. If* Φ *is bounded from below, then* $c = \inf_{\mathfrak{E}} \Phi$ *is a critical value of* Φ.

3. Proof of Theorem 1.1

In this section, we are going to prove our main theorem. For that purpose, we note that associated to problem (1.1) we have the functional $I_\lambda : X_\lambda \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}} W(t, u)\mathrm{d}t.$$

Under our assumptions we can prove that the functional I_λ is of class C^1 in X_λ, and

$$I_\lambda'(u)\varphi = \int_{\mathbb{R}} \left[{}_{-\infty}D_t^\alpha u \cdot {}_{-\infty}D_t^\alpha \varphi + \lambda(L(t)u(t), \varphi(t))\right]\mathrm{d}t - \int_{\mathbb{R}} (\nabla W(t, u), \varphi)\mathrm{d}t.$$

Furthermore, critical points of the functional I_λ are weak solutions of problem (1.1). We begin our analysis by consider some useful lemmas.

Lemma 3.1. *Assume that* (L_1), (L_2), (W_1) *and* (W_2) *hold. Then, for all* $\lambda \geq \Lambda$, I_λ *is bounded from below in* X_λ.

Proof. By (W_1), (2.13) and Hölder inequality, we get

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{p}\int_{\mathbb{R}} \xi(t)|u(t)|^p \mathrm{d}t \\ &\geq \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{p}\|\xi\|_{L^{\frac{2}{2-p}}}\|u\|_{L^2}^p \\ &\geq \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{p\theta_0^{p/2}}\|\xi\|_{L^{\frac{2}{2-p}}}\|u\|_\lambda^p, \end{aligned}$$

which implies that $I_\lambda(u) \to +\infty$ as $\|u\|_\lambda \to +\infty$, since $1 < p < 2$. Thereofore I_λ is a functional bounded from below in X_λ. □

Lemma 3.2. *Suppose that L_1, L_2, (W_1) and (W_2) are satisfied. Then I_λ satisfies the (PS)-condition for each $\lambda \geq \Lambda$.*

Proof. Let $\{u_n\} \in X_\lambda$ be a sequence such that $I_\lambda(u_n)$ is bounded and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. By the previous lemma, it is clear that $\{u_n\}$ is bounded in X_λ. Thus, there exists a constant $\mathfrak{C} > 0$ such that

$$\|u_n\|_{L^r} \leq \frac{1}{\theta_0^{1/2}|\{L < k\}|^{\frac{r-2}{2r}}}\|u_n\|_\lambda \leq \mathfrak{C}, \qquad \text{for all } \lambda \geq \Lambda, \tag{3.1}$$

where $r \in [2, \infty]$. Then up to a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ weakly in X_λ. For any $\epsilon > 0$, since $\xi \in L^{\frac{2}{2-p}}(\mathbb{R})$, we can choose $T > 0$ such that

$$\left(\int_{|t|>T} |\xi(t)|^{\frac{2}{2-p}}\mathrm{d}t\right)^{\frac{2-p}{2}} < \epsilon. \tag{3.2}$$

Moreover, since $u_n \to u$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$, we get $u_n \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$. Hence

$$\lim_{n\to\infty}\int_{|t|\leq T} |u_n(t) - u(t)|^2 \mathrm{d}t = 0. \tag{3.3}$$

Therefore, from (3.3), there exists $n_0 \in \mathbb{N}$ such that

$$\int_{|t|<T} |u_n(t) - u(t)|^2 \mathrm{d}t < \epsilon^2, \qquad \text{for } n \geq n_0. \tag{3.4}$$

Hence, by (W_1), (3.1), (3.4) and the Hölder inequality, for any $n \geq n_0$, we have

$$\begin{aligned}
&\int_{|t|\leq T} |\nabla W(t, u_n(t)) - \nabla W(t, u(t))||u_n(t) - u_n(t)|\mathrm{d}t \\
&\leq \left(\int_{|t|\leq T} |\nabla W(t, u_n(t)) - \nabla W(t, u(t))|^2\right)^{1/2} \left(\int_{|t|\leq T} |u_n(t) - u(t)|^2 \mathrm{d}t\right)^{1/2} \\
&\leq \epsilon \left(\int_{|t|\leq T} 2(|\nabla W(t, u_n(t))|^2 + |\nabla W(t, u(t))|^2)\mathrm{d}t\right)^{1/2} \\
&\leq 2\epsilon \left(\int_{|t|\leq T} |\xi(t)|^2 \left(|u_n(t)|^{2(p-1)} + |u(t)|^{2(p-1)}\right) \mathrm{d}t\right)^{1/2} \\
&\leq 2\epsilon [\|\xi\|_{L^{\frac{2}{2-p}}} (\|u_n\|_{L^2}^{2(p-2)} + \|u\|_{L^2}^{2(p-1)})]^{1/2} \\
&\leq 2\epsilon [\|\xi\|^2_{L^{\frac{2}{2-p}}} (\mathfrak{C}^{2(p-1)} + \|u\|_{L^2}^{2(p-1)})]^{1/2}.
\end{aligned} \tag{3.5}$$

On the other hand, by (3.1), (3.2), (3.4) and (W_1), we have

$$\begin{aligned}
&\int_{|t|>T} |\nabla W(t, u_n(t)) - \nabla W(t, u(t))||u_n(t) - u(t)|\mathrm{d}t \\
&\leq 2\int_{|t|>T} |\xi(t)|(|u_n(t)|^p + |u(t)|^p)\mathrm{d}t \\
&\leq 2\epsilon \frac{1}{\theta_0^{p/2}} (\|u_n\|_\lambda^p + \|u\|_\lambda^p) \\
&\leq \frac{2\epsilon}{\theta_0^{p/2}} (\mathfrak{K}^p + \|u\|_\lambda^p).
\end{aligned} \tag{3.6}$$

Since ϵ is arbitrary, combining (3.5) and (3.6), we have

$$\int_{\mathbb{R}} |\nabla W(t, u_n(t)) - \nabla W(t, u(t))||u_n(t) - u(t)|\mathrm{d}t < \epsilon, \tag{3.7}$$

as $n \to \infty$. Hence, since $\langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle \to 0$ and the following identity hold:

$$\begin{aligned}
\langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle = \|u_n - u\|_\lambda^2 + \int_{\mathbb{R}} &(\nabla W(t, u_n(t)) \\
&- \nabla W(t, u(t)))(u_n(t) - u(t))\mathrm{d}t,
\end{aligned} \tag{3.8}$$

we get that $u_n \to u$ strongly in X_λ which implies that I_λ satisfies the (PS)-condition. $\square$

Proof of Theorem 1.1. From Lemmas 2.3, 3.1, 3.2, we know that

$$c_\lambda = \inf_{X_\lambda} I_\lambda(u)$$

is a critical value of functional I_λ, namely, there exists a critical point $u_\lambda \in X_\lambda$ such that $I_\lambda(u_\lambda) = c_\lambda$.

Finally, we show that $u_\lambda \neq 0$. Let $u_0 \in (W_0^{1,2}(\mathbb{I}) \cap X_\lambda) \setminus \{0\}$ and $\|u_0\|_\infty \leq 1$. Then by (W_2), we have

$$\begin{aligned} I_\lambda(su_0) &= \frac{1}{2}\|su_0\|_\lambda^2 - \int_{\mathbb{R}} W(t, su_0(t))\mathrm{d}t \\ &\leq \frac{s^2}{2}\|u_0\|_\lambda^2 - \int_{\mathbb{I}} W(t, su_0(t))\mathrm{d}t \\ &\leq \frac{s^2}{2}\|u_0\|_\lambda^2 - \eta s^\nu \int_{\mathbb{I}} |u_0(t)|^\nu \mathrm{d}t, \qquad 0 < s < \delta. \end{aligned} \tag{3.9}$$

Since $1 < \nu < 2$, it follows from (3.9) that $I_\lambda(su_0) < 0$ for $s > 0$ small enough. Hence $I_\lambda(u_\lambda) = c_\lambda < 0$, therefore, u_λ is a nontrivial critical point of I_λ and so u_λ is a nontrivial weak solution of problem (1.1). The proof is complete. □

4. Concentration of solutions

In the following, we study the concentration of solutions for problem (1.1) as $\lambda \to \infty$. Define

$$\tilde{c} = \inf_{w \in E_0^\alpha} I_\lambda|_{E_0^\alpha}(w),$$

where $I_\lambda|_{E_0^\alpha}$ is a restriction of I_λ on E_0^α; that is,

$$I_\lambda|_{E_0^\alpha}(w) = \frac{1}{2}\int_0^{\mathbb{T}} |{}_0D_t^\alpha w(t)|\mathrm{d}t - \int_0^{\mathbb{T}} W(t, w(t))\mathrm{d}t,$$

for $w \in H^\alpha(\mathbb{R}, \mathbb{R}^n)$. Following the same way as in the proof of Theorem 1.1, we can show that $\tilde{c} < 0$ can be achieved. Since $E_0^\alpha \subset X_\lambda$ for all $\lambda > 0$, we get

$$c_\lambda \leq \tilde{c} < 0, \quad \text{for all } \lambda > \Lambda.$$

Proof of Theorem 1.2. We follow the arguments in [1]. For any sequence $\lambda_n \to \infty$, let $u_n = u_{\lambda_n}$ be the critical point of I_{λ_n} obtained in Theorem 1.1. Thus

$$I_{\lambda_n}(u_n) \leq \tilde{c} < 0 \tag{4.1}$$

and

$$\begin{aligned} I_{\lambda_n}(u_n) &= \frac{1}{2}\|u_n\|_{\lambda_n}^2 - \int_{\mathbb{R}} W(t, u_n(t))\mathrm{d}t \\ &\geq \frac{1}{2}\|u_n\|_{\lambda_n}^2 - \frac{1}{p\theta_0^{p/2}}\|\xi\|_{L^{\frac{2}{2-p}}}\|u\|_{\lambda_n}^p, \end{aligned}$$

which implies that

$$\|u_n\|_{\lambda_n} \leq C, \tag{4.2}$$

where the constant $C > 0$ is independent of λ_n. Therefore, we may assume that $u_n \rightharpoonup \tilde{u}$ in X_λ and $u_n \to \tilde{u}$ in $L^p_{\mathrm{loc}}(\mathbb{R})$ for $2 \le p \le \infty$. By Fatou's lemma, we have

$$\begin{aligned}\int_{\mathbb{R}} l(t)|\tilde{u}(t)|^2 \mathrm{d}t &\le \liminf_{n\to\infty} \int_{\mathbb{R}} l(t)u_n^2(t)\mathrm{d}t \\ &\le \liminf_{n\to\infty} \int_{\mathbb{R}} (L(t)u_n(t), u_n(t))\mathrm{d}t \\ &\le \liminf_{n\to\infty} \frac{\|u_n\|^2_{\lambda_n}}{\lambda_n} = 0,\end{aligned}$$

thus $\tilde{u} = 0$ a.e. in $\mathbb{R} \setminus \mathbb{J}$, $\tilde{u} \in E_0^\alpha$ by (L_2). Now for any $\varphi \in C_0^\infty((0, \mathbb{T}), \mathbb{R}^n)$, since $\langle I'_{\lambda_n}(u_n), \varphi\rangle = 0$, it is easy to check that

$$\int_0^{\mathbb{T}} {}_0D_t^\alpha \tilde{u} \cdot {}_0D_t^\alpha \varphi \mathrm{d}t - \int_0^{\mathbb{T}} (\nabla W(t, \tilde{u}(t)), \varphi(t))\mathrm{d}t = 0,$$

that is, $\tilde{u}$ is a weak solution of (1.5) by the density of $C_0^\infty((0, \mathbb{T}), \mathbb{R}^n)$ in E_0^α.

Next, we show that $u_n(t) \to \tilde{u}(t)$ in $L^p(\mathbb{R})$ for $2 \le p < \infty$. Otherwise, by the vanishing lemma (see Lemma 2.1 in [19]), there exists $\delta > 0$, $R_0 > 0$ and $t_n \in \mathbb{R}$ such that

$$\int_{t_n - R_0}^{t_n + R_0} (u_n - \tilde{u})^2 \mathrm{d}t \ge \delta.$$

Moreover, $t_n \to \infty$, hence $|(t_n - R_0, t_n + R_0) \cap \{l < k\}| \to 0$. By the Hölder inequality, we have

$$\begin{aligned}&\int_{(t_n - R_0, t_n + R_0)\cap\{l<k\}} |u_n - \tilde{u}|^2 \mathrm{d}t \\ &\quad \le |(t_n - R_0, t_n + R_0) \cap \{l < k\}| \|u_n - \tilde{u}\|^2_\infty \to 0.\end{aligned}$$

Therefore

$$\begin{aligned}\|u_n\|^2_{\lambda_n} &\ge \lambda_n k \int_{(t_n - R_0, t_n + R_0)\cap\{l\ge k\}} |u_n(t)|^2 \mathrm{d}t \\ &= \lambda_n k \int_{(t_n - R_0, t_n + R_0)\cap\{l\ge k\}} |u_n(t) - \tilde{u}(t)|^2 \mathrm{d}t \\ &= \lambda_n k \left(\int_{(t_n - R_0, t_n + R_0)} |u_n(t) - \tilde{u}(t)|^2 \mathrm{d}t \right. \\ &\quad \left. - \int_{(t_n - R_0, t_n + R_0)\cap\{l<k\}} |u_n(t) - \tilde{u}(t)|^2 \mathrm{d}t\right) + o(1) \\ &\to \infty,\end{aligned}$$

which contradicts (4.2). By virtue of $\langle I'_{\lambda_n}(u_n), u_n\rangle = \langle I'_{\lambda_n}(u_n), \tilde{u}\rangle = 0$ and the fact that $u_n \to \tilde{u}$ strongly in $L^p(\mathbb{R})$ for $2 \le p < \infty$, we have

$$\lim_{n\to\infty} \|u_n\|^2_{\lambda_n} = \|\tilde{u}\|^2_{\lambda_n}.$$

Hence, $u_n \to \tilde{u}$ strongly in X_λ. Moreover, from (4.1), we have $\tilde{u} \ne 0$. This completes the proof. □

5. Remark about the multiplicity of weak solutions

In this section, we comment about the multiplicity result of weak solutions for (1.1). Our main result in this section is

Theorem 5.1. *Let conditions* (L_1), (L_3), (W_1) *and* (W_2) *hold, and* W *satisfies the following condition*:

(W_3) $W(t,u) = -W(t,-u)$ *for all* $(t,u) \in \mathbb{R} \times \mathbb{R}^n$.

Then problem (1.1) *possesses infinitely many nontrivial solutions.*

In order to find infinitely many solutions of (1.1) under the assumptions of Theorem 5.1, we shall use the 'genus' properties. Therefore, we recall the following definition and result (see [15]).

Let $\mathcal{B}$ be a Banach space, $I \in C^1(\mathcal{B}, \mathbb{R})$ and $c \in \mathbb{R}$. We set

$$\Sigma = \{A \subset \mathcal{B} \setminus \{0\} : \ A \ \text{ is closed in } \ \mathcal{B} \ \text{ and symmetric with respect to } 0\},$$
$$K_c = \{u \in \mathcal{B} : \ I(u) = c, I'(u) = 0\}, \quad I^c = \{u \in \mathcal{B} : \quad I(u) \le c\}.$$

DEFINITION 5.1

For $A \in \Sigma$, we say genus of A is j (denote by $\gamma(A) = j$) if there is an odd map $\psi \in C(A, \mathbb{R}^j \setminus \{0\})$, and j is the smallest integer with this property.

Lemma 5.1. *Let* I *be an even* C^1 *functional on* $\mathcal{B}$ *and it satisfies the* (*PS*) *condition. For any* $j \in \mathbb{N}$, *set*

$$\Sigma_j = \{A \in \Sigma : \quad \gamma(A) \ge j\}, \quad c_j = \inf_{A \in \Sigma_j} \sup_{u \in A} I(u).$$

(1) *If* $\Sigma_j \ne \phi$ *and* $c_j \in \mathbb{R}$, *then* c_j *is a critical value of* I.
(2) *If there exists* $r \in \mathbb{N}$ *such that*

$$c_j = c_{j+1} = \ldots = c_{j+r} = c \in \mathbb{R}$$

and $c \ne I(0)$, *then* $\gamma(K_c) \ge r + 1$.

Remark 5.1. From Remark 7.3 of [15], we know that if $K_c \in \Sigma$ and $\gamma(K_c) > 1$, then K_c contains infinitely many distinct points, that is, I has infinitely many distinct critical points in $\mathcal{B}$.

Proof of Theorem 5.1. From Lemma 3.1 and 3.2, we know that $I_\lambda \in C^1(X_\lambda, \mathbb{R})$ is bounded from below and satisfies the (PS)-condition. Furthermore, from (W_3), I_λ is even and $I_\lambda(0) = 0$. In order to apply Lemma 5.1, we prove that

$$\text{for any} \quad j \in \mathbb{N}, \quad \text{there exists} \ \ \epsilon > 0 \ \text{ such that } \ \gamma(I_\lambda^{-\epsilon}) \ge j. \tag{5.1}$$

Let $\{e_j\}_{j=1}^{\infty}$ be the standard orthogonal basis of X_λ, that is,

$$\|e_j\|_\lambda = 1 \ \text{ and } \ \langle e_i, e_k \rangle_\lambda = 0, \quad 1 \le i \ne k. \tag{5.2}$$

For any $j \in \mathbb{N}$, define

$$X_\lambda^j = \mathrm{span}\{e_1, e_2, \ldots, e_j\}, \quad S_j = \{u \in X_\lambda^j : \ \ \|u\|_\lambda = 1\}.$$

Then, for any $u \in X_\lambda^j$, there exists $\kappa_i \in \mathbb{R}$, $i = 1, 2, \ldots, j$, such that

$$u(t) = \sum_{i=1}^{j} \kappa_i e_i(t) \ \text{ for } \ t \in \mathbb{R}, \tag{5.3}$$

which implies that

$$\|u\|_{L^\theta} = \left(\int_{\mathbb{R}} |u(t)|^\theta\right)^{1/\theta} = \left(\int_{\mathbb{R}} \left|\sum_{i=1}^{j} \kappa_i e_i(t)\right|^\theta \mathrm{d}t\right)^{1/\theta} \tag{5.4}$$

and

$$\begin{aligned}
\|u\|_\lambda^2 &= \int_{\mathbb{R}} \left[|{}_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t), u(t))\right]\mathrm{d}t \\
&= \sum_{i=1}^{j} \kappa_i^2 \int_{\mathbb{R}} \left[|{}_{-\infty}D_t^\alpha e_i(t)|^2 + (L(t)e_i(t), e_i(t))\right]\mathrm{d}t \\
&= \sum_{i=1}^{j} \kappa_i^2 \|e_i\|_\lambda^2 = \sum_{i=1}^{j} \kappa_i^2.
\end{aligned} \tag{5.5}$$

On the other hand, in view of (L_3) and (W_2), for any bounded open set $\mathbb{I}$, there exists $\eta > 0$ (dependent on $\mathbb{I}$) such that

$$W(t, u) \geq a(t)|u|^\theta \geq \eta|u|^\theta, \quad (t, u) \in \mathbb{I} \times \mathbb{R}^n. \tag{5.6}$$

As a result, for any $u \in S_j$, we can take some $\mathbb{I}_0 \subset \mathbb{R}$ such that

$$\int_{\mathbb{R}} W(t, u(t))\mathrm{d}t = \int_{\mathbb{R}} W\left(t, \sum_{i=1}^{j} \kappa_i e_i(t)\right)\mathrm{d}t \geq \eta \int_{\mathbb{I}_0} \left|\sum_{i=1}^{j} \kappa_i e_i(t)\right|^\theta \mathrm{d}t = \varrho > 0. \tag{5.7}$$

Indeed, if not, for any bounded open set $\mathbb{I} \subset \mathbb{R}$, there exists $\{u_n\}_{n\in\mathbb{N}} \in S_j$ such that

$$\int_{\mathbb{I}} |u_n(t)|^\theta \mathrm{d}t = \int_{\mathbb{I}} \left|\sum_{i=1}^{j} \kappa_{in} e_i(t)\right|^\theta \mathrm{d}t \to 0, \ \text{ as } \ n \to 0,$$

where $u_n = \sum_{i=1}^{j} \kappa_{in} e_i$ such that $\sum_{i=1}^{j} \kappa_{in}^2 = 1$. Because $\sum_{i=1}^{j} \kappa_{in}^2 = 1$, we have

$$\lim_{n\to+\infty} \kappa_{in} =: \kappa_{i0} \in [0,1] \quad \text{and} \quad \sum_{i=1}^{j} \kappa_{i0}^2 = 1.$$

Hence, for any bounded open set $\mathbb{I} \subset \mathbb{R}$,

$$\int_{\mathbb{I}} \left| \sum_{i=1}^{j} \kappa_{i0} e_i(t) \right|^{\theta} \mathrm{d}t = 0.$$

The fact that $\mathbb{I}$ is arbitrary yields that $u_0 = \sum_{i=1}^{j} \kappa_{i0} e_i(t) = 0$ a.e. on $\mathbb{R}$ which contradicts the fact that $\|u_0\|_\lambda = 1$. Hence, (5.7) holds true.

In addition, since all norms of a finite dimensional norm space are equivalent, there is a constant $\tilde{c} > 0$ such that

$$\tilde{c}\|u\|_\lambda \le \|u\|_{L^\theta}, \quad \forall u \in X_\lambda^j. \tag{5.8}$$

Consequently, according to (W_3) and (5.3)–(5.8), we have

$$\begin{aligned} I_\lambda(su) &= \frac{s^2}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}} W(t, s\sum_{i=1}^{j} \kappa_i e_i(t))\mathrm{d}t \\ &\le \frac{s^2}{2}\|u\|_\lambda^2 - s^\theta \int_{\mathbb{R}} a(t) \left| \sum_{i=1}^{j} \kappa_i e_i(t) \right|^{\theta} \mathrm{d}t \\ &\le \frac{s^2}{2}\|u\|_\lambda^2 - \eta s^\theta \int_{\mathbb{I}_0} \left| \sum_{i=1}^{j} \kappa_i e_i(t) \right|^{\theta} \mathrm{d}t \\ &\le \frac{s^2}{2}\|u\|_\lambda^2 - \varrho s^\theta = \frac{s^2}{2} - \varrho s^\theta, \quad u \in S_j, \end{aligned}$$

which implies that there exists $\epsilon > 0$ and $\delta > 0$ such that

$$I_\lambda(\delta u) < -\epsilon \ \text{ for } \ u \in S_j. \tag{5.9}$$

Let

$$S_j^\delta = \{\delta u / \ u \in S_j\}, \quad \Omega = \left\{ (\kappa_1, \kappa_2, \ldots, \kappa_j) : \ \sum_{i=1}^{j} \kappa_i^2 < \delta^2 \right\}.$$

Then it follows from (5.9) that

$$I_\lambda(u) < -\epsilon, \quad \forall u \in S_j^\delta,$$

which, together with the fact that $I_\lambda \in C^1(X^\alpha, \mathbb{R})$ and is even, yields that

$$S_j^\delta \subset I_\lambda^{-\epsilon} \in \Sigma.$$

On the other hand, it follows from (5.3) and (5.5) that there exists an odd homeomorphism mapping $\varphi \in C(S_j^\delta, \partial\Omega)$. By some properties of the genus, we obtain

$$\gamma(I_\lambda^{-\epsilon}) \geq \gamma(S_j^\delta) = j, \tag{5.10}$$

so (5.1) follows. Set

$$c_j = \inf_{A \in \Sigma_j} \sup_{u \in A} I(u),$$

then, from (5.10) and the fact that I_λ is bounded from below on X_λ, we have

$$-\infty < c_j \leq -\epsilon < 0,$$

that is, for any $j \in \mathbb{N}$, c_j is a real negative number. By Lemma 5.1 and Remark 5.1, I_λ has infinitely many nontrivial critical points, and consequently, (1.1) possesses infinitely many solutions. □

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