ISSN 0037–4466, Siberian Mathematical Journal, 2019, Vol. 60, No. 3, pp. 429–449. © Pleiades Publishing, Inc., 2019.
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ON THE SOLVABILITY OF SOME DYNAMIC POROELASTIC PROBLEMS

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UDC 517.958

Abstract: We consider the direct problems for poroelasticity equations. In the low-frequency approximation we prove existence and uniqueness theorems for the solution to a certain mixed problem. In the high-frequency approximation we establish the uniqueness of a weak solution to the mixed problem and its continuous dependence on the data in the cases of bounded and unbounded temporal intervals and for however many spatial variables.

DOI: 10.1134/S0037446619030078

Keywords: poroelasticity equations, dynamic permeability, mixed problem, existence, uniqueness, stability

1. Introduction

In the applied questions of analyzing the propagation of elastic waves we often have to account for the porosity of the medium. In particular, the questions of this type arise in seismology, exploration geophysics, and studies of the properties of oil and gas fields. As a rule, the two frequency regimes are distinguished while studying poroelastic phenomena: low-frequency [1] and high-frequency [2]. In order to describe poroelastic phenomena, the classical Biot model [1] is normally used in the low-frequency approximation; while in the high-frequency approximation, the generalization of the model of [2] to the case of dynamic permeability depending on the square root of temporal frequency [3].

2. Low-Frequency Approximation

In this section we prove existence and uniqueness theorems for a solution to a certain mixed problem for the low-frequency approximation of poroelasticity equations. Our results generalize those obtained by Santos in the two-dimensional case; see [4].

Statement of the problem. Consider $Q_T = (0,T) \times \Omega$ with $T > 0$, where Ω is a bounded open domain in $\mathbb{R}^3$ associated with some isotropic inhomogeneous porous body with smooth boundary $\partial\Omega$. Consider the vector functions $\mathbf{u}_s, \mathbf{u}_f : \overline{Q}_T \to \mathbb{R}^3$ characterizing the *absolute* displacement of both solid (s) and fluid (f) phases respectively. In the low-frequency approximation $\mathbf{u}_s$ and $\mathbf{u}_f$ satisfy in Q_T the classical Biot equations [1]

$$\begin{gathered}\rho_{11}\partial_t^2\mathbf{u}_s + \rho_{12}\partial_t^2\mathbf{u}_f + b\partial_t(\mathbf{u}_s - \mathbf{u}_f) - \nabla\cdot\sigma(\mathbf{u}) = \mathbf{f}_s,\\ \rho_{12}\partial_t^2\mathbf{u}_s + \rho_{22}\partial_t^2\mathbf{u}_f - b\partial_t(\mathbf{u}_s - \mathbf{u}_f) - \nabla s(\mathbf{u}) = \mathbf{f}_f,\end{gathered} \tag{2.1}$$

where in the case of isotropic media

$$\sigma(\mathbf{u}) = (\lambda\nabla\cdot\mathbf{u}_s + q\nabla\cdot\mathbf{u}_f)\, I + \mu\big(\nabla\mathbf{u}_s + \nabla\mathbf{u}_s^T\big), \quad s(\mathbf{u}) = q\nabla\cdot\mathbf{u}_s + r\nabla\cdot\mathbf{u}_f. \tag{2.2}$$

Here $\mathbf{u} = (\mathbf{u}_s, \mathbf{u}_f)$, while $\sigma(\mathbf{u})$ is the general stress tensor of the spatial material, the scalar $s(\mathbf{u})$ is related to the pressure p in the fluid phase as $s = -\phi p$, where ϕ stands for the effective porosity; $\mathbf{f}_s, \mathbf{f}_f : \overline{Q}_T \to \mathbb{R}^3$ characterize the spatial densities of exterior forces acting in the solid and fluid phases respectively. The

The authors were partially supported by the National Institute of Science and Technology of Petroleum Geophysics (INCT-GP/MEC/CNPq), Salvador, Brazil.

Original article submitted October 10, 2017; revised October 10, 2017; accepted October 17, 2018.

coefficient $\rho_{12} : \overline{\Omega} \to \mathbb{R}_-$ amounts to a parameter describing the mass interaction between the fluid and solid phases, ρ_{11}, ρ_{22}: $\overline{\Omega} \to \mathbb{R}_+$ characterize the inertia of the two phases and are related to the densities $\rho_s, \rho_f : \overline{\Omega} \to \mathbb{R}_+$ of the fluid and solid phases as $\rho_{11} + \rho_{12} = \rho_s$ and $\rho_{22} + \rho_{12} = \rho_f$, while $b = \nu\phi^2/k$ is the dissipation coefficient, where $\nu : \overline{\Omega} \to \mathbb{R}_+$ is the viscosity of the fluid, $k : \overline{\Omega} \to \mathbb{R}_+$ is the permeability; and I is the identity matrix of size 3×3. The coefficients $\lambda, \mu : \overline{\Omega} \to \mathbb{R}_+$ and $q, r : \overline{\Omega} \to \mathbb{R}_+$ are respectively the Lamé and Biot parameters [1]. The physical properties of the fluid/solid system enable us to assume that

$$\begin{gathered} 0 < m_1 \le b(\mathbf{x}), \lambda(\mathbf{x}), \mu(\mathbf{x}), r(\mathbf{x}) \le M_1 < \infty, \quad \mathbf{x} \in \overline{\Omega}, \\ 0 \le q_m \le q(\mathbf{x}) \le q_M < \infty, \quad \mathbf{x} \in \overline{\Omega}, \\ r(\mathbf{x})(\lambda(\mathbf{x}) + \mu(\mathbf{x})) - q^2(\mathbf{x}) > 0, \quad \mathbf{x} \in \overline{\Omega}. \end{gathered} \tag{2.3}$$

Assume also that

$$0 < m_2 \le \rho_{11}(\mathbf{x}), \rho_{22}(\mathbf{x}) \le M_2 < \infty, \quad \mathbf{x} \in \overline{\Omega},$$

$$-\infty < \rho_m \le \rho_{12}(\mathbf{x}) \le \rho_M \le 0, \quad \mathbf{x} \in \overline{\Omega},$$

$$\rho_{11}(\mathbf{x})\rho_{22}(\mathbf{x}) - \rho_{12}^2(\mathbf{x}) > 0, \quad \mathbf{x} \in \overline{\Omega}.$$

Consider the matrices $\mathscr{A}$ and $\mathscr{E}$ defined as

$$\mathscr{A} = \begin{pmatrix} \rho_{11} I & \rho_{12} I \\ \rho_{12} I & \rho_{22} I \end{pmatrix}, \quad \mathscr{E} = b \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Define the differential operator

$$\mathscr{L} = (\nabla \cdot \sigma_1(\mathbf{u}), \nabla \cdot \sigma_2(\mathbf{u}), \nabla \cdot \sigma_3(\mathbf{u}), \nabla s(\mathbf{u})),$$

where $\sigma_i(\mathbf{u}) = (\sigma_{i1}(\mathbf{u}), \sigma_{i2}(\mathbf{u}), \sigma_{i3}(\mathbf{u}))$ for $i = 1, 2, 3$. Using the notations, express (2.1) and (2.2) in the more concise form

$$\mathscr{A}\partial_t^2 \mathbf{u} + \mathscr{E}\partial_t \mathbf{u} - \mathscr{L}(\mathbf{u}) = \mathbf{f},$$

where $\mathbf{f} = (\mathbf{f}_s, \mathbf{f}_f)$. Thus, we can state our problem as follows: *Find a function $\mathbf{u} : \overline{Q}_T \to \mathbb{R}^6$ such that*

$$\begin{gathered} \mathscr{A}\partial_t^2 \mathbf{u} + \mathscr{E}\partial_t \mathbf{u} - \mathscr{L}(\mathbf{u}) = \mathbf{f}, \quad (t, \mathbf{x}) \in Q_T, \\ \mathbf{u}(0, \cdot) = \mathbf{u}^0, \quad \partial_t \mathbf{u}(0, \cdot) = \mathbf{v}^0, \quad \mathbf{x} \in \Omega, \\ (\sigma_1(\mathbf{u}) \cdot \mathbf{n}, \sigma_2(\mathbf{u}) \cdot \mathbf{n}, \sigma_3(\mathbf{u}) \cdot \mathbf{n}) = \boldsymbol{\psi}, \quad s(\mathbf{u}) = \xi, \quad (t, \mathbf{x}) \in \Gamma_T, \end{gathered} \tag{2.4}$$

where $\Gamma_T = (0, T) \times \partial\Omega$ and $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the outer normal to $\partial\Omega$.

Assume that the coefficients λ, μ, q, r, b, ρ_{11}, ρ_{12}, ρ_{22} and $\mathbf{f} : Q_T \to \mathbb{R}^6$, $\mathbf{u}^0, \mathbf{v}^0 : \Omega \to \mathbb{R}^6$, $\boldsymbol{\psi} : \Gamma_T \to \mathbb{R}^3$, $\xi : \Gamma_T \to \mathbb{R}$ are prescribed and possess the required smoothness.

Function spaces, and notation. Denote by $\Omega \in \mathbb{R}^3$ a bounded open set with smooth boundary $\partial\Omega$. Given a nonnegative integer index m, denote by $H^m(\Omega)$ the ordinary Sobolev space with the norm

$$\|v\|_m = \left(\sum_{|\alpha| \le m} \int_\Omega |D^\alpha v(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2}.$$

The norm of the vector function $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $[H^m(\Omega)]^n$ with $n \in \mathbb{N}$ is introduced as

$$\|\mathbf{v}\|_m = \left(\sum_{i=1}^n \|v_i\|_m^2 \right)^{1/2}.$$

Denote the inner product and norm of $\mathbf{v}, \mathbf{w} \in [L^2(\Omega)]^n$ by

$$(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^{n} \int_{\Omega} v_i w_i \, d\mathbf{x}, \quad \|\mathbf{v}\|_0 = (\mathbf{v}, \mathbf{v})^{1/2}.$$

The inner product and norm of vector functions $\mathbf{v}, \mathbf{w} \in [L^2(\partial\Omega')]^n$, with $\partial\Omega' \subseteq \partial\Omega$, are introduced as

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} \int_{\partial\Omega'} v_i w_i \, d\sigma, \quad \|\mathbf{v}\|_0 = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2},$$

where $d\sigma$ is the surface measure on $\partial\Omega'$.

Denote the dual space of X by X'. Then $[H^{-m}(\Omega)]^n = [H^m(\Omega)']^n$ with the norm

$$\|\boldsymbol{\zeta}\|_{-m} = \sup_{\mathbf{0} \neq \mathbf{v} \in [H^m(\Omega)]^n} \frac{(\boldsymbol{\zeta}, \mathbf{v})}{\|\mathbf{v}\|_m},$$

where $(\cdot, \cdot)$ is the duality between $[H^{-m}(\Omega)]^n$ and $[H^m(\Omega)]^n$.

Recall that $\boldsymbol{\zeta}_v = \mathbf{v}/\partial\Omega \in [L^2(\partial\Omega)]^n$ for every $\mathbf{v} \in [H^1(\Omega)]^n$, and $|\boldsymbol{\zeta}_v|_0 \leq C\|\mathbf{v}\|_0^{1/2}\|\mathbf{v}\|_1^{1/2}$ with some positive constant C. Then $[H^{1/2}(\partial\Omega)]^n$ is defined as the image of $[H^1(\Omega)]^n$ under $\boldsymbol{\zeta}$; see [5]. The norm of $\boldsymbol{\vartheta} \in [H^{1/2}(\partial\Omega)]^n$ is defined as

$$|\boldsymbol{\vartheta}|_{1/2} = \inf_{\substack{\mathbf{v} \in [H^1(\Omega)]^n, \\ \boldsymbol{\zeta}_v = \boldsymbol{\vartheta}}} \|\mathbf{v}\|_1.$$

Denote by $[H^{-1/2}(\partial\Omega)]^n$ the dual space of $[H^{1/2}(\partial\Omega)]^n$. Denote the duality between $[H^{-1/2}(\partial\Omega)]^n$ and $[H^{1/2}(\partial\Omega)]^n$ by $\langle \cdot, \cdot \rangle$. Introduce the norm in $[H^{-1/2}(\partial\Omega)]^n$ as

$$|\mathbf{v}|_{-1/2} = \sup_{\mathbf{0} \neq \boldsymbol{\zeta} \in [H^{1/2}(\partial\Omega)]^n} \frac{\langle \mathbf{v}, \boldsymbol{\zeta} \rangle}{|\boldsymbol{\zeta}|_{1/2}}.$$

Consider the space $H(\Omega; \mathrm{div}) = \{\mathbf{v} \in [L^2(\Omega)]^3 : \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$ equipped with the norm

$$\|\mathbf{v}\|_{H(\Omega;\mathrm{div})} = \left(\|\mathbf{v}\|_0^2 + \|\nabla \cdot \mathbf{v}\|_0^2\right)^{1/2}.$$

Observe that we can identify $H(\Omega; \mathrm{div})$ with the closed subspace of $[L^2(\Omega)]^4$ consisting of the elements (v_1, v_2, v_3, v_4), where $v_4 = \nabla \cdot \mathbf{v}$ with $\mathbf{v} \in [L^2(\Omega)]^3$ and $v_4 \in L^2(\Omega)$.

Denote by $\mathbf{n} = \mathbf{n}(\mathbf{x})$ the outer normal to $\partial\Omega$. It is known that $\mathbf{v} \cdot \mathbf{n} \in H^{-1/2}(\partial\Omega)$ for every $\mathbf{v} \in H^{-1/2}(\partial\Omega)$ and $|\mathbf{v} \cdot \mathbf{n}|_{-1/2} \leq C\|\mathbf{v}\|_{H(\Omega;\mathrm{div})}$ with a positive constant C.

Recall the integration-by-parts formula [6]:

$$(\nabla \cdot \mathbf{v}, w) + (\mathbf{v}, \nabla w) = \langle \mathbf{v} \cdot \mathbf{n}, w \rangle, \quad \mathbf{v} \in H(\Omega; \mathrm{div}), \; w \in H^1(\Omega). \tag{2.5}$$

Put $V = [H^1(\Omega)]^3 \times H(\Omega; \mathrm{div})$. Define the norm $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in V$ as $\|\mathbf{v}\|_V = \left(\|\mathbf{v}_1\|_1^2 + \|\mathbf{v}_2\|_{H(\Omega;\mathrm{div})}^2\right)^{1/2}$. Since $H(\Omega; \mathrm{div})$ can be identified with the closed subspace $[L^2(\Omega)]^4$, we can represent every element of V' as $\mathbf{z} = (z_1, z_2, \ldots, z_7)$, where $z_1, z_2, z_3 \in H^{-1}(\Omega)$ and $z_4, z_5, z_6, z_7 \in L^2(\Omega)$.

Take $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2) \in V$ with $\mathbf{w}_i = (w_{i1}, w_{i2}, w_{i3})$ for $i = 1, 2$. Denote by $[\cdot, \cdot]$ the duality between V' and V; i.e.,

$$[\mathbf{z}, \mathbf{w}] = ((z_1, z_2, z_3), \mathbf{w}_1) + \int_{\Omega} ((z_4, z_5, z_6) \cdot \mathbf{w}_2 + z_7 \nabla \cdot \mathbf{w}_2) \, d\mathbf{x}.$$

Then

$$|[\mathbf{z}, \mathbf{w}]| \leq \|(z_1, z_2, z_3)\|_{-1}\|\mathbf{w}_1\|_1 + \|(z_4, z_5, z_6, z_7)\|_0\|\mathbf{w}_2\|_{H(\Omega;\mathrm{div})} \leq \|\mathbf{z}\|_{V'}\|\mathbf{w}\|_V.$$

For a real Banach space X with the norm $\|\cdot\|_X$, the $L^p(0,T;X)$ space consists of all strictly measurable functions $\mathbf{v} : (0,T) \to X$ such that $\|\mathbf{v}\|_{L^p(0,T;X)} < \infty$, where

$$\|\mathbf{v}\|_{L^p(0,T;X)} = \begin{cases} \left[\int\limits_0^T \|\mathbf{v}(t)\|_X^p \, dt\right]^{1/p}, & 1 \le p < \infty; \\ \operatorname*{ess\,sup}\limits_{t\in(0,T)} \|\mathbf{v}(t)\|_X, & p = \infty. \end{cases}$$

Finally, recall that $\mathscr{D}'(0,T)$ stands for the space of distributions in $(0,T)$.

Weak statement of problem (2.4). Given $\mathbf{u}, \mathbf{v} \in V$, where $\mathbf{u} = (\mathbf{u}_s, \mathbf{u}_f)$ and $\mathbf{v} = (\mathbf{v}_s, \mathbf{v}_f)$, with $\mathbf{u}_k = (u_{k1}, u_{k2}, u_{k3})$ and $\mathbf{v}_k = (v_{k1}, v_{k2}, v_{k3})$ for $k = s, f$, define the bilinear forms

$$M(\mathbf{u}_s, \mathbf{v}_s) = \int\limits_\Omega \left[\lambda \nabla \cdot \mathbf{u}_s \nabla \cdot \mathbf{v}_s + 2\mu \sum_{i,j=1}^3 \epsilon_{ij}(\mathbf{u}_s)\epsilon_{ij}(\mathbf{v}_s)\right] d\mathbf{x},$$

$$B(\mathbf{u}, \mathbf{v}) = M(\mathbf{u}_s, \mathbf{v}_s) + (q\nabla \cdot \mathbf{u}_f, \nabla \cdot \mathbf{v}_s) + (q\nabla \cdot \mathbf{u}_s + r\nabla \cdot \mathbf{u}_f, \nabla \cdot \mathbf{v}_f),$$

where $\epsilon_{ij}(\cdot)$ are the components of the standard deformation tensor ϵ. Note that $B(\mathbf{u}, \mathbf{v})$ is symmetric and

$$|B(\mathbf{u}, \mathbf{v})| \le C(\|\mathbf{u}_s\|_1 + \|\nabla \cdot \mathbf{u}_f\|_0)\big(\|\mathbf{v}_s\|_1 + \|\nabla \cdot \mathbf{v}_f\|_0\big) \le C\|\mathbf{u}\|_V \|\mathbf{v}\|_V$$

with a positive constant C. Furthermore, recall Korn's inequality [7]

$$\int\limits_\Omega \sum_{i,j=1}^3 \epsilon_{ij}^2(\mathbf{u}) \, d\mathbf{x} + \|\mathbf{u}\|_0^2 \ge C_1 \|\mathbf{u}\|_1^2, \quad \mathbf{u} \in [H^1(\Omega)]^3,$$

where C_1 is some positive constant.

Take the matrix

$$D = \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 & q \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 & q \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 & q \\ 0 & 0 & 0 & 4\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4\mu & 0 \\ q & q & q & 0 & 0 & 0 & r \end{pmatrix}$$

in $\mathbb{R}^{7\times 7}$ and put $\mathbf{z}(\mathbf{u}) = (\epsilon_{11}(\mathbf{u}_s),\ \epsilon_{22}(\mathbf{u}_s),\ \epsilon_{33}(\mathbf{u}_s),\ \epsilon_{12}(\mathbf{u}_s),\ \epsilon_{13}(\mathbf{u}_s),\ \epsilon_{23}(\mathbf{u}_s),\ \nabla \cdot \mathbf{u}_f)$. Then

$$B(\mathbf{u}, \mathbf{u}) = \int\limits_\Omega (D\mathbf{z}(\mathbf{u}), \mathbf{z}(\mathbf{u})) \, d\mathbf{x},$$

where $(\cdot, \cdot)$ stands for the ordinary dot product in $\mathbb{R}^7$. By (2.3) the matrix D is positive definite. Therefore, if we denote by $\lambda_{\min}$ the minimal eigenvalue of D and put $C_2 = \min\big(\frac{\lambda_{\min} C_1}{2}, \lambda_{\min}\big)$ then, applying Korn's inequality, we find that

$$B(\mathbf{u}, \mathbf{u}) \ge C_2 \|\mathbf{u}\|_V^2 - \lambda_{\min} \|\mathbf{u}\|_0^2, \quad \mathbf{u} \in V.$$

Denote a fixed constant satisfying $\gamma \ge \lambda_{\min}$ by γ, and define the bilinear form $B_\gamma(\cdot, \cdot)$ as

$$B_\gamma(\mathbf{u}, \mathbf{v}) = B(\mathbf{u}, \mathbf{v}) + \gamma(\mathbf{u}, \mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in V.$$

Then B_γ is symmetric and

$$|B_\gamma(\mathbf{u}, \mathbf{v})| \le C\|\mathbf{u}\|_V \|\mathbf{v}\|_V, \quad B_\gamma(\mathbf{u}, \mathbf{u}) \ge C_2 \|\mathbf{u}\|_V^2, \quad \mathbf{u}, \mathbf{v} \in V. \tag{2.6}$$

Multiply the first relation in (2.4) by some test function $\mathbf{v} \in V$ and then integrate over Ω. Applying the integration-by-parts formula (2.5) to $(\mathscr{L}(\mathbf{u}), \mathbf{v})$ and using the boundary conditions, arrive at the equality

$$\big(\mathscr{A}\partial_t^2\mathbf{u}, \mathbf{v}\big) + (\mathscr{E}\partial_t\mathbf{u}, \mathbf{v}) + B(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) + \langle \boldsymbol{\psi}, \mathbf{v}_s\rangle + \langle\, \mathbf{v}_f \cdot \mathbf{n}, \xi\rangle \quad \text{for almost all } t \in (0, T).$$

Thus, we can state the weak form of the mixed problem (2.4) as follows: *Given* $\mathbf{f}$, $\mathbf{u}^0$, $\mathbf{v}^0$, $\boldsymbol{\psi}$, *and* ξ, *find a vector-function* $\mathbf{u} : Q_T \to \mathbb{R}^6$ *satisfying*

$$\mathbf{u}, \partial_t\mathbf{u} \in L^\infty(0, T; V), \quad \partial_t^2\mathbf{u} \in L^\infty(0, T; [L^2(\Omega)]^6) \tag{2.7}$$

and

$$\big(\mathscr{A}\partial_t^2\mathbf{u}, \mathbf{v}\big) + (\mathscr{E}\partial_t\mathbf{u}, \mathbf{v}) + B(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) + \langle \boldsymbol{\psi}, \mathbf{v}_s\rangle + \langle \mathbf{v}_f \cdot \mathbf{n}, \xi\rangle \quad \textit{for all } \mathbf{v} \in V \textit{ in } \mathscr{D}'(0, T), \tag{2.8}$$

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}^0, \quad \partial_t\mathbf{u}(0, \mathbf{x}) = \mathbf{v}^0, \quad \mathbf{x} \in \Omega. \tag{2.9}$$

Refer to $\mathbf{u}$ as a *weak solution* to (2.4).

Solution of problem (2.4). Put

$$\begin{gathered}
\mathscr{G}_0^2 = \|\mathbf{u}^0\|_2^2 + \|\mathbf{v}^0\|_1^2 + \|\mathbf{f}(0)\|_0^2 + 1, \\
N_s^2 = \big\|\partial_t^s\boldsymbol{\psi}\big\|^2_{L^\infty(0,T;[H^{-1/2}(\partial\Omega)]^3)} \\
+\big\|\partial_t^{s+1}\boldsymbol{\psi}\big\|^2_{L^2(0,T;[H^{-1/2}(\partial\Omega)]^3)} + \big\|\partial_t^s\xi\big\|^2_{L^\infty(0,T;H^{1/2}(\partial\Omega))} \\
+\big\|\partial_t^{s+1}\xi\big\|^2_{L^2(0,T;H^{1/2}(\partial\Omega))} + \big\|\partial_t^s\mathbf{f}\big\|^2_{L^2(0,T;[L^2(\partial\Omega)]^6)}, \quad s = 0, 1.
\end{gathered}$$

Theorem 1.1. *Given* $\mathbf{f}, \boldsymbol{\psi}, \xi, \mathbf{u}^0$, *and* $\mathbf{v}^0$, *if* $\max\{\mathscr{G}_0, N_0, N_1\} < \infty$ *then a weak solution* $\mathbf{u}(t, \mathbf{x})$ *to problem* (2.4) *exists and satisfies* (2.7).

Proof. Consider some sequence $\{\mathbf{v}_i\}_{i=1}^\infty$, $\mathbf{v}_i = (\mathbf{v}_{si}, \mathbf{v}_{fi}) \in [H^2(\Omega)]^6$ such that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ are linearly independent for all m and whose span is dense in $[H^2(\Omega)]^6$. This sequence exists because $[H^2(\Omega)]^6$ is a separable space. Put $\mathbf{u}_m = (\mathbf{u}_{sm}, \mathbf{u}_{fm}) \in S_m$ and $S_m = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m)$, where

$$\mathbf{u}_m(t, \mathbf{x}) = \sum_{i=1}^m g_{mi}(t)\mathbf{v}_i(\mathbf{x}). \tag{2.10}$$

Define $\mathbf{u}_m$ as a solution to the finite-dimensional $(1 \le i \le m)$ problem

$$\big(\mathscr{A}\partial_t^2\mathbf{u}_m, \mathbf{v}_i\big) + (\mathscr{E}\partial_t\mathbf{u}_m, \mathbf{v}_i) + B(\mathbf{u}_m, \mathbf{v}_i) = (\mathbf{f}, \mathbf{v}_i) + \langle \boldsymbol{\psi}, \mathbf{v}_{si}\rangle + \langle \mathbf{v}_{fi} \cdot \mathbf{n}, \xi\rangle, \quad t \in (0, T), \tag{2.11}$$

$$\mathbf{u}_m(0, \cdot) = \mathbf{u}_m^0, \quad \partial_t\mathbf{u}_m(0, \cdot) = \mathbf{v}_m^0, \quad \mathbf{x} \in \Omega, \tag{2.12}$$

where $\mathbf{u}_m^0$, $\mathbf{v}_m^0 \in S_m$ and

$$\mathbf{u}_m^0 \xrightarrow[m\to\infty]{} \mathbf{u}^0 \text{ in } [H^2(\Omega)]^6, \quad \mathbf{v}_m^0 \xrightarrow[m\to\infty]{} \mathbf{v}^0 \text{ in } [H^1(\Omega)]^6. \tag{2.13}$$

Given $m \in \mathbb{N}$, there is a unique function $\mathbf{u}_m$ of the form (2.10) with smooth g_{mi} satisfying (2.11)–(2.13) for almost all $t \in [0, T]$. Observe that

$$\frac{d}{dt}\|\mathbf{u}_m\|_0^2 = 2\int\limits_\Omega \mathbf{u}_m(t, \mathbf{x})\partial_t\mathbf{u}_m(t, \mathbf{x})\, d\mathbf{x} \le \|\mathbf{u}_m\|_0^2 + \|\partial_t\mathbf{u}_m\|_0^2.$$

Multiplying (2.11) by $g'_{mi}(t)$, where the prime stands for the usual derivative with respect to t, and then summing over i from 1 to m, we find that

$$\begin{aligned}&\left(\mathscr{A}\partial_t^2\mathbf{u}_m,\partial_t\mathbf{u}_m\right)+(\mathscr{E}\partial_t\mathbf{u}_m,\partial_t\mathbf{u}_m)+B(\mathbf{u}_m,\partial_t\mathbf{u}_m)\\&\quad=(\mathbf{f},\partial_t\mathbf{u}_m)+\langle\boldsymbol{\psi},\partial_t\mathbf{u}_{sm}\rangle+\langle\partial_t\mathbf{u}_{fm}\cdot\mathbf{n},\xi\rangle,\end{aligned}$$

and so

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|\mathscr{A}^{1/2}\partial_t\mathbf{u}_m\|_0^2+B_\gamma(\mathbf{u}_m,\mathbf{u}_m)\right]+(\mathscr{E}\partial_t\mathbf{u}_m,\partial_t\mathbf{u}_m)\\&\le C\left[\|\mathbf{f}\|_0^2+\|\partial_t\mathbf{u}_m\|_0^2+\|\mathbf{u}_m\|_0^2\right]+\langle\psi,\partial_t\mathbf{u}_{sm}\rangle+\langle\partial_t\mathbf{u}_{fm}\cdot\mathbf{n},\xi\rangle\end{aligned}\tag{2.14}$$

with a positive constant C independent of m. Applying the integration-by-parts formula, we find that

$$\begin{aligned}\left|\int_0^t\langle\boldsymbol{\psi},\partial_t\mathbf{u}_{sm}\rangle(s)\,ds\right|&=\left|\langle\boldsymbol{\psi},\mathbf{u}_{sm}\rangle(t)-\langle\boldsymbol{\psi},\mathbf{u}_{sm}\rangle(0)-\int_0^t\langle\partial_t\boldsymbol{\psi},\mathbf{u}_{sm}\rangle(s)\,ds\right|\\&\le\epsilon\|\mathbf{u}_{sm}\|_1^2+C\left[N_0^2+\|\mathbf{u}_{sm}(0)\|_0^2+\int_0^t\|\mathbf{u}_{sm}\|_1^2(s)\,ds\right],\end{aligned}$$

$$\begin{aligned}\left|\int_0^t\langle\partial_t\mathbf{u}_{fm}\cdot\mathbf{n},\xi\rangle(s)\,ds\right|&=\left|\langle\mathbf{u}_{fm}\cdot\mathbf{n},\xi\rangle(t)-\langle\mathbf{u}_{fm}\cdot\mathbf{n},\xi\rangle(0)-\int_0^t\langle\mathbf{u}_{fm}\cdot\mathbf{n},\partial_t\xi\rangle(s)\,ds\right|\\&\le\epsilon\|\mathbf{u}_{fm}\|^2_{H(\Omega;\mathrm{div})}+C\left[N_0^2+\|\mathbf{u}_{fm}(0)\|^2_{H(\Omega;\mathrm{div})}+\int_0^t\|\mathbf{u}_{fm}\|^2_{H(\Omega;\mathrm{div})}(s)\,ds\right],\end{aligned}$$

where ϵ is a positive constant. Integrating (2.14) over the interval $(0,t)$ and using the above estimates and (2.6), we arrive at

$$\begin{aligned}&\|\mathscr{A}^{1/2}\partial_t\mathbf{u}_m(t)\|_0^2+C_2\|\mathbf{u}_m(t)\|_V^2+\int_0^t(\mathscr{E}\partial_t\mathbf{u}_m,\partial_t\mathbf{u}_m)(s)\,ds\le\epsilon\|\mathbf{u}_m(t)\|_V^2\\&+C\left[N_0^2+\|\mathscr{A}^{1/2}\partial_t\mathbf{u}_m(0)\|_0^2+\|\mathbf{u}_m(0)\|_0^2+\int_0^t\left(\|\partial_t\mathbf{u}_m\|_0^2(s)+\|\mathbf{u}_m(s)\|_V^2\right)ds\right].\end{aligned}\tag{2.15}$$

Observe that $\|\mathscr{A}^{1/2}\partial_t\mathbf{u}_m(t)\|_0$ is equivalent to $\|\partial_t\mathbf{u}_m(t)\|_0$ and (2.13) yields

$$\|\mathscr{A}^{1/2}\partial_t\mathbf{u}_m(0)\|_0^2+\|\mathbf{u}_m(0)\|_0^2\le C\mathscr{G}_0^2.$$

Since $\mathscr{E}$ is a nonnegative matrix, putting $2\epsilon=C_2$ in (2.15) and applying Grönwall's lemma, we obtain the first a priori estimate

$$\|\partial_t\mathbf{u}_m\|_{L^\infty(0,T;[L^2(\Omega)]^6)}+\|\mathbf{u}_m\|_{L^\infty(0,T;V)}\le C(\mathscr{G}_0+N_0).\tag{2.16}$$

Differentiating (2.11) yields

$$\begin{aligned}&\left(\mathscr{A}\partial_t^3\mathbf{u}_m,\mathbf{v}_i\right)+\left(\mathscr{E}\partial_t^2\mathbf{u}_m,\mathbf{v}_i\right)+B(\partial_t\mathbf{u}_m,\mathbf{v}_i)\\&=(\partial_t\mathbf{f},\mathbf{v}_i)+\langle\partial_t\boldsymbol{\psi},\mathbf{v}_{si}\rangle+\langle\,\mathbf{v}_{fi}\cdot\mathbf{n},\partial_t\xi\rangle,\quad t\in(0,T),\ 1\le i\le m.\end{aligned}$$

Multiplying the last equality by $g''_{mi}(t)$ and then summing over i from 1 to m, we see that

$$\big(\mathscr{A}\partial_t^3\mathbf{u}_m, \partial_t^2\mathbf{u}_m\big) + \big(\mathscr{E}\partial_t^2\mathbf{u}_m, \partial_t^2\mathbf{u}_m\big) + B\big(\partial_t\mathbf{u}_m, \partial_t^2\mathbf{u}_m\big)$$
$$= \big(\partial_t\mathbf{f}, \partial_t^2\mathbf{u}_m\big) + \big\langle\partial_t\boldsymbol{\psi}, \partial_t^2\mathbf{u}_{sm}\big\rangle + \big\langle\partial_t^2\mathbf{u}_{fm}\cdot\mathbf{n}, \partial_t\xi\big\rangle, \quad t\in(0,T).$$

Repeating the procedure that led to (2.15), we arrive at

$$\big\|\partial_t^2\mathbf{u}_m(t)\big\|_0^2 + \|\partial_t\mathbf{u}_m(t)\|_V^2 \le C\Bigg[N_1^2 + \mathscr{G}_0^2 + \big\|\partial_t^2\mathbf{u}_m(0)\big\|_0^2 + \int\limits_0^t \big(\big\|\partial_t^2\mathbf{u}_m(s)\big\|_0^2 + \|\partial_t\mathbf{u}_m(s)\|_V^2\big)\,ds\Bigg]. \tag{2.17}$$

Let us estimate $\big\|\partial_t^2\mathbf{u}_m(0)\big\|_0^2$. Using the integration-by-parts formula (2.5), the initial data (2.12), and the equality

$$\mathscr{L}(\mathbf{u}_m(0), \mathbf{v}_i) = -B(\mathbf{u}_m(0), \mathbf{v}_i) + \langle\boldsymbol{\psi}, \mathbf{v}_{si}\rangle + \langle\mathbf{v}_{fi}\cdot\mathbf{n}, \xi\rangle,$$

we find that

$$\big(\mathscr{A}\partial_t^2\mathbf{u}_m(0), \mathbf{v}_i\big) = (\mathbf{f}(0), \mathbf{v}_i) - (\mathscr{E}\partial_t\mathbf{u}_m(0), \mathbf{v}_i) + (\mathscr{L}(\mathbf{u}_m(0)), \mathbf{v}_i). \tag{2.18}$$

Multiplying (2.18) by $g''_{mi}(0)$ and then summing over i from 1 to m, we find that $\big\|\partial_t^2\mathbf{u}_m(0)\big\|_0^2 \le C\mathscr{G}_0^2$. Finally, using the Grönwall inequality, we arrive at the second a priori estimate

$$\big\|\partial_t^2\mathbf{u}_m\big\|_{L^\infty(0,T;[L^2(\Omega)]^6)} + \|\partial_t\mathbf{u}_m\|_{L^\infty(0,T;V)} \le C(\mathscr{G}_0 + N_1). \tag{2.19}$$

The Dunford–Pettis Theorem implies that

$$[L^1(0,T;V')]' = L^\infty(0,T;V), \quad [L^1(0,T;[L^2(\Omega)]^6)]' = L^\infty(0,T;[L^2(\Omega)]^6).$$

From the sequence $\{\mathbf{u}_m\}_{m=1}^\infty$ we can refine a subsequence, also denoted by $\{\mathbf{u}_m\}_{m=1}^\infty$, such that

$$\begin{aligned} \mathbf{u}_m &\xrightarrow[m\to\infty]{} \mathbf{u} \quad *\text{-weakly in } L^\infty(0,T;V),\\ \partial_t\mathbf{u}_m &\xrightarrow[m\to\infty]{} \partial_t\mathbf{u} \quad *\text{-weakly in } L^\infty(0,T;V),\\ \partial_t^2\mathbf{u}_m &\xrightarrow[m\to\infty]{} \partial_t^2\mathbf{u} \quad *\text{-weakly in } L^\infty(0,T;[L^2(\Omega)]^6). \end{aligned} \tag{2.20}$$

From (2.20) we conclude that

$$\begin{aligned} &\int\limits_0^T \big(\partial_t^k\mathbf{u}_m, \mathbf{v}_i\big)(t)\,dt \xrightarrow[m\to\infty]{} \int\limits_0^T \big(\partial_t^k\mathbf{u}, \mathbf{v}_i\big)(t)\,dt, \quad k=0,1,2;\\ &\big(\mathscr{A}\partial_t^2\mathbf{u}_m, \mathbf{v}_i\big) \xrightarrow[m\to\infty]{} \big(\mathscr{A}\partial_t^2\mathbf{u}, \mathbf{v}_i\big) \quad *\text{-weakly in } L^\infty(0,T);\\ &(\mathscr{E}\partial_t\mathbf{u}_m, \mathbf{v}_i) \xrightarrow[m\to\infty]{} (\mathscr{E}\partial_t\mathbf{u}, \mathbf{v}_i) \quad *\text{-weakly in } L^\infty(0,T) \end{aligned} \tag{2.21}$$

for all $\mathbf{v}_i$. Suppose that $g(t)\in L^1(0,T)$. Using (2.20) once and the integration-by-parts formula (2.5) twice, we arrive at

$$\lim_{m\to\infty}\int\limits_0^T B(\mathbf{u}_m, \mathbf{v}_i)(t)g(t)\,dt$$
$$= \lim_{m\to\infty}\int\limits_0^T [-(\mathscr{L}(\mathbf{v}_i), \mathbf{u}_m) + \langle(\tau_1(\mathbf{v}_i)\cdot\mathbf{n}, \tau_2(\mathbf{v}_i)\cdot\mathbf{n}, \tau_3(\mathbf{v}_i)\cdot\mathbf{n}), \mathbf{u}_{sm}\rangle$$
$$+\langle\mathbf{u}_{fm}\cdot\mathbf{n}, s(\mathbf{v}_i)\rangle](t)g(t)\,dt = \int\limits_0^T B(\mathbf{u}, \mathbf{v}_i)(t)g(t)\,dt.$$

Therefore,

$$B(\mathbf{u}_m, \mathbf{v}_i) \xrightarrow[m\to\infty]{} B(\mathbf{u}, \mathbf{v}_i) \quad *\text{-weakly in } L^\infty(0,T)$$

for every $\mathbf{v}_i$. Passing in (2.11) to the limit with respect to m, then using (2.20) and (2.21), we infer that

$$\big(\mathscr{A}\partial_t^2\mathbf{u}, \mathbf{v}_i\big) + (\mathscr{E}\partial_t\mathbf{u}, \mathbf{v}_i) + B(\mathbf{u}, \mathbf{v}_i) = (\mathbf{f}, \mathbf{v}_i) + \langle \boldsymbol{\psi}, \mathbf{v}_{si}\rangle + \langle \xi, \mathbf{v}_{fi}\rangle.$$

But the span of the collection of $\mathbf{v}_i$ is dense in $[H^2(\Omega)]^6$. So,

$$\mathscr{A}\partial_t^2\mathbf{u} + \mathscr{E}\partial_t\mathbf{u} - \mathscr{L}(\mathbf{u}) = \mathbf{f} \quad \text{in } [D'(\Omega)]^6 \text{ for almost all } t \in (0,T).$$

As the next step, we verify the fulfillment of the initial and boundary conditions of (2.4). By (2.7) and Lemma 1.2 of [8, Chapter 1] it follows from

$$\mathbf{u}_m(0) \xrightarrow[m\to\infty]{} \mathbf{u}^0 \;\; *\text{-weakly in } [H^1(\Omega)]^6, \quad \partial_t\mathbf{u}_m(0) \xrightarrow[m\to\infty]{} \mathbf{v}^0 \;\; *\text{-weakly in } [L^2(\Omega)]^6$$

that the initial conditions of our problem are fulfilled in the weak sense (as in [8]). The book mainly considers the first boundary problem, and the fulfillment of boundary conditions follows because the solution belongs to $H_0^1(\Omega)$. In our case, integration by parts yields

$$B(\mathbf{u}, \mathbf{v}_i) = -(\mathscr{L}(\mathbf{u}), \mathbf{v}_i) + \langle(\tau_1(\mathbf{u})\cdot\mathbf{n}, \tau_2(\mathbf{u})\cdot\mathbf{n}, \tau_3(\mathbf{u})\cdot\mathbf{n}), \mathbf{v}_{si}\rangle + \langle \mathbf{v}_{fi}\cdot\mathbf{n}, s(\mathbf{u})\rangle, \quad i \in \mathbb{N}.$$

Since the span of the collection of $\mathbf{v}_i$ is dense in $[H^2(\Omega)]^6$, for almost all $t \in (0,T)$

$$\langle(\tau_1(\mathbf{u})\cdot\mathbf{n}, \tau_2(\mathbf{u})\cdot\mathbf{n}, \tau_3(\mathbf{u})\cdot\mathbf{n}), \mathbf{v}_s\rangle = \langle\boldsymbol{\psi}, \mathbf{v}_s\rangle, \quad \mathbf{v}_s \in [H^2(\Omega)]^3,$$
$$\langle \mathbf{v}_f\cdot\mathbf{n}, s(\mathbf{u})\rangle = \langle \mathbf{v}_f\cdot\mathbf{n}, \xi\rangle, \quad \mathbf{v}_f \in H(\Omega; \operatorname{div}).$$

In the last formula the first equality means the fulfillment of the first boundary condition of (2.4), while the second means that for every $p \in H^{-1/2}(\partial\Omega)$ there is $q \in H(\Omega;\operatorname{div})$ with $q\cdot\mathbf{n} = p$; i.e., $\langle p, s(\mathbf{u})\rangle = \langle p, \xi\rangle$ for every $p \in H^{-1/2}(\partial\Omega)$. Consequently, the constructed solution satisfies all boundary conditions of (2.4). This completes the proof of the existence theorem. $\square$

Theorem 1.2. *Under the assumptions of Theorem* 1.1 *the solution is unique.*

Proof. Suppose that there are two solutions $\mathbf{u}_k$, for $k = 1, 2$, to (2.4) corresponding to the same initial and boundary conditions and take $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$. Then

$$\mathbf{u}, \partial_t\mathbf{u} \in L^\infty(0,T;V), \quad \partial_t^2\mathbf{u} \in L^\infty(0,T;[L^2(\Omega)]^6)$$

and $\mathbf{u}$ satisfies

$$\mathscr{A}\partial_t^2\mathbf{u} + \mathscr{E}\partial_t\mathbf{u} - \mathscr{L}(\mathbf{u}) = \mathbf{0}, \quad (t,\mathbf{x}) \in Q_T,$$
$$\mathbf{u}(0,\cdot) = \partial_t\mathbf{u}(0,\cdot) = \mathbf{0}, \quad \mathbf{x} \in \Omega,$$
$$(\tau_1(\mathbf{u})\cdot\mathbf{n}, \tau_2(\mathbf{u})\cdot\mathbf{n}, \tau_3(\mathbf{u})\cdot\mathbf{n}) = \mathbf{0}, \quad s(\mathbf{u}) = 0, \;\; (t,\mathbf{x}) \in \Gamma_T.$$

Therefore, for every vector function $\mathbf{v} \in V(\Omega)$ we have

$$\big(\mathscr{A}\partial_t^2\mathbf{u}, \mathbf{v}\big) + (\mathscr{E}\partial_t\mathbf{u}, \mathbf{v}) + B(\mathbf{u}, \mathbf{v}) = 0 \quad \text{for almost all } t \in (0,T).$$

Since $\mathbf{v} \in V(\Omega)$, put $\mathbf{v} = \partial_t\mathbf{u}$ in the previous relation to reduce it to

$$\frac{1}{2}\frac{d}{dt}\big[\|\mathscr{A}^{1/2}\partial_t\mathbf{u}\|_0^2 + B(\mathbf{u}, \mathbf{u})\big] + (\mathscr{E}\partial_t\mathbf{u}, \partial_t\mathbf{u}) = 0.$$

Applying the inequality

$$\frac{d}{dt}\|\mathbf{u}\|_0^2 \le \|\mathbf{u}\|_0^2 + \|\partial_t\mathbf{u}\|_0^2$$

to the previous formula and integrating from 0 to t, we see that

$$\|\mathbf{u}\|_V^2 + \|\partial_t\mathbf{u}\|_0^2 \le C\int_0^t \big(\|\mathbf{u}(s,\cdot)\|_0^2 + \|\partial_t\mathbf{u}(s,\cdot)\|_0^2\big)\,ds$$

with some positive constant C. The latter means that $\|\mathbf{u}\|_V = \|\partial_t\mathbf{u}\|_0 = 0$; i.e., $\mathbf{u}_1 = \mathbf{u}_2$ almost everywhere in Q_T. $\square$

3. High-Frequency Approximation

Using the results of [3], we can express the high-frequency analog of Biot's equations as [9]

$$\begin{gathered}\rho\partial_t^2\mathbf{u}_s + \rho_f\partial_t^2\mathbf{u}_f = \nabla\cdot\sigma + \mathbf{f}_s,\\ \rho_f\partial_t^2\mathbf{u}_s + \rho_w\partial_t^2\mathbf{u}_f + \frac{\eta}{\kappa\sqrt{\omega}}h * \left[\partial_t^2\mathbf{u}_f + \omega\partial_t\mathbf{u}_f\right] = -\nabla p + \mathbf{f}_f\end{gathered} \tag{3.1}$$

with the state equations (the case of an isotropic medium)

$$\sigma = (\lambda_f\nabla\cdot\mathbf{u}_s + \beta m\nabla\cdot\mathbf{u}_f)I + \mu\big(\nabla\mathbf{u}_s + \nabla\mathbf{u}_s^T\big), \quad p = -m(\beta\nabla\cdot\mathbf{u}_s + \nabla\cdot\mathbf{u}_f). \tag{3.2}$$

This model uses the following functions and physical parameters: the *relative* displacement vector $\mathbf{u}_f$ of the fluid phase, the spatial density $\mathbf{f}_f$ of the exterior force acting in the fluid phase, the density ρ_f and dynamic viscosity η of the fluid; the *absolute* displacement vector $\mathbf{u}_s$ of the solid phase, the spatial density $\mathbf{f}_s$ of the exterior force acting in solid phase, the density ρ_s and the shear modulus μ of the elastic skeleton; the porosity $0 < \phi < 1$, the tortuosity $a \geq 1$, the absolute permeability κ, the Lamé coefficient λ_f, and two Biot's coefficients β and m of a saturated rock; the elastic stress tensor σ and the acoustic pressure p. Moreover, we use the notation

$$\begin{gathered}\rho_w = \frac{a}{\phi}\rho_f, \quad \rho = \phi\rho_f + (1-\phi)\rho_s, \quad \omega = \frac{2\pi f_c}{P},\\ h(t) = \frac{e^{-\omega t}}{\sqrt{\pi t}}, \quad h * z(t,\cdot) = \int_0^t h(t-s)z(s,\cdot)\,ds,\end{gathered} \tag{3.3}$$

where f_c is the transition frequency and $P > 0$ is the Pride number; see [9]. Observe that, in accordance with the definitions (3.3),

$$\rho\rho_w - \rho_f^2 = (\phi\rho_f + (1-\phi)\rho_s)\frac{a}{\phi}\rho_f - \rho_f^2 = (a-1)\rho_f^2 + \frac{a(1-\phi)}{\phi}\rho_s\rho_f > 0. \tag{3.4}$$

We will deal with the N-dimensional case, where $N = 1, 2, 3$, although the results hold for $N \geq 4$; i.e., they are independent of the dimension N. We assume also that

$$\lambda_0 + 2\mu > 0, \tag{3.5}$$

where $\lambda_0 \equiv \lambda_f - m\beta^2$ is the Lamé coefficient of a dry rock.

Remark 3.1. By the definition of h, the convolution term in (3.1) amounts to the order $1/2$ fractional derivative of $\partial_t^2\mathbf{u}_f + \omega\partial_t\mathbf{u}_f$.

Statement of the problem. Denote by $\Omega \subset \mathbb{R}^N$ an arbitrary bounded open set with the boundary $\partial\Omega$ of class C^2 and put $\Delta\mathbf{u}_s = (\Delta u_{s,1}, \Delta u_{s,2}, \Delta u_{s,3})$. In the domain $Q_+ = (0,\infty)\times\Omega$ consider the mixed problem

$$\begin{gathered}\rho\partial_t^2\mathbf{u}_s + \rho_f\partial_t^2\mathbf{u}_f - \mu\Delta\mathbf{u}_s - \nabla[(\lambda_f+\mu)\nabla\cdot\mathbf{u}_s + \beta m\nabla\cdot\mathbf{u}_f] = \mathbf{f}_s, \quad (t,\mathbf{x}) \in Q_+,\\ \rho_f\partial_t^2\mathbf{u}_s + \rho_w\partial_t^2\mathbf{u}_f + \frac{\eta}{\kappa\sqrt{\omega}}h * \left[\partial_t^2\mathbf{u}_f + \omega\partial_t\mathbf{u}_f\right]\\ -m\nabla(\beta\nabla\cdot\mathbf{u}_s + \nabla\cdot\mathbf{u}_f) = \mathbf{f}_f, \quad (t,\mathbf{x}) \in Q_+,\\ \mathbf{u}_s(0,\cdot) = \mathbf{v}^0, \quad \partial_t\mathbf{u}_s(0,\cdot) = \mathbf{v}^1, \quad \mathbf{u}_f(0,\cdot) = \mathbf{w}^0, \quad \partial_t\mathbf{u}_f(0,\cdot) = \mathbf{w}^1, \quad \mathbf{x} \in \Omega,\\ \mathbf{u}_s = \mathbf{0}, \quad \mathbf{n}\cdot\mathbf{u}_f = 0, \quad (t,\mathbf{x}) \in \Gamma_+,\end{gathered} \tag{3.6}$$

where $\mathbf{n}$ is the outer unit vector orthogonal to $\partial\Omega$, and $\Gamma_+ = \mathbb{R}_+ \times \partial\Omega$. The differential equations in (3.6) are obtained by inserting (3.2) into (3.1).

Function spaces, and notation. Consider the following spaces related to some Hilbert space V:

$$L^2_\xi(\mathbb{R}; V) = \{\mathbf{u} \in L^2_{\mathrm{loc}}(\mathbb{R}; V) : \operatorname{supp} \mathbf{u} \subset \overline{Q}_+,\ e^{-t\xi}\mathbf{u} \in L^2(\mathbb{R}; V)\},$$

$$H^1_\xi(\mathbb{R}; V) = \{\mathbf{u} \in L^2(\mathbb{R}; V) : e^{-t\xi}\partial_t \mathbf{u} \in L^2(\mathbb{R}; V)\},$$

$$\mathscr{S}'_\xi(\mathbb{R}; V) = \{\mathbf{u} \in \mathscr{D}'(\mathbb{R}; V) : \operatorname{supp} \mathbf{u} \subset \overline{Q}_+,\ e^{-t\xi}\mathbf{u} \in \mathscr{S}'(\mathbb{R}; V)\},$$

where $\xi \in \mathbb{R}$ and $\mathscr{S}'(\mathbb{R}; V)$ is the space of temporal distributions with values in V. Define the spaces depending on a parameter $\xi_0 \in \mathbb{R}_+$:

$$\begin{gathered}
\mathscr{V}(\xi_0) = \big\{\mathbf{v} \in H^1_{\xi_0}(\mathbb{R}; L^2(\Omega)^N) : \mathbf{v} \in L^2_{\xi_0}\big(\mathbb{R}; H^1_0(\Omega)^N\big), \\
\partial_t \mathbf{v} \in L^2_{\xi_0}(\mathbb{R}; L^2(\Omega)^N) \cap C(\mathbb{R}; L^2(\Omega)^N),\ \partial_t^2 \mathbf{v} \in \mathscr{S}'_{\xi_0}(\mathbb{R}; V)\big\}, \\
\mathscr{W}(\xi_0) = \big\{\mathbf{w} \in H^1_{\xi_0}(\mathbb{R}; L^2(\Omega)^N) : \mathbf{w} \in L^2_{\xi_0}(\mathbb{R}; L^2(\Omega; \operatorname{div})),\ \mathbf{n} \cdot \mathbf{w} = 0 \text{ on } \partial\Omega, \\
\partial_t \mathbf{w} \in L^2_{\xi_0}(\mathbb{R}; L^2(\Omega)^N) \cap C(\mathbb{R}; L^2(\Omega)^N),\ \partial_t^2 \mathbf{w} \in \mathscr{S}'_{\xi_0}(\mathbb{R}; V)\big\},
\end{gathered}$$

where $L^2(\Omega; \operatorname{div}) = \{\mathbf{u} \in L^2(\Omega)^N : \operatorname{div} \mathbf{u} \in L^2(\Omega)\}$ is equipped with the norm

$$\|\mathbf{u}\|_{L^2(\Omega;\operatorname{div})} = \big[\|\mathbf{u}\|^2_{L^2(\Omega)^N} + \|\operatorname{div} \mathbf{u}\|^2_{L^2(\Omega)}\big]^{1/2}.$$

Observe that the choice of ξ_0 means some weakening conditions on the behavior of $(\mathbf{v}, \mathbf{w})$ as $t \to +\infty$.

Finally, recall the definition of Laplace transform used in the special vector subspace of distributions; see [10, Chapter 10]. Take a distribution $\mathbf{z} \in \mathscr{D}'(\mathbb{R}; V)$ with the following properties:

(i) $\operatorname{supp} \mathbf{z} \subset [0, +\infty)$;

(ii) there is $\xi_0 \in \mathbb{R}$ such that the distribution $e^{-t\xi_0}\mathbf{z}$ belongs to the vector subspace $\mathscr{S}'(\mathbb{R}; V)$. Under these assumptions the Laplace transform of $\mathbf{z}$ is the holomorphic function $\hat{\mathbf{z}}$ defined as

$$\hat{\mathbf{z}}(\tau) = \langle e^{-t(\tau-\xi_0)}, e^{-t\xi_0}\mathbf{z}\rangle, \quad \operatorname{Re}\tau > \xi_0.$$

Moreover, we have

$$\widehat{\big(\partial_t^j \mathbf{z}\big)}(\tau) = \tau^j \hat{\mathbf{z}}(\tau), \quad \widehat{(h * \mathbf{z})}(\tau) = \widehat{h}(\tau)\hat{\mathbf{z}}(\tau), \quad \operatorname{Re}\tau > \xi_0,\ j \in \mathbb{N},$$

for every scalar function $h : \mathbb{R}_+ \to \mathbb{R}$ with $e^{-t\xi_0}h \in L^1(\mathbb{R}_+)$.

Integrodifferential problem. Consider a more general version of (3.1). Replacing the kernel of the convolution operator $\eta(\kappa\sqrt{\omega})^{-1}h$ by a more general function k, we will deal with the following generalization of problem (3.6):

$$\begin{gathered}
\rho_1\partial_t^2\mathbf{u}_s + \rho_2\partial_t^2\mathbf{u}_f - \alpha_1\Delta\mathbf{u}_s - \nabla[\alpha_2\nabla\cdot\mathbf{u}_s + \alpha_3\nabla\cdot\mathbf{u}_f] = \mathbf{f}_s, \quad (t,\mathbf{x}) \in Q_+, \\
\rho_2\partial_t^2\mathbf{u}_s + \rho_3\partial_t^2\mathbf{u}_f + k * \big[\partial_t^2\mathbf{u}_f + \omega\partial_t\mathbf{u}_f\big] \\
-\nabla(\alpha_3\nabla\cdot\mathbf{u}_s + \alpha_4\nabla\cdot\mathbf{u}_f) = \mathbf{f}_f, \quad (t,\mathbf{x}) \in Q_+, \\
\mathbf{u}_s(0,\cdot) = \mathbf{v}^0, \quad \partial_t\mathbf{u}_s(0,\cdot) = \mathbf{v}^1, \quad \mathbf{u}_f(0,\cdot) = \mathbf{w}^0, \quad \partial_t\mathbf{u}_f(0,\cdot) = \mathbf{w}^1, \quad \mathbf{x} \in \Omega, \\
\mathbf{u}_s = \mathbf{0}, \quad \mathbf{n}\cdot\mathbf{u}_f = 0, \quad (t,\mathbf{x}) \in \Gamma_+,
\end{gathered} \tag{3.7}$$

where $\rho_j \in \mathbb{R}$ for $j = 1, 2, 3$ and $\alpha_j \in \mathbb{R}$ for $j = 1, 2, 3, 4$ are prescribed constants.

Make the change of the unknowns functions:

$$\mathbf{u}_s = \mathbf{v}^0 + t\mathbf{v}^1 + \tilde{\mathbf{v}}, \quad \mathbf{u}_f = \mathbf{w}^0 + t\mathbf{w}^1 + \widetilde{\mathbf{w}}. \tag{3.8}$$

Using (3.7) and (3.8), we obtain the problem for determining $(\tilde{\mathbf{v}}, \widetilde{\mathbf{w}})$:

$$\begin{gathered}
\rho_1 \partial_t^2 \tilde{\mathbf{v}} + \rho_2 \partial_t^2 \widetilde{\mathbf{w}} - \alpha_1 \Delta \tilde{\mathbf{v}} - \nabla(\alpha_2 \nabla \cdot \tilde{\mathbf{v}} + \alpha_3 \nabla \cdot \widetilde{\mathbf{w}}) = \tilde{\mathbf{f}}^1 + \tilde{\mathbf{f}}^3, \quad (t, \mathbf{x}) \in Q_+, \\
\rho_2 \partial_t^2 \tilde{\mathbf{v}} + \rho_3 \partial_t^2 \widetilde{\mathbf{w}} + k * \left[\partial_t^2 \widetilde{\mathbf{w}} + \omega \partial_t \widetilde{\mathbf{w}}\right] - \nabla(\alpha_3 \nabla \cdot \tilde{\mathbf{v}} + \alpha_4 \nabla \cdot \widetilde{\mathbf{w}}) \\
= \tilde{\mathbf{g}}^1 + \tilde{\mathbf{g}}^2 + \tilde{\mathbf{g}}^3, \quad (t, \mathbf{x}) \in Q_+, \\
\tilde{\mathbf{v}}(0, \cdot) = \widetilde{\mathbf{w}}(0, \cdot) = \mathbf{0}, \quad \partial_t \tilde{\mathbf{v}}(0, \cdot) = \partial_t \widetilde{\mathbf{w}}(0, \cdot) = \mathbf{0}, \quad \mathbf{x} \in \Omega, \\
\tilde{\mathbf{v}} = \mathbf{0}, \quad \mathbf{n} \cdot \widetilde{\mathbf{w}} = 0, \quad (t, \mathbf{x}) \in \Gamma_+,
\end{gathered} \tag{3.9}$$

where

$$\begin{gathered}
\tilde{\mathbf{f}}^1 = \alpha_1 \Delta \mathbf{v}^0 + \nabla\big(\alpha_2 \nabla \cdot \mathbf{v}^0 + \alpha_3 \nabla \cdot \mathbf{w}^0\big), \\
\tilde{\mathbf{f}}^3 = t\alpha_1 \Delta \mathbf{v}^1 + t\nabla\big(\alpha_2 \nabla \cdot \mathbf{v}^1 + \alpha_3 \nabla \cdot \mathbf{w}^1\big) + \tilde{\mathbf{f}}_s, \\
\tilde{\mathbf{g}}^1 = \nabla(\alpha_3 \nabla \cdot \mathbf{v}^0 + \alpha_4 \nabla \cdot \mathbf{w}^0), \quad \tilde{\mathbf{g}}^2 = -\omega(1 * k)\mathbf{w}^1, \\
\tilde{\mathbf{g}}^3 = t\nabla(\alpha_3 \nabla \cdot \mathbf{v}^1 + \alpha_4 \nabla \cdot \mathbf{w}^1) + \tilde{\mathbf{f}}_f,
\end{gathered} \tag{3.10}$$

with $\tilde{\mathbf{f}}_s = \mathbf{f}_s$ and $\tilde{\mathbf{f}}_f = \mathbf{f}_f$. Extend the unknowns $(\tilde{\mathbf{v}}, \widetilde{\mathbf{w}})$ to $\mathbb{R}_- \times \Omega$ by zero and denote this continuation by $(\mathbf{v}, \mathbf{w})$. Do the same with our data and remove the tildes over the extended functions.

Let us state the problem: *Find a pair of functions* $(\mathbf{v}, \mathbf{w}) \in \mathscr{V}(\xi_0) \times \mathscr{W}(\xi_0)$ *satisfying the equations:*

$$\rho_1 \partial_t^2 \mathbf{v} + \rho_2 \partial_t^2 \mathbf{w} - \alpha_1 \Delta \mathbf{v} - \nabla(\alpha_2 \nabla \cdot \mathbf{v} + \alpha_3 \nabla \cdot \mathbf{w}) = \mathbf{f}^1 + \mathbf{f}^3, \quad (t, \mathbf{x}) \in Q, \tag{3.11}$$

$$\rho_2 \partial_t^2 \mathbf{v} + \rho_3 \partial_t^2 \mathbf{w} + k * \left[\partial_t^2 \mathbf{w} + \omega \partial_t \mathbf{w}\right] - \nabla(\alpha_3 \nabla \cdot \mathbf{v} + \alpha_4 \nabla \cdot \mathbf{w}) = \mathbf{g}^1 + \mathbf{g}^2 + \mathbf{g}^3, \ (t, \mathbf{x}) \in Q, \tag{3.12}$$

$$\mathbf{v} = \mathbf{0}, \quad \mathbf{n} \cdot \mathbf{w} = 0, \quad (t, \mathbf{x}) \in \Gamma, \tag{3.13}$$

where $Q = \mathbb{R} \times \Omega$ *and* $\Gamma = \mathbb{R} \times \partial\Omega$.

We assume that the constants ρ_j for $j = 1, 2, 3$ and α_j for $j = 1, 2, 3, 4$ satisfy

$$\rho_1 > 0, \ \rho_3 > 0, \ \rho_1\rho_3 - \rho_2^2 > 0, \alpha_1 + \alpha_2 > 0, \ \alpha_4 > 0, \ (\alpha_1 + \alpha_2)\alpha_4 - \alpha_3^2 > 0. \tag{3.14}$$

Finally, we assume that problem (3.11)–(3.13) satisfies the smoothness conditions

$$\mathbf{v}^0 \in H^2(\Omega)^N \cap H_0^1(\Omega)^N, \quad \mathbf{v}^1 \in H^2(\Omega)^N, \quad \mathbf{w}^0, \mathbf{w}^1 \in L^2(\Omega)^N, \tag{3.15}$$

$$\operatorname{div} \mathbf{v}^0, \operatorname{div} \mathbf{w}^0 \in H^1(\Omega), \quad \mathbf{n} \cdot \mathbf{w}^0 = 0, \quad \mathbf{x} \in \partial\Omega, \quad \tilde{\mathbf{f}}_s, \tilde{\mathbf{f}}_f \in L^2_{\xi_0}(\mathbb{R}_+; L^2(\Omega)^N), \tag{3.16}$$

$$k \in L^1_{\xi_0}(\mathbb{R}_+; \mathbb{R}), \quad \operatorname{Re}[(\tau + \omega)\hat{k}(\tau)] \geq c, \quad \operatorname{Re} \tau = \xi_0, \tag{3.17}$$

where $\hat{k}$ and c stand respectively for the Laplace transform of the kernel k and some nonnegative constant.

Remark 3.2. If $k \in L^1_{\xi_0}(\mathbb{R}_+; \mathbb{R})$ then $1 * k \in L^2_{\xi_0}(\mathbb{R}_+; \mathbb{R})$.

Moreover, recall that $L^p_{\xi_0}(\mathbb{R}_+; X) \hookrightarrow L^p_{\xi}(\mathbb{R}_+; X)$ for all $p \in [1, +\infty]$, $\xi > \xi_0$, and every Banach space X.

Remark 3.3. The expression $k(t) = c(\gamma, \omega)t^{-1+\gamma}e^{-t\omega}$ with $\gamma \in (0, 1)$ yields simple examples of kernels k with weak singularity at $t = 0$ satisfying (3.17). More general functions k are of the form

$$k(t) = \sum_{j=1}^{+\infty} c_j t^{-1+\gamma_j} e^{-t\omega_j}, \tag{3.18}$$

where $\{\gamma_j\}_{j=1}^{+\infty}$ is a sequence in $[\gamma_0, 1]$ with $\gamma_0 > 0$ and $\{\omega_j\}_{j=1}^{+\infty}$ is a sequence in $\big(0, \omega\big]$ with $\omega > 0$, while $\{c_j\}_{j=1}^{+\infty}$ is a *nonnegative* real sequence such that

$$\sum_{j=1}^{+\infty} c_j \Gamma(\gamma_j)(\xi_0 + \omega_j)^{-\gamma_j} < +\infty.$$

This condition ensures that $k \in L^1_{\xi_0}(\mathbb{R}_+; \mathbb{R})$. Since the Laplace transform of k is given by the formula

$$\hat{k}(\tau) = \sum_{j=1}^{+\infty} c_j \Gamma(\gamma_j)(\tau + \omega_j)^{-\gamma_j}$$

for $\operatorname{Re}\tau = \xi_0 > 0$, we have

$$\begin{aligned} &\operatorname{Re}[(\tau+\omega)\hat{k}(\tau)] = \operatorname{Re}[(\tau+\omega_j)\hat{k}(\tau)] + (\omega - \omega_j)\operatorname{Re}\hat{k}(\tau) \\ &\geq \sum_{j=1}^{+\infty} c_j \Gamma(\gamma_j)(\xi + \omega_j)^{1-\gamma_j} \cos[(1-\gamma_0)\pi/2] =: c, \quad \operatorname{Re}\tau = \xi \geq \xi_0. \end{aligned} \tag{3.19}$$

Observe that the sequence in (3.19) converges for all $\tau = \xi + iy$, where $y \in \mathbb{R}$ and $\xi \geq \xi_0$.

Uniqueness and continuous dependence.

Theorem 3.1. *Assume that* (3.14)–(3.17) *are satisfied. Then the solution to* (3.11)–(3.13) *is unique in* $\mathscr{V}(\xi_0) \times \mathscr{W}(\xi_0)$ *and satisfies the following estimate of continuous dependence for all* $\xi \geq \xi_0$*:*

$$\begin{aligned} &\int_0^{+\infty} e^{-2\xi t}\big[\|\partial_t \mathbf{v}(t,\cdot)\|^2_{L^2(\Omega)^N} + \|\partial_t \mathbf{w}(t,\cdot)\|^2_{L^2(\Omega)^N}\big]\,dt \\ &+ \int_0^{+\infty} e^{-2\xi t}\big[\|\nabla \mathbf{v}(t,\cdot)\|^2_{L^2(\Omega)^N} + \|\nabla\cdot \mathbf{w}(t,\cdot)\|^2_{L^2(\Omega)}\big]\,dt \\ &\leq C\bigg\{\|\mathbf{v}^0\|^2_{H^2(\Omega)^N} + \|\mathbf{v}^1\|^2_{H^2(\Omega)^N} + \|\mathbf{w}^0\|^2_{L^2(\Omega)^N} + \|\mathbf{w}^1\|^2_{L^2(\Omega)^N} + \|\operatorname{div}\mathbf{w}^0\|^2_{H^1(\Omega)} \\ &+\|\operatorname{div}\mathbf{w}^1\|^2_{H^1(\Omega)} + \bigg[\int_0^{+\infty} e^{-\xi t}[\|\mathbf{f}_s(t,\cdot)\|_{L^2(\Omega)^N} + \|\mathbf{f}_f(t,\cdot)\|_{L^2(\Omega)^N}]\,dt\bigg]^2\bigg\}, \end{aligned} \tag{3.20}$$

where $N = 1, 2, 3$ *and the positive constant* C *depends only on* k, ξ, ω, ρ_1, ρ_2, ρ_3, α_1, α_2, α_3, *and* α_4.

Theorem 3.1 directly implies the uniqueness and continuous dependence of the solution to problem (3.6). In this case $k(t) = \eta\kappa^{-1}(\omega\pi t)^{-1/2}e^{-\omega t}$; hence, it suffices to put $\rho_1 = \rho$, $\rho_2 = \rho_f$, $\rho_3 = \rho_w$, $\alpha_1 = \mu$, $\alpha_2 = \lambda_f + \mu$, $\alpha_3 = m\beta$, and $\alpha_4 = m$.

Theorem 3.2. *Under conditions* (3.5) and (3.15)–(3.17) *the solution to problem* (3.6) *is unique in* $\mathscr{V}(\xi_0) \times \mathscr{W}(\xi_0)$ *and satisfies* (3.20).

Proof of Theorem 3.1. Take a solution $(\mathbf{v}, \mathbf{w}) \in \mathscr{V}(\xi_0) \times \mathscr{W}(\xi_0)$ to (3.11)–(3.13). Applying the Laplace transform to both sides of (3.11)–(3.13), for $\xi > \xi_0$ we obtain

$$\begin{aligned} &\tau[\rho_1\widehat{(\partial_t\mathbf{v})}(\tau,\mathbf{x}) + \rho_2\widehat{(\partial_t\mathbf{w})}(\tau,\mathbf{x})] - \alpha_1\Delta\hat{\mathbf{v}}(\tau,\mathbf{x}) - \alpha_2\nabla(\nabla\cdot\hat{\mathbf{v}})(\tau,\mathbf{x}) \\ &-\alpha_3\nabla(\nabla\cdot\widehat{\mathbf{w}})(\tau,\mathbf{x}) = \hat{\mathbf{f}}^1(\tau,\mathbf{x}) + \hat{\mathbf{f}}^3(\tau,\mathbf{x}), \quad \operatorname{Re}\tau = \xi,\ \mathbf{x}\in\Omega, \end{aligned} \tag{3.21}$$

$$\begin{aligned} &\tau[\rho_2\widehat{(\partial_t\mathbf{v})}(\tau,\mathbf{x}) + \rho_3\widehat{(\partial_t\mathbf{w})}(\tau,\mathbf{x})] + (\tau+\omega)\hat{k}(\tau)\widehat{(\partial_t\mathbf{w})}(\tau,\mathbf{x}) - \alpha_3\nabla(\nabla\cdot\hat{\mathbf{v}})(\tau,\mathbf{x}) \\ &-\alpha_4\nabla\big(\nabla\cdot\widehat{\mathbf{w}}\big)(\tau,\mathbf{x}) = \hat{\mathbf{g}}^1(\tau,\mathbf{x}) + \hat{\mathbf{g}}^2(\tau,\mathbf{x}) + \hat{\mathbf{g}}^3(\tau,\mathbf{x}), \quad \operatorname{Re}\tau = \xi,\ \mathbf{x}\in\Omega, \end{aligned} \tag{3.22}$$

$$\hat{\mathbf{v}}(\tau,\mathbf{x}) = \mathbf{0}, \quad \mathbf{n}\cdot\widehat{\mathbf{w}}(\tau,\mathbf{x}) = 0, \quad \operatorname{Re}\tau = \xi,\ \mathbf{x}\in\partial\Omega. \tag{3.23}$$

Observe that the transformed equations involve only ordinary functions.

Take the inner product of (3.21) and (3.22) with $(\partial_t \mathbf{v})\widehat{\ }(\tau, \mathbf{x}) = \tau \hat{\mathbf{v}}(\tau, \mathbf{x})$ and $(\partial_t \mathbf{w})\widehat{\ }(\tau, \mathbf{x}) = \tau \widehat{\mathbf{w}}(\tau, \mathbf{x})$ respectively, then integrate over $\mathbf{x} \in \Omega$. Adding up termwise and using the boundary conditions (3.23), we arrive at the following expression (recall that now the inner product is taken in $\mathbb{C}^N$):

$$\begin{aligned}
\tau \int_\Omega &\{\rho_1 |(\partial_t \mathbf{v})\widehat{\ }(\tau, \mathbf{x})|^2 + 2\rho_2 \operatorname{Re}[(\partial_t \mathbf{v})\widehat{\ }(\tau, \mathbf{x}) \cdot (\partial_t \mathbf{w})\widehat{\ }(\tau, \mathbf{x})] + \rho_3 |(\partial_t \mathbf{w})\widehat{\ }(\tau, \mathbf{x})|^2\}\, d\mathbf{x} \\
&+ \bar{\tau} \int_\Omega [\alpha_1 |\nabla \hat{\mathbf{v}}(\tau, \mathbf{x})|^2 + \alpha_2 |\nabla \cdot \hat{\mathbf{v}}(\tau, \mathbf{x})|^2 + \alpha_4 |\nabla \cdot \widehat{\mathbf{w}}(\tau, \mathbf{x})|^2]\, d\mathbf{x} \\
&+ 2\alpha_3 \bar{\tau} \int_\Omega \operatorname{Re}[\nabla \cdot \hat{\mathbf{v}}(\tau, \mathbf{x}) \nabla \cdot \overline{\widehat{\mathbf{w}}(\tau, \mathbf{x})}]\, d\mathbf{x} + (\tau + \omega)\hat{\mathbf{k}}(\tau) \int_\Omega |(\partial_t \mathbf{w})\widehat{\ }(\tau, \mathbf{x})|^2\, d\mathbf{x} \\
&= \int_\Omega [(\hat{\mathbf{f}}^1 + \hat{\mathbf{f}}^3)(\tau, \mathbf{x}) \cdot (\partial_t \mathbf{v})\widehat{\ }(\tau, \mathbf{x}) + (\hat{\mathbf{g}}^1 + \hat{\mathbf{g}}^2 + \hat{\mathbf{g}}^3)(\tau, \mathbf{x}) \cdot (\partial_t \mathbf{w})\widehat{\ }(\tau, \mathbf{x})]\, d\mathbf{x}. \qquad (3.24)
\end{aligned}$$

Observe that

$$\operatorname{Re}[\nabla \cdot \hat{\mathbf{v}}(\tau, \mathbf{x}) \nabla \cdot \overline{\widehat{\mathbf{w}}(\tau, \mathbf{x})}] \geq -|\nabla \cdot \hat{\mathbf{v}}(\tau, \mathbf{x})||\nabla \cdot \widehat{\mathbf{w}}(\tau, \mathbf{x})|.$$

Considering only the real part of (3.24) and using the assumption on k, we arrive at the integral inequality

$$\begin{aligned}
\xi \int_\Omega &[\rho_1 |(\partial_t \mathbf{v})\widehat{\ }(\tau, \mathbf{x})|^2 + 2\rho_2 \operatorname{Re}[(\partial_t \mathbf{v})\widehat{\ }(\tau, \mathbf{x}) \cdot (\partial_t \mathbf{w})\widehat{\ }(\tau, \mathbf{x})] + \rho_3 |(\partial_t \mathbf{w})\widehat{\ }(\tau, \mathbf{x})|^2]\, d\mathbf{x} \\
&+ \xi \int_\Omega [\alpha_1 |\nabla \hat{\mathbf{v}}(\tau, \mathbf{x})|^2 + \alpha_2 |\nabla \cdot \hat{\mathbf{v}}(\tau, \mathbf{x})|^2 + \alpha_4 |\nabla \cdot \widehat{\mathbf{w}}(\tau, \mathbf{x})|^2]\, d\mathbf{x} \\
&- 2\alpha_3 \xi \int_\Omega |\nabla \cdot \hat{\mathbf{v}}(\tau, \mathbf{x})||\nabla \cdot \widehat{\mathbf{w}}(\tau, \mathbf{x})|\, d\mathbf{x} \\
&\leq \operatorname{Re} \int_\Omega [(\hat{\mathbf{f}}^1 + \hat{\mathbf{f}}^3)(\tau, \mathbf{x}) \cdot (\partial_t \mathbf{v})\widehat{\ }(\tau, \mathbf{x}) + (\hat{\mathbf{g}}^1 + \hat{\mathbf{g}}^2 + \hat{\mathbf{g}}^3)(\tau, \mathbf{x}) \cdot (\partial_t \mathbf{w})\widehat{\ }(\tau, \mathbf{x})]\, d\mathbf{x} \\
&\leq 2\sqrt{3} \int_\Omega [|\hat{\mathbf{f}}^1(\tau, \mathbf{x})|^2 + |\hat{\mathbf{f}}^3(\tau, \mathbf{x})|^2 + |\hat{\mathbf{g}}^1(\tau, \mathbf{x})|^2 + |\hat{\mathbf{g}}^2(\tau, \mathbf{x})|^2 + |\hat{\mathbf{g}}^3(\tau, \mathbf{x})|^2]^{1/2} \\
&\qquad \times [|(\partial_t \mathbf{v})\widehat{\ }(\tau, \mathbf{x})|^2 + |(\partial_t \mathbf{w})\widehat{\ }(\tau, \mathbf{x})|^2]^{1/2}\, d\mathbf{x}. \qquad (3.25)
\end{aligned}$$

Using (3.14), we arrive at the inequality

$$\begin{aligned}
\rho_1 |(\partial_t \mathbf{v})\widehat{\ }(\tau, \mathbf{x})|^2 &+ 2\rho_2 \operatorname{Re}[(\partial_t \mathbf{v})\widehat{\ }(\tau, \mathbf{x}) \cdot (\partial_t \mathbf{w})\widehat{\ }(\tau, \mathbf{x})] + \rho_3 |(\partial_t \mathbf{v})\widehat{\ }(\tau, \mathbf{x})|^2 \\
&\geq \lambda [|(\partial_t \mathbf{v})\widehat{\ }(\tau, \mathbf{x})|^2 + (\partial_t \mathbf{w})\widehat{\ }(\tau, \mathbf{x})|^2], \qquad (3.26)
\end{aligned}$$

where λ is the smallest positive solution to the equation $\rho_1 \lambda^2 - 2\rho_2 \lambda + \rho_3 = 0$; i.e.,

$$\lambda = \frac{1}{2}\big\{\rho_1 + \rho_3 - \big[(\rho_1 - \rho_3)^2 + 4\rho_2^2\big]^{1/2}\big\} > 0.$$

Define δ_N for $N = 1, 2, 3$ in accordance with the following rule (see (3.4)):

$$\delta_1 = 0, \quad \delta_N \in \big(0, \min\{\alpha_1, \alpha_1 + \alpha_2 - \alpha_3^2 \alpha_4^{-1}\}\big), \quad N = 2, 3. \qquad (3.27)$$

Recalling that

$$\|\nabla\cdot\mathbf{z}\|^2_{L^2(\Omega)} + \|\operatorname{rot}\mathbf{z}\|^2_{L^2(\Omega)^N} = \|\nabla\mathbf{z}\|^2_{L^2(\Omega)^{N^2}}, \quad \mathbf{z}\in H^1_0(\Omega)^{N^2},\ N=2,3, \tag{3.28}$$

we see that

$$\begin{aligned}
&\int_\Omega [(\alpha_1-\delta_N)|\nabla\hat{\mathbf{v}}(\tau,\mathbf{x})|^2 + \alpha_2|\nabla\cdot\hat{\mathbf{v}}(\tau,\mathbf{x})|^2 + \alpha_4|\nabla\cdot\widehat{\mathbf{w}}(\tau,\mathbf{x})|^2]\,d\mathbf{x} \\
&-2\alpha_3\int_\Omega |\nabla\cdot\hat{\mathbf{v}}(\tau,\mathbf{x})||\nabla\cdot\widehat{\mathbf{w}}(\tau,\mathbf{x})|\,d\mathbf{x} \ge \int_\Omega [(\alpha_1+\alpha_2-\delta_N)|\nabla\cdot\hat{\mathbf{v}}(\tau,\mathbf{x})|^2 \\
&\qquad -2\alpha_3|\nabla\cdot\hat{\mathbf{v}}(\tau,\mathbf{x})||\nabla\cdot\widehat{\mathbf{w}}(\tau,\mathbf{x})| + \alpha_4|\nabla\cdot\widehat{\mathbf{w}}(\tau,\mathbf{x})|^2]\,d\mathbf{x} \\
&\qquad \ge \zeta_N\int_\Omega [|\nabla\cdot\hat{\mathbf{v}}(\tau,\mathbf{x})|^2 + |\nabla\cdot\widehat{\mathbf{w}}(\tau,\mathbf{x})|^2]\,d\mathbf{x},
\end{aligned}$$

where ζ_N is the smallest positive solution to the equation

$$\zeta^2 - 2(\alpha_1+\alpha_2+\alpha_4-\delta_N)\zeta + \left[(\alpha_1+\alpha_2-\delta_N)\alpha_4 - \alpha_3^2\right] = 0;$$

see (3.14) and (3.27). Consequently, since $2ab \le \varepsilon a^2 + \varepsilon^{-1}b^2$ with $\varepsilon = \xi\lambda/(2\sqrt{3})$ we infer from (3.25) and (3.26) that

$$\begin{aligned}
&\xi\lambda\int_\Omega [|\widehat{(\partial_t\mathbf{v})}(\tau,\mathbf{x})|^2 + |\widehat{(\partial_t\mathbf{w})}(\tau,\mathbf{x})|^2]\,d\mathbf{x} \\
&+\xi\int_\Omega [\delta_N|\nabla\hat{\mathbf{v}}(\tau,\mathbf{x})|^2 + \zeta_N|\nabla\cdot\hat{\mathbf{v}}(\tau,\mathbf{x})|^2 + \zeta_N|\nabla\cdot\widehat{\mathbf{w}}(\tau,\mathbf{x})|^2]\,d\mathbf{x} \\
&\le 6(\xi\lambda)^{-1}\int_\Omega [|\hat{\mathbf{f}}^1(\tau,\mathbf{x})|^2 + |\hat{\mathbf{f}}^3(\tau,\mathbf{x})|^2 + |\hat{\mathbf{g}}^1(\tau,\mathbf{x})|^2 + |\hat{\mathbf{g}}^2(\tau,\mathbf{x})|^2 + |\hat{\mathbf{g}}^3(\tau,\mathbf{x})|^2]\,d\mathbf{x} \\
&\qquad +\frac{\xi\lambda}{2}\int_\Omega [|\widehat{(\partial_t\mathbf{v})}(\tau,\mathbf{x})|^2 + |\widehat{(\partial_t\mathbf{w})}(\tau,\mathbf{x})|^2]\,d\mathbf{x}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\xi\lambda}{2}\int_\Omega [|\widehat{(\partial_t\mathbf{v})}(\tau,\mathbf{x})|^2 + |\widehat{(\partial_t\mathbf{w})}(\tau,\mathbf{x})|^2]\,d\mathbf{x} \\
&+\xi\int_\Omega [\delta_N|\nabla\hat{\mathbf{v}}(\tau,\mathbf{x})|^2 + \zeta_N|\nabla\cdot\hat{\mathbf{v}}(\tau,\mathbf{x})|^2 + \zeta_N|\nabla\cdot\widehat{\mathbf{w}}(\tau,\mathbf{x})|^2]\,d\mathbf{x} \\
&\le 6(\xi\lambda)^{-1}\int_\Omega [|\hat{\mathbf{f}}^1(\tau,\mathbf{x})|^2 + |\hat{\mathbf{f}}^3(\tau,\mathbf{x})|^2 + |\hat{\mathbf{g}}^1(\tau,\mathbf{x})|^2 + |\hat{\mathbf{g}}^2(\tau,\mathbf{x})|^2 + |\hat{\mathbf{g}}^3(\tau,\mathbf{x})|^2]\,d\mathbf{x} \\
&\qquad =: 6(\xi\lambda)^{-1}\Phi(\hat{\mathbf{f}}_s,\hat{\mathbf{f}}_f)(\tau).
\end{aligned}$$

Integrating the previous inequality along $\operatorname{Re}\tau = \xi \geq \xi_0$, we arrive at

$$\frac{\xi\lambda}{2}\int\limits_{\xi-i\infty}^{\xi+i\infty} d\tau \int\limits_{\Omega} [|(\partial_t \mathbf{v})\widehat{\ }(\tau,\mathbf{x})|^2 + |(\partial_t \mathbf{w})\widehat{\ }(\tau,\mathbf{x})|^2]\, d\mathbf{x}$$

$$+\xi \int\limits_{\xi-i\infty}^{\xi+i\infty} d\tau \int\limits_{\Omega} [\delta_N|\nabla\hat{\mathbf{v}}(\tau,\mathbf{x})|^2 + \zeta_N|\nabla\cdot\hat{\mathbf{v}}(\tau,\mathbf{x})|^2 + \zeta_N|\nabla\cdot\widehat{\mathbf{w}}(\tau,\mathbf{x})|^2]\, d\mathbf{x}$$

$$\leq 6(\xi\lambda)^{-1} \int\limits_{\xi-i\infty}^{\xi+i\infty} \Phi(\hat{\mathbf{f}}_s, \hat{\mathbf{g}}_f)(\tau)\, d\tau. \tag{3.29}$$

The last step consists in estimating the vector functions $\hat{\tilde{\mathbf{f}}}^j$ for $j = 1,3$ and $\hat{\tilde{\mathbf{g}}}^j$ for $j = 1,2,3$ in $L^2((\xi - i\infty, \xi + i\infty); L^2(\Omega)^N)$ in terms of the data. To this end, from (3.15)–(3.17) we infer for all τ with $\operatorname{Re}\tau = \xi$ that

$$\|\hat{\tilde{\mathbf{f}}}^1(\tau,\cdot)\|_{L^2(\Omega)^N} \leq |\tau|^{-1}\|\tilde{\mathbf{f}}^1\|_{L^2(\Omega)^N}, \|\hat{\tilde{\mathbf{f}}}^3(\tau,\cdot)\|_{L^2(\Omega)^N}$$
$$\leq |\tau|^{-2}\|\mathbf{v}^2\|_{L^2(\Omega)^N} + \|\hat{\tilde{\mathbf{f}}}_s(\tau,\cdot)\|_{L^2(\Omega)^N},$$

$$\|\hat{\tilde{\mathbf{g}}}^1(\tau,\cdot)\|_{L^2(\Omega)^N} \leq |\tau|^{-1}\|\tilde{\mathbf{g}}^1\|_{L^2(\Omega)^N},$$

$$\|\hat{\tilde{\mathbf{g}}}^2(\tau,\cdot)\|_{L^2(\Omega)^N} \leq \omega|\tau|^{-1}|\hat{k}(\tau)|\|\mathbf{w}^1\|_{L^2(\Omega)^N},$$

$$\|\hat{\tilde{\mathbf{g}}}^3(\tau,\cdot)\|_{L^2(\Omega)^N} \leq |\tau|^{-2}\|\mathbf{w}^2\|_{L^2(\Omega)^N} + \|\hat{\tilde{\mathbf{f}}}_f(\tau,\cdot)\|_{L^2(\Omega)^N},$$

where

$$\mathbf{v}^2 = \alpha_1\Delta\mathbf{v}^1 + \nabla(\alpha_2\nabla\cdot\mathbf{v}^1 + \alpha_3\nabla\cdot\mathbf{w}^1), \quad \mathbf{w}^2 = \nabla(\alpha_3\nabla\cdot\mathbf{v}^1 + \alpha_4\nabla\cdot\mathbf{w}^1).$$

Consequently,

$$\int\limits_{\xi-i\infty}^{\xi+i\infty} \Phi(\hat{\mathbf{f}}_s, \hat{\mathbf{f}}_f)(\tau)\, d\tau \leq \frac{\xi^{-1}}{2}\left[\|\mathbf{f}^1\|^2_{L^2(\Omega)^N} + \|\mathbf{g}^1\|^2_{L^2(\Omega)^N}\right]$$

$$+\omega^2\|\mathbf{w}^1\|^2_{L^2(\Omega)^N} \int\limits_{\xi-i\infty}^{\xi+i\infty} |\tau|^{-2}|\hat{k}(\tau)|^2\, d\tau + \frac{\xi^{-3}}{4}\left[\|\mathbf{v}^2\|^2_{L^2(\Omega)^N} + \|\mathbf{w}^2\|^2_{L^2(\Omega)^N}\right]$$

$$+ \int\limits_{\xi-i\infty}^{\xi+i\infty} \left[\|\hat{\tilde{\mathbf{f}}}_s(\tau,\cdot)\|^2_{L^2(\Omega)^N} + \|\hat{\tilde{\mathbf{f}}}_f(\tau,\cdot)\|^2_{L^2(\Omega)^N}\right] d\tau.$$

Apply Parseval's equality,

$$\int\limits_0^{+\infty} e^{-2\tau t}|\mathbf{z}(t,\mathbf{x})|^2\, dt = (2\pi)^{-1}\int\limits_{-\infty}^{+\infty} |\hat{\mathbf{z}}(\xi_0 + iy, \mathbf{x})|^2\, dy.$$

The assumption $\tilde{\mathbf{f}}_s, \tilde{\mathbf{f}}_f \in L^2_{\xi_0}(\mathbb{R}_+; L^2(\Omega)^N)$ implies that $\tilde{\mathbf{f}}_s, \tilde{\mathbf{f}}_f \in L^2_{\xi}(\mathbb{R}_+; L^2(\Omega)^N)$ for all $\xi \geq \xi_0$. The same holds for the extended vector functions $\mathbf{f}_s$ and $\mathbf{f}_f$. Therefore, for all $\xi > \xi_0$ we have

$$\begin{aligned}
&\int_{\xi-i\infty}^{\xi+i\infty} \left[\|\hat{\tilde{\mathbf{f}}}_s(\tau,\cdot)\|^2_{L^2(\Omega)^N} + \|\hat{\tilde{\mathbf{f}}}_f(\tau,\cdot)\|^2_{L^2(\Omega)^N}\right] d\tau \\
&\leq \int_0^{+\infty} e^{-2\xi t}\left[\|\mathbf{f}_s(t,\cdot)\|^2_{L^2(\Omega)^N} + \|\mathbf{f}_f(t,\cdot)\|^2_{L^2(\Omega)^N}\right] dt \\
&\leq \int_0^{+\infty} e^{-2\xi_0 t}\left[\|\mathbf{f}_s(t,\cdot)\|^2_{L^2(\Omega)^N} + \|\mathbf{f}_f(t,\cdot)\|^2_{L^2(\Omega)^N}\right] dt. \qquad (3.30)
\end{aligned}$$

In particular, we conclude from (3.29) and (3.30) that

$$(\partial_t \mathbf{v})\widehat{\ }, (\partial_t \mathbf{w})\widehat{\ } \in L^2_\xi((\xi - i\infty, \xi + i\infty); L^2(\Omega)^N)$$

for all $\xi \geq \xi_0$. By Parseval's equality, (3.29) and (3.30) yield

$$\begin{aligned}
&\frac{\xi\lambda}{2}\int_0^{+\infty} e^{-2\xi t}\left[\|\partial_t \mathbf{v}(t,\cdot)\|^2_{L^2(\Omega)^N} + \|\partial_t \mathbf{w}(t,\cdot)\|^2_{L^2(\Omega)^N}\right] dt \\
&+\xi \int_0^{+\infty} e^{-2\xi t} dt \int_\Omega [\delta_N |\nabla \mathbf{v}(t,\mathbf{x})|^2 + \zeta_N |\nabla\cdot \mathbf{v}(t,\mathbf{x})|^2 + \zeta_N |\nabla \cdot \mathbf{w}(t,\mathbf{x})|^2]\, d\mathbf{x} \\
&\leq 3\xi^{-2}\lambda^{-1}\left[\|\mathbf{f}^1\|^2_{L^2(\Omega)^N} + \|\mathbf{g}^1\|^2_{L^2(\Omega)^N}\right] \\
&+6\omega^2(\xi\lambda)^{-1}\|\mathbf{w}^1\|^2_{L^2(\Omega)^N}\int_0^{+\infty} e^{-2\xi t}|1 * \hat{k}(t)|^2\, dt + \frac{3}{2}\xi^{-4}\lambda^{-1}\left[\|\mathbf{v}^2\|^2_{L^2(\Omega)^N} + \|\mathbf{w}^2\|^2_{L^2(\Omega)^N}\right] \\
&+6(\xi\lambda)^{-1}\int_0^{+\infty} e^{-2\xi t}\left[\|\mathbf{f}_s(t,\cdot)\|^2_{L^2(\Omega)^N} + \|\mathbf{f}_f(t,\cdot)\|^2_{L^2(\Omega)^N}\right] dt. \qquad (3.31)
\end{aligned}$$

Passing to the limit as $\xi \to \xi_0+$ in (3.31), we conclude that this estimate holds with ξ replaced by ξ_0. Consequently, (3.30) holds for all $\xi \geq \xi_0$. Now it is easy to obtain (3.20), which implies the uniqueness and continuous dependence of the solution on the data of the problem. $\square$

Uniqueness and continuous dependence in a bounded temporal interval. In this subsection we need stronger restrictions on the kernel k. Assume that k is a *totally monotone function* on $\mathbb{R}_+$ and belongs to some L^p-space, namely

$$\begin{gathered}
k \in C^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+) \cap L^p(0,T) \quad \text{for some } p \in (1, +\infty], \\
(-1)^n k^{(n)}(t) \geq 0, \quad t \in \mathbb{R}_+,\ n \in \mathbb{N}.
\end{gathered} \qquad (3.32)$$

Then the function

$$l(t) = \int_t^{+\infty} k(s)\, ds, \quad t \in \mathbb{R}_+, \qquad (3.33)$$

belongs to $C^\infty(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ and is totally monotone.

REMARK 3.4. The kernels defined in (3.11) are totally monotone. Moreover, suppose that $\{c_j\}_{j=0}^{+\infty}$ is a *nonnegative* sequence defining the entire function $\phi(z) = \sum_{j=0}^{+\infty} c_j z^j$ and that k is a totally monotone function. Then $\phi \circ k$ is also totally monotone; see [11, Theorem 3].

For more general functions k defined in (3.12), we need the following assumption on the *nonnegative* constants c_j valid for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$:

$$\sum_{j=1}^{+\infty} c_j \left[\omega^n + \sum_{l=1}^{n} (1-\gamma_j)\dots(l-\gamma_j) \binom{n}{l} t^{-l}\omega^{n-l}\right] < +\infty.$$

This condition is satisfied when either $c_j = 0$ for all $j \geq j_0$ or

$$\sum_{j=1}^{+\infty} c_j (1-\gamma_j)\dots(l-\gamma_j) < +\infty, \quad l \in \mathbb{N}.$$

Let us give a weak statement of problem (3.17)–(3.19): *Find a pair of vector functions* $(\mathbf{v}, \mathbf{w})$ *satisfying for all* $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in H_0^1(\Omega)^N \times L_0^2(\Omega; \mathrm{div})$ *the equality*

$$\begin{gathered}
\rho_1\big(\partial_t^2 \mathbf{v}(t,\cdot), \boldsymbol{\phi}\big)_{0,2} + \rho_2\big(\partial_t^2 \mathbf{w}(t,\cdot), \boldsymbol{\phi}\big)_{0,2} + \rho_2\big(\partial_t^2 \mathbf{v}(t,\cdot), \boldsymbol{\psi}\big)_{0,2} \\
+\rho_3\big(\partial_t^2 \mathbf{w}(t,\cdot), \boldsymbol{\psi}\big)_{0,2} + \big[k * \big(\partial_t^2 \mathbf{w} + \omega\partial_t \mathbf{w}, \boldsymbol{\psi}\big)_{0,2}\big](t) + \alpha_1(\nabla \mathbf{v}(t,\cdot), \nabla\boldsymbol{\phi})_{0,2} \\
+\alpha_2(\nabla\cdot\mathbf{v}(t,\cdot), \nabla\cdot\boldsymbol{\phi})_{0,2} + \alpha_3(\nabla\cdot\mathbf{w}(t,\cdot), \nabla\cdot\boldsymbol{\phi})_{0,2} + \alpha_3(\nabla\cdot\mathbf{v}(t,\cdot), \nabla\cdot\boldsymbol{\psi})_{0,2} \\
+\alpha_4(\nabla\cdot\mathbf{w}(t,\cdot), \nabla\cdot\boldsymbol{\psi})_{0,2} = (\mathbf{f}^1(t,\cdot) + \mathbf{f}^3(t,\cdot), \boldsymbol{\phi})_{0,2} \\
+(\mathbf{g}^1(t,\cdot) + \mathbf{g}^2(t,\cdot) + \mathbf{g}^3(t,\cdot), \boldsymbol{\psi})_{0,2} \quad \textit{for almost all } t \in (0,T),
\end{gathered} \tag{3.34}$$

where $(\cdot,\cdot)_{0,2} = (\cdot,\cdot)_{L^2(\Omega)^N}$.

Make the following assumptions:

$$\|k\|_{L^1(0,T)} < \rho_3, \;\; \rho_1(\rho_3 - \|k\|_{L^1(0,T)}) - \rho_2^2 > 0, \quad (\alpha_1+\alpha_2)\alpha_4 - \alpha_3^2 > 0. \tag{3.35}$$

REMARK 3.5. The first two inequalities in (3.35) presume that k has to be small in the norm of $L^1(0,T)$.

Theorem 3.3. *Under conditions* (3.8)–(3.11) *and* (3.35), *the solution to problem* (3.34) *in the space*

$$\big[H^2(0,T; L^2(\Omega)^N) \cap L^2\big(0,T; H_0^1(\Omega)^N\big)\big] \times \big[H^2(0,T; L^2(\Omega)^N) \cap L^2\big(0,T; L_0^2(\Omega; \mathrm{div})\big)\big]$$

is unique and satisfies the estimate

$$\begin{gathered}
\|\partial_t \mathbf{v}(t,\cdot)\|^2_{L^2(\Omega)^N} + \|\partial_t \mathbf{w}(t,\cdot)\|^2_{L^2(\Omega)^N} + \|\nabla \mathbf{v}(t,\cdot)\|^2_{L^2(\Omega)^N} + \|\nabla\cdot\mathbf{w}(t,\cdot)\|^2_{L^2(\Omega)} \\
\leq C\Big\{\|\mathbf{v}^0\|^2_{H^2(\Omega)^N} + \|\mathbf{v}^1\|^2_{H^2(\Omega)^N} + \|\mathbf{w}^0\|^2_{L^2(\Omega)^N} + \|\mathbf{w}^1\|^2_{L^2(\Omega)^N} + \|\operatorname{div}\mathbf{w}^0\|^2_{H^1(\Omega)} \\
+\|\operatorname{div}\mathbf{w}^1\|^2_{H^1(\Omega)} + \left[\int_0^T [\|\mathbf{f}_s(t,\cdot)\|_{L^2(\Omega)^N} + \|\mathbf{f}_f(t,\cdot)\|_{L^2(\Omega)^N}]\, dt\right]^2\Big\}
\end{gathered} \tag{3.36}$$

for all $t \in [0,T]$, *where the positive constant* C *depends only on* T, k, ω, ρ_1, ρ_2, ρ_3, α_1, α_2, α_3, *and* α_4.

Suppose that

$$\frac{\eta}{\kappa\omega} < \min\left\{\rho, \frac{\rho\rho_w - \rho_f^2}{\rho}\right\}. \tag{3.37}$$

Theorem 3.4. *Under conditions* (3.8)–(3.11), (3.5), *and* (3.37), *the solution to problem* (3.6) *in the space*

$$\big[H^2(0,T;L^2(\Omega)^N)\cap L^2\big(0,T;H_0^1(\Omega)^N\big)\big]\times\big[H^2(0,T;L^2(\Omega)^N)\cap L^2\big(0,T;L_0^2(\Omega;\mathrm{div})\big)\big]$$

is unique and satisfies the estimate (3.36) *for all* $t\in[0,T]$.

Proof of Theorem 3.4. By hypothesis $k(t)=\eta\kappa^{-1}(\pi\omega t)^{-1/2}e^{-\omega t}$, which implies that (3.32) holds for every $p\in(1,2)$. Moreover, since $\rho_1=\rho$, $\rho_2=\rho_f$, $\rho_3=\rho_w$, $\alpha_1=\mu$, $\alpha_2=\lambda_f+\mu$, $\alpha_3=m\beta$, and $\alpha_4=m$, it follows that k satisfies (3.35) by (3.37), while the third inequality in (3.35) amounts to (3.5). $\square$

In order to prove Theorem 3.3, we need several easy propositions.

Lemma 3.1. *If* $\mathbf{z}\in H^1(0,T;L^2(\Omega)^N)$ *and* $\mathbf{z}(0,\mathbf{x})=\mathbf{0}$ *then*

$$\int_0^\tau \mathbf{z}(t,\mathbf{x})\cdot\left[\int_0^t k(t-s)\partial_t\mathbf{z}(s,\mathbf{x})\,ds\right]dt$$
$$\geq-\left[\frac{1}{2}l(0)+\varepsilon\right]|\mathbf{z}(\tau,\mathbf{x})|^2-\frac{1}{4\varepsilon}\|k\|_{L^1(0,T)}\int_0^\tau k(\tau-s)|\mathbf{z}(s,\mathbf{x})|^2\,ds$$

for all $\tau\in(0,T]$, *all* $\varepsilon\in\mathbb{R}_+$, *and almost all* $\mathbf{x}\in\Omega$, *where* $\cdot$ *stands for the inner product in* $\mathbb{R}^N$.

Lemma 3.2. *If* $k_1\in L^p(0,T)$ *with* $p\in(1,+\infty]$ *and* $q\in L^1(0,T)$ *then every nonnegative function* $\psi\in L^\infty(0,T)$ *satisfying the integral inequality*

$$\psi(\tau)\leq k_1*\psi(\tau)+\int_0^\tau q(t)\psi(t)^{1/2}\,dt,\quad \tau\in[0,T], \tag{3.38}$$

also satisfies

$$\psi(\tau)\leq\frac{1}{4}[1+\|h\|_{L^1(0,T)}]^2\left[\int_0^\tau q(s)\,ds\right]^2,\quad \tau\in[0,T],$$

with $h=\sum_{j=0}^{+\infty}(k_1*)^jk_1$.

Proof of Theorem 3.3. Putting in (3.34) $\boldsymbol{\phi}=\partial_t\mathbf{v}(t,\cdot)$ and $\boldsymbol{\psi}=\partial_t\mathbf{w}(t,\cdot)$ for $t\in[0,T]$, we arrive at the identity valid for almost all $t\in(0,T)$:

$$\begin{aligned}
&\rho_1\big(\partial_t^2\mathbf{v}(t,\cdot),\partial_t\mathbf{v}(t,\cdot)\big)_{0,2}+\rho_2\big(\partial_t^2\mathbf{w}(t,\cdot),\partial_t\mathbf{v}(t,\cdot)\big)_{0,2}\\
&+\rho_2\big(\partial_t^2\mathbf{v}(t,\cdot),\partial_t\mathbf{w}(t,\cdot)\big)_{0,2}+\rho_3\big(\partial_t^2\mathbf{w}(t,\cdot),\partial_t\mathbf{w}(t,\cdot)\big)_{0,2}\\
&+\big[k*\big(\partial_t^2\mathbf{w}+\omega\partial_t\mathbf{w},\partial_t\mathbf{w}(t,\cdot)\big)_{0,2}\big](t)+\alpha_1\big(\nabla\mathbf{v}(t,\cdot),\partial_t\nabla\mathbf{v}(t,\cdot)\big)_{0,2}\\
&+\alpha_2(\nabla\cdot\mathbf{v}(t,\cdot),\partial_t\nabla\cdot\mathbf{v}(t,\cdot))_{0,2}+\alpha_3(\nabla\mathbf{w}(t,\cdot),\partial_t\nabla\cdot\mathbf{v}(t,\cdot))_{0,2}\\
&+\alpha_3(\nabla\cdot\mathbf{v}(t,\cdot),\partial_t\nabla\cdot\mathbf{w}(t,\cdot))_{0,2}+\alpha_4(\nabla\cdot\mathbf{w}(t,\cdot),\partial_t\nabla\cdot\mathbf{w}(t,\cdot))_{0,2}\\
&=(\mathbf{f}^1(t,\cdot)+\mathbf{f}^3(t,\cdot),\partial_t\mathbf{v}(t,\cdot))_{0,2}\\
&+(\mathbf{g}^1(t,\cdot)+\mathbf{g}^2(t,\cdot)+\mathbf{g}^3(t,\cdot),\partial_t\mathbf{w}(t,\cdot))_{0,2}.
\end{aligned}\tag{3.39}$$

Integrate both sides of (3.39) over $(0,\tau)$, with $\tau \in (0,T]$. Standard calculations lead us to the identity valid for all $\tau \in [0,T]$:

$$
\begin{aligned}
&\frac{1}{2}\rho_1\|\partial_t\mathbf{v}(\tau,\cdot)\|_{0,2}^2 + \rho_2(\partial_t\mathbf{v}(\tau,\cdot),\partial_t\mathbf{w}(\tau,\cdot))_{0,2} + \frac{1}{2}\rho_3\|\partial_t\mathbf{w}(\tau,\cdot)\|_{0,2}^2 \\
&\quad + \int_\Omega d\mathbf{x}\int_0^\tau \partial_t\mathbf{w}(t,\mathbf{x})\,dt\int_0^t k(t-s)\partial_t^2\mathbf{w}(s,\mathbf{x})\,ds \\
&\quad +\omega\int_\Omega d\mathbf{x}\int_0^\tau \partial_t\mathbf{w}(t,\mathbf{x})\,dt\int_0^t k(t-s)\partial_t\mathbf{w}(s,\mathbf{x})\,ds + \frac{1}{2}\alpha_1\|\nabla\mathbf{v}(\tau,\cdot)\|_{0,2}^2 \\
&\quad +\frac{1}{2}\alpha_2\|\nabla\cdot\mathbf{v}(\tau,\cdot)\|_{0,2}^2 + \alpha_3(\nabla\cdot\mathbf{v}(\tau,\cdot),\nabla\cdot\mathbf{w}(\tau,\cdot))_{0,2} + \frac{1}{2}\alpha_4\|\nabla\cdot\mathbf{w}(\tau,\cdot)\|_{0,2}^2 \\
&= \int_0^\tau \Big[(\mathbf{f}^1(t,\cdot)+\mathbf{f}^3(t,\cdot),\partial_t\mathbf{v}(t,\cdot))_{0,2} + \sum_{j=1}^3(\mathbf{g}^j(t,\cdot),\partial_t\mathbf{w}(t,\cdot))_{0,2}\Big]\,dt, \qquad (3.40)
\end{aligned}
$$

where $\|\cdot\|_{0,2} = \|\cdot\|_{L^2(\Omega)^N}$.

Using (3.28), (3.33), (3.40), and Lemma 3.1 with $\mathbf{z} = \partial_t\mathbf{w}$, we obtain the integral inequality valid for all $\tau \in (0,T]$:

$$
\begin{aligned}
&\frac{1}{2}\rho_1\|\partial_t\mathbf{v}(\tau,\cdot)\|_{0,2}^2 - \rho_2\|\partial_t\mathbf{v}(\tau,\cdot)\|_{0,2}\|\partial_t\mathbf{w}(\tau,\cdot)\|_{0,2} \\
&\quad +\frac{1}{2}(\rho_3 - \|k\|_{L^1(0,T)} - 2\varepsilon)\|\partial_t\mathbf{w}(\tau,\cdot)\|_{0,2}^2 + \varepsilon\|\nabla\mathbf{v}(\tau,\cdot)\|_{0,2}^2 \\
&\quad +\frac{1}{2}(\alpha_1+\alpha_2-2\varepsilon)\|\nabla\cdot\mathbf{v}(\tau,\cdot)\|_{0,2}^2 + \alpha_3(\nabla\cdot\mathbf{v}(\tau,\cdot),\nabla\cdot\mathbf{w}(\tau,\cdot))_{0,2} \\
&\quad +\frac{1}{2}\alpha_4\|\nabla\cdot\mathbf{w}(\tau,\cdot)\|_{0,2}^2 \le \frac{1}{4\varepsilon}\|k\|_{L^1(0,T)}\int_0^\tau k(\tau-s)|\partial_t\mathbf{w}(s,\mathbf{x})|^2\,ds \\
&\quad + \int_0^\tau \Big[(\tilde{\mathbf{f}}^1(t,\cdot)+\tilde{\mathbf{f}}^3(t,\cdot),\partial_t\mathbf{v}(t,\cdot))_{0,2} + \sum_{j=1}^3(\tilde{\mathbf{g}}^j(t,\cdot),\partial_t\mathbf{w}(t,\cdot))_{0,2}\Big]\,dt. \qquad (3.41)
\end{aligned}
$$

Choose ε so that

$$
0 < \varepsilon < \frac{1}{2}\min\{\rho_3 - \|k\|_{L^1(0,T)}, \alpha_1+\alpha_2\}, \quad \rho_1(\rho_3 - \|k\|_{L^1(0,T)} - 2\varepsilon) - \rho_2^2 > 0; \qquad (3.42)
$$

see (3.35). Denote by λ and ζ the smallest positive solutions to the equations

$$
\begin{gathered}
\lambda^2 - (\rho_1 + \rho_3 - \|k\|_{L^1(0,T)} - 2\varepsilon)\lambda + \rho_1(\rho_3 - \|k\|_{L^1(0,T)} - 2\varepsilon) - \rho_2^2 = 0, \\
\zeta^2 - (\alpha_1+\alpha_2+\alpha_4-2\varepsilon)\zeta + (\alpha_1+\alpha_2-2\varepsilon)\alpha_4 - \alpha_3^2 = 0.
\end{gathered} \qquad (3.43)
$$

Therefore, we infer from (3.41)–(3.43) the integral inequality

$$
\begin{aligned}
&\frac{\lambda}{2}\big[\|\partial_t\mathbf{v}(\tau,\cdot)\|_{0,2}^2 + \|\partial_t\mathbf{w}(\tau,\cdot)\|_{0,2}^2\big] + \frac{\zeta}{2}\big[\|\nabla\cdot\mathbf{v}(\tau,\cdot)\|_{0,2}^2 + \|\nabla\cdot\mathbf{w}(\tau,\cdot)\|_{0,2}^2\big] \\
&\quad +\varepsilon\|\nabla\mathbf{v}(\tau,\cdot)\|_{0,2}^2 \le \frac{1}{4\varepsilon}\|k\|_{L^1(0,T)}\int_0^\tau k(\tau-s)|\partial_t\mathbf{w}(s,\mathbf{x})|^2\,ds \\
&\quad + \int_0^\tau p(t)\big[\|\partial_t\mathbf{v}(t,\cdot)\|_{0,2}^2 + \|\partial_t\mathbf{w}(t,\cdot)\|_{0,2}^2\big]^{1/2}\,dt, \quad \tau \in [0,T], \qquad (3.44)
\end{aligned}
$$

where

$$p(t)=\sqrt{3}\Bigg[\|\tilde{\mathbf{f}}^1(t,\cdot)\|_{0,2}^2+\|\tilde{\mathbf{f}}^3(t,\cdot)\|_{0,2}^2+\sum_{j=1}^{3}\|\tilde{\mathbf{g}}^j(t,\cdot)\|_{0,2}^2\Bigg]^{1/2},\quad t\in(0,T). \tag{3.45}$$

Introduce the auxiliary function

$$\psi(\tau)=\frac{\lambda}{2}\big[\|\partial_t\mathbf{v}(\tau,\cdot)\|_{0,2}^2+\|\partial_t\mathbf{w}(\tau,\cdot)\|_{0,2}^2\big],\quad \tau\in(0,T).$$

Then (3.44) implies the integral inequality

$$\psi(\tau)\le\frac{1}{2\varepsilon\lambda}\|k\|_{L^1(0,T)}\int\limits_0^\tau k(\tau-s)\psi(s)\,ds+\frac{2}{\lambda}\int\limits_0^\tau p(t)\psi(t)^{1/2}\,dt,\quad \tau\in[0,T].$$

Lemma 3.2 shows that

$$\psi(\tau)^{1/2}\le\frac{1}{\lambda}[1+\|h_\varepsilon\|_{L^1(0,T)}]\int\limits_0^\tau p(s)\,ds,\quad \tau\in[0,T],$$

where h_ε is a solution to the convolution equation (3.38) with $k_1=k(2\varepsilon\lambda)^{-1}$.

Consequently, from (3.44) we obtain the final estimate

$$\begin{aligned}
&\frac{\lambda}{2}\big[\|\partial_t\mathbf{v}(\tau,\cdot)\|_{0,2}^2+\|\partial_t\mathbf{w}(\tau,\cdot)\|_{0,2}^2\big]\\
&+\varepsilon\|\nabla\mathbf{v}(\tau,\cdot)\|_{0,2}^2+\frac{\zeta}{2}\big[\|\nabla\cdot\mathbf{v}(\tau,\cdot)\|_{0,2}^2+\|\nabla\cdot\mathbf{w}(\tau,\cdot)\|_{0,2}^2\big]\\
&\le\frac{1}{4\varepsilon\lambda^2}\|k\|_{L^1(0,T)}[1+\|h_\varepsilon\|_{L^1(0,T)}]^2\int\limits_0^\tau k(\tau-s)\Bigg[\int\limits_0^s p(r)\,dr\Bigg]^2ds\\
&+\frac{1}{2\lambda}[1+\|h_\varepsilon\|_{L^1(0,T)}]\Bigg[\int\limits_0^\tau p(t)\,dt\Bigg]^2,\quad \tau\in[0,T].
\end{aligned} \tag{3.46}$$

Moreover, (3.10) yields

$$\begin{gathered}
\|\tilde{\mathbf{f}}^1\|_{0,2}\le|\alpha_1|\|\Delta\mathbf{v}^0\|_{0,2}+|\alpha_2|\|\nabla(\nabla\cdot\mathbf{v}^0)\|_{0,2}+|\alpha_3|\|\nabla(\nabla\cdot\mathbf{w}^0)\|_{0,2},\\
\|\tilde{\mathbf{f}}^3(t,\cdot)\|_{0,2}\le t|\alpha_1|\|\Delta\mathbf{v}^1\|_{0,2}+t|\alpha_2|\|\nabla(\nabla\cdot\mathbf{v}^1)\|_{0,2}\\
+t|\alpha_3|\|\nabla(\nabla\cdot\mathbf{w}^1)\|_{0,2}+\|\tilde{\mathbf{f}}_s(t,\cdot)\|_{0,2},\\
\|\tilde{\mathbf{g}}^1(t,\cdot)\|_{0,2}\le|\alpha_3|\|\nabla(\nabla\cdot\mathbf{v}^0)\|_{0,2}+\alpha_4\|\nabla(\nabla\cdot\mathbf{w}^0)\|_{0,2},\\
\|\tilde{\mathbf{g}}^2(t,\cdot)\|_{0,2}\le\omega(k*1)(t)\|\mathbf{w}^1\|_{0,2},\\
\|\tilde{\mathbf{g}}^3(t,\cdot)\|_{0,2}\le t|\alpha_3|\|\nabla(\nabla\cdot\mathbf{v}^1)\|_{0,2}+t\alpha_4\|\nabla(\nabla\cdot\mathbf{w}^1)\|_{0,2}+\|\tilde{\mathbf{f}}_f(t,\cdot)\|_{0,2}.
\end{gathered} \tag{3.47}$$

Using (3.45)–(3.47), we arrive at (3.36). $\square$

REMARK 3.6. Presenting the results for the high-frequency approximation, we were largely guided by the article [12].

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