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FINITE GROUPS WHOSE n-MAXIMAL SUBGROUPS ARE MODULAR

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Abstract: Let G be a finite group. If $M_n < M_{n-1} < \dots < M_1 < M_0 = G$ with M_i a maximal subgroup of M_{i-1} for all $i = 1, \dots, n$, then M_n ($n > 0$) is an *n-maximal subgroup* of G. A subgroup M of G is called *modular* provided that (i) $\langle X, M \cap Z\rangle = \langle X, M\rangle \cap Z$ for all $X \le G$ and $Z \le G$ such that $X \le Z$, and (ii) $\langle M, Y \cap Z\rangle = \langle M, Y\rangle \cap Z$ for all $Y \le G$ and $Z \le G$ such that $M \le Z$. In this paper, we study finite groups whose n-maximal subgroups are modular.

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1. Introduction

Throughout this paper, all groups are finite and G always denotes a group. Moreover, $\mathbb{P}$ is the set of all primes and the symbol $\pi(G)$ stands for the set of prime divisors of the order of G.

We say that G is: *nearly nilpotent* if G is supersoluble and G induces on its every non-Frattini chief factor H/K (i.e., $H/K \nleq \Phi(G/K)$) an automorphism group of order dividing a prime; *strongly supersoluble* if G is supersoluble and G induces on its every chief factor H/K an automorphism group of square free order. We use $\mathfrak{N}_n$ and $\mathfrak{U}_s$ to denote the classes of all nearly nilpotent and of all strongly supersoluble groups, respectively. Nearly nilpotent and strongly supersoluble groups were studied respectively in [1] and [2, 3].

It is clear that the group $C_7 \rtimes \mathrm{Aut}(C_7)$ is strongly supersoluble but not nearly nilpotent; the group $C_{13} \rtimes \mathrm{Aut}(C_{13})$ is supersoluble but not strongly supersoluble; and the group S_3 is nearly nilpotent but not nilpotent.

A subgroup M of G is called *modular* if M is a modular element (in the sense of Kurosh [4, p. 43]) of the lattice $\mathscr{L}(G)$ of all subgroups of G, i.e.,

(i) $\langle X, M \cap Z\rangle = \langle X, M\rangle \cap Z$ for all $X \le G$ and $Z \le G$ such that $X \le Z$;

(ii) $\langle M, Y \cap Z\rangle = \langle M, Y\rangle \cap Z$ for all $Y \le G$ and $Z \le G$ such that $M \le Z$.

Recall that a subgroup H of G is called a *2-maximal* (*second maximal*) subgroup of G whenever H is a maximal subgroup of some maximal subgroup M of G. Similarly we can define *3-maximal* subgroups, and so on.

The relationship between n-maximal subgroups (where $n > 1$) of G and the structure of G was studied by many authors (see, in particular, the recent papers [5–12] and Chapter 4 of the book [13]). One of the earliest results in this line of research was obtained by Huppert in the article [14] who established the supersolubility of the group whose all 2-maximal subgroups are normal. In the same article Huppert proved that if all 3-maximal subgroups of G are normal in G, then the commutator subgroup G' of G is a nilpotent group and the principal rank of G is at most 2. These two results were developed by many authors. In particular, Schmidt proved in [1] that if all 2-maximal subgroups of G are modular in G, then G is nearly nilpotent; if all 3-maximal subgroups of G are modular in G and G is not supersoluble, then either G is a group of order pq^2 for primes p and q or $G = Q \rtimes P$, where $Q = C_G(Q)$ is a quaternion

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group of order 8 and $|P| = 3$. Mann proved in [15] that if all n-maximal subgroups of a soluble group G are subnormal and $n < |\pi(G)|$, then G is nilpotent; but if $n \leq |\pi(G)| + 1$, then G is ϕ-dispersive for some ordering ϕ of $\mathbb{P}$. Finally, in the case $n \leq |\pi(G)|$ Mann described G completely.

In this paper, we prove the following modular analogs of the above-mentioned Mann's results.

Theorem A. *Suppose that G is soluble and every n-maximal subgroup of G is modular. If $n \leq |\pi(G)|$, then G is strongly supersoluble and G induces on its every non-Frattini chief factor H/K an automorphism group of order $p_1 \dots p_m$ where $m \leq n$ and $p_1, \dots, p_m$ are distinct primes.*

We use $G^{\mathfrak{U}_s}$ to denote the intersection of all normal subgroups N of G with strongly supersoluble quotient G/N.

Theorem B. *Suppose that G is soluble and every n-maximal subgroup of G is modular. If $n \leq |\pi(G)| + 1$, then $G^{\mathfrak{U}_s}$ is a nilpotent Hall subgroup of G.*

Finally, note that the restrictions on $|\pi(G)|$ in Theorems A and B cannot be weakened (see Section 4 below).

2. Proof of Theorem A

A normal subgroup A of G is said to be *hypercyclically embedded* in G [4, p. 217] if either $A = 1$ or $A \neq 1$ and every chief factor of G below A is cyclic. We use $Z_{\mathfrak{U}}(G)$ to denote the product of all normal hypercyclically embedded subgroups of G. It is clear that a normal subgroup A of G is hypercyclically embedded in G if and only if $A \leq Z_{\mathfrak{U}}(G)$.

Recall that G is said to be a *P-group* [4, p. 49] if $G = A \rtimes \langle t \rangle$ with an elementary abelian p-group A and an element t of prime order $q \neq p$ induces a nontrivial power automorphism on A.

The following two lemmas collect the properties of modular subgroups which we use in our proofs.

Lemma 2.1 [4, Theorems 5.1.14 and 5.2.5]. *Let M be a modular subgroup of G.*

(i) *M/M_G is nilpotent and $M^G/M_G \leq Z_{\mathfrak{U}}(G/M_G)$.*

(ii) *If $M_G = 1$, then*

$$G = S_1 \times \cdots \times S_r \times K,$$

where $0 \leq r \in \mathbb{Z}$ and for all $i, j \in \{1, \dots, r\}$

(a) *S_i is a nonabelian P-group,*

(b) *$(|S_i|, |S_j|) = 1 = (|S_i|, |K|)$ for all $i \neq j$,*

(c) *$M = Q_1 \times \cdots \times Q_r \times (M \cap K)$ and Q_i is a nonnormal Sylow subgroup of S_i,*

(d) *$M \cap K$ is quasinormal in G.*

Lemma 2.2 [4, p. 201]. *Let A, B, and N be subgroups of G, where A is modular in G and N is normal in G.*

(1) *If B is modular in G, then $\langle A, B \rangle$ is modular in G.*

(2) *AN/N is modular in G/N.*

(3) *N is modular in G.*

(4) *If $A \leq B$, then A is modular in B.*

(5) *If φ is an isomorphism of G onto $\overline{G}$, then A^{φ} is modular in $\overline{G}$.*

A subgroup H of G is said to be *quasinormal* (*S-quasinormal*) in G if $HP = PH$ for all subgroups (all Sylow subgroups) P of G.

Lemma 2.3 [16, Chapter 1]. *Let $H \leq K \leq G$.*

(1) *If H is S-quasinormal in G, then H is S-quasinormal in K.*

(2) *Suppose that H is normal in G. Then K/H is S-quasinormal in G/H if and only if K is S-quasinormal in G.*

(3) *If H is S-quasinormal in G, then H is subnormal in G and H^G/H_G is nilpotent.*

Lemma 2.4. *Suppose that G is soluble, and let $N \neq G$ be a minimal normal subgroup of G. Suppose also that every n-maximal subgroup of G is either modular or S-quasinormal in G, where $n \leq |\pi(G)| + r$ for some integer r. Then there is a natural number $m \leq n$ such that every m-maximal subgroup of G/N is either modular or S-quasinormal in G/N and $m \leq |\pi(G/N)| + r$.*

Proof. Assume first that N is not a Sylow subgroup of G. Then $|\pi(G/N)| = |\pi(G)|$. Moreover, if H/N is an n-maximal subgroup of G/N, then H is an n-maximal subgroup of G, and so H is either modular or S-quasinormal in G by hypothesis. Consequently, H/N is either modular or S-quasinormal in G/N by Lemmas 2.2(2) and 2.3(2). On the other hand, if G/N includes no n-maximal subgroups, then, by the solubility of G, the trivial subgroup of G/N is modular in G/N and is a unique m-maximal subgroup of G/N for some $m < n$ with $m < |\pi(G/N)|$. Hence $m < |\pi(G/N)| + r$. Thus the conclusion of the lemma is fulfilled for G/N.

Finally, consider the case that N is a Sylow p-subgroup of G. Let E be a Hall p'-subgroup of G. It is clear that $|\pi(E)| = |\pi(G)| - 1$ and E is a maximal subgroup of G. Therefore, every $(n-1)$-maximal subgroup of E is either modular or S-quasinormal in E by Lemmas 2.2(4) and 2.3(1). Thus, by the isomorphism $G/N \simeq E$, Lemma 2.2(5) implies that every $(n-1)$-maximal subgroup of G/N is either modular or S-quasinormal in G/N, and also we have $n - 1 \leq |\pi(G/N)| + r$. The lemma is proved.

A *formation* is a class $\mathfrak{F}$ of groups with the properties: (i) Every homomorphic image of each group in $\mathfrak{F}$ belongs to $\mathfrak{F}$; (ii) $G/N \cap R \in \mathfrak{F}$ whenever $G/N \in \mathfrak{F}$ and $G/R \in \mathfrak{F}$. A formation $\mathfrak{F}$ is said to be *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$; *hereditary* if $H \in \mathfrak{F}$ whenever $H \leq G \in \mathfrak{F}$.

Lemma 2.5 [3, Theorem A]. *The class of all strongly supersoluble groups is a hereditary saturated formation.*

Let $\mathfrak{X}$ be a class of groups. A group G is called a *minimal non-$\mathfrak{X}$-group* [13] or *$\mathfrak{X}$-critical group* [17] if G is not in $\mathfrak{X}$ but all proper subgroups of G are in $\mathfrak{X}$. A $\mathfrak{N}$-critical group is also called a *Schmidt group*.

Fix some ordering ϕ of $\mathbb{P}$. The record $p\phi q$ means that p precedes q in ϕ and $p \neq q$. Recall that a group G of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$ is called *ϕ-dispersive* whenever $p_1 \phi p_2 \phi \dots \phi p_n$ and for every i there is a normal subgroup of G of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_i^{\alpha_i}$. Furthermore, if ϕ is such that $p\phi q$ always implies $p > q$ then every ϕ-dispersive group is called *Ore dispersive*.

Lemma 2.6 [13, I, Propositions 1.8, 1.11, and 1.12]. *The following hold for every $\mathfrak{U}$-critical group G:*

(1) *G is soluble and $|\pi(G)| \leq 3$.*

(2) *If G is not a Schmidt group, then G is Ore dispersive.*

(3) *$G^{\mathfrak{U}}$ is a unique normal Sylow subgroup of G.*

(4) *If S is a complement to $G^{\mathfrak{U}}$ in G, then $S/S \cap \Phi(G)$ is either a cyclic prime power order group or a Miller–Moreno (i.e., a minimal nonabelian) group.*

(5) *$G^{\mathfrak{U}}/\Phi(G^{\mathfrak{U}})$ is a noncyclic chief factor of G.*

(6) *If $G^{\mathfrak{U}}$ is nonabelian, then the center, commutator subgroup, and Frattini subgroup of $G^{\mathfrak{U}}$ coincide with one another.*

(7) *If $p > 2$, then $G^{\mathfrak{U}}$ is of exponent p; for $p = 2$ the exponent of $G^{\mathfrak{U}}$ is at most 4.*

Lemma 2.7 [18, Lemma 12.8]. *If H/K is an abelian chief factor of G and M is a maximal subgroup of G such that $K \leq M$ and $MH = G$, then*

$$G/M_G \simeq (H/K) \rtimes (G/C_G(H/K)) \simeq (HM_G/M_G) \rtimes (G/C_G(HM_G/M_G)).$$

The following lemma is evident.

Lemma 2.8. *If H/K and T/L are G-isomorphic chief factors of G, then $(H/K) \rtimes (G/C_G(H/K)) \simeq (T/L) \rtimes (G/C_G(T/L))$.*

Recall that a class of soluble groups $\mathfrak{X}$ is a *Schunck class* [17, III, 2.7] if $G \in \mathfrak{X}$ whenever $G/M_G \in \mathfrak{X}$ for all maximal subgroups M of G.

Proposition 2.9. *The class of all nearly nilpotent groups $\mathfrak{N}_n$ is a Schunck class, and $\mathfrak{N}_n \subseteq \mathfrak{U}_s$. Hence every homomorphic image of each nearly nilpotent group is nearly nilpotent, and G is nearly nilpotent whenever $G/\Phi(G)$ is nearly nilpotent.*

Proof. Suppose that $G/M_G \in \mathfrak{N}_n$ for every maximal subgroup M of G. Then $G/\Phi(G)$ is supersoluble, and so G is supersoluble. If H/K is a non-Frattini chief factor of G and M is a maximal subgroup of G such that $K \leq M$ and $MH = G$, then $G/M_G \simeq (H/K) \rtimes (G/C_G(H/K))$ by Lemma 2.7. Since clearly $C_{(H/K)\rtimes(G/C_G(H/K))}(H/K) = H/K$, it follows that $|G/C_G(H/K)| = p$ is a prime. Hence $G \in \mathfrak{N}_n$. Therefore $\mathfrak{N}_n$ is a Schunck class, and so every homomorphic image of each nearly nilpotent group is nearly nilpotent, and G is nearly nilpotent whenever $G/\Phi(G)$ is nearly nilpotent by [17, III, 2.7].

We show now that every nearly nilpotent group G is strongly supersoluble. Assume this false and let G be a counterexample of minimal order. Let R be a minimal normal subgroup of G. Then G/R is strongly supersoluble by the choice of G since G/R is nearly nilpotent. Moreover, if $R \leq \Phi(G)$, then G is strongly supersoluble by Lemma 2.5, contrary to the choice of G. Therefore $R \nleq \Phi(G)$, and so $G/C_G(R)$ is of prime order since G is nearly nilpotent. Therefore G is strongly supersoluble by the Jordan–Hölder Theorem. This contradiction completes the proof of the proposition.

Lemma 2.10. *Let $G = R \rtimes M$ be a soluble primitive group, where $R = C_G(R)$ is a minimal normal subgroup of G. Let $T \neq 1$ be a subgroup of G. Suppose that G is not nearly nilpotent.*

(1) *If $T < M$, then T is neither modular nor S-quasinormal in G.*

(2) *If $T < R$ and $|M|$ is a prime, then some subgroup V of R with $|V| = |T|$ is neither modular nor S-quasinormal in G.*

Proof. (1) Assume first that T is modular in G but not S-quasinormal in G. Then T is not quasinormal in G, and so Lemma 2.1(ii) implies that G is a nonabelian P-group since $T_G \leq M_G = 1$. But then G is supersoluble. This contradiction shows that T is S-quasinormal in G, and so T is subnormal in G by Lemma 2.3(3). Hence $1 < T^G = T^{RM} = T^M \leq M_G = 1$ by [17, A, 14.3]; a contradiction. Hence we have (1).

(2) Let V be a subgroup of R with $|V| = |T|$ such that V is normal in a Sylow p-subgroup of G. If V is S-quasinormal in G, then for every Sylow q-subgroup Q of G, where $q \neq p$, we have $VQ = QV$ and so $V = R \cap VQ$. Hence $Q \leq N_G(V)$. Thus V is normal in G; a contradiction. Hence V is modular in G, which implies that $1 < V \leq R \cap Z_{\mathfrak{U}}(G)$ by Lemma 2.1(i) and so $R \leq Z_{\mathfrak{U}}(G)$. But then $|R| = p$, which implies that G is nearly nilpotent; a contradiction.

The lemma is proved.

Proposition 2.11. *If every maximal subgroup of G or every 2-maximal subgroup of G is either modular or S-quasinormal in G, then G is nearly nilpotent. Hence G is strongly supersoluble.*

Proof. Assume the proposition false and let G be a counterexample of minimal order.

We show first that G is soluble. Indeed, if M is a maximal subgroup of G and either M is modular in G or M is S-quasinormal in G, then $|G : M|$ is a prime by Lemmas 2.1(i) and 2.3(3). Therefore if every maximal subgroup of G is either modular or S-quasinormal in G, then G is supersoluble. On the other hand, if every 2-maximal subgroup of G is either modular or S-quasinormal in G, then every maximal subgroup of G is supersoluble by Lemmas 2.2(4) and 2.3(1) and so G is soluble by Lemma 2.6(1).

Therefore, in view of Proposition 2.9, we need to show only that $G/M_G \in \mathfrak{N}_n$ for every maximal subgroup M of G. If $M_G \neq 1$, then the choice of G and Lemmas 2.2(2) and 2.3(2) imply that $G/M_G \in \mathfrak{N}_n$. Assume now that $M_G = 1$, and so there is a minimal normal subgroup R of G such that $G = R \rtimes M$ and $R = C_G(R)$ by [17, A, 15.6]. Then M is not S-quasinormal in G by Lemma 2.3(3). On the other hand, if M is modular in G, then $G = M^G$ is a nonabelian P-group by Lemma 2.1(ii). It follows that G is nearly nilpotent; a contradiction. Hence every 2-maximal subgroup of G is either modular or S-quasinormal in G.

Now, let T be any maximal subgroup of M. Then T is either modular or S-quasinormal in G, and so $T = 1$ and $|M| = q$ for some prime q. Therefore R is a maximal subgroup of G. Then every maximal subgroup of R is either modular or S-quasinormal in G and so $|R| = p$ by Lemma 2.10(2), which implies that $|G| = pq$. Hence G is nearly nilpotent; a contradiction.

The proposition is proved.

In fact, Theorem A is a special case of the following

Theorem 2.12. *Suppose that G is soluble and every n-maximal subgroup of G is either modular or S-quasinormal in G. If $n \leq |\pi(G)|$, then G is strongly supersoluble and G induces on its every non-Frattini chief factor H/K an automorphism group of order $p_1 \dots p_m$, where $m \leq n$ and $p_1, \dots, p_m$ are distinct primes.*

Proof. Assume the theorem false and let G be a counterexample of minimal order.

We show first that G is strongly supersoluble. Suppose this false. Let R be a minimal normal subgroup of G.

(1) G/R is strongly supersoluble. Hence G is primitive and so $R \nleq \Phi(G)$ and $R = C_G(R) = O_p(G)$ for some prime p.

Lemma 2.4 implies that the hypothesis holds for G/R, and so the choice of G implies that G/R is strongly supersoluble. Therefore, again by the choice of G, R is a unique minimal normal subgroup of G and $R \nleq \Phi(G)$ by Lemma 2.5. Hence G is primitive and so $R = C_G(R) = O_p(G)$ for some prime p by [17, A, 15.6].

(2) Every maximal subgroup M of G is strongly supersoluble.

By hypothesis every $(n-1)$-maximal subgroup T of M is either modular or S-quasinormal in G. Hence T is modular in M by Lemma 2.2(4) in the former case, and it is S-quasinormal in M by Lemma 2.3(1) in the second case. Since the solubility of G implies that either $|\pi(M)| = |\pi(G)|$ or $|\pi(M)| = |\pi(G)| - 1$, the hypothesis holds for M. It follows that M is strongly supersoluble by the choice of G.

(3) G is supersoluble.

Suppose this false. Since every maximal subgroup M of G is strongly supersoluble by Claim (2), G is a minimal nonsupersoluble group. Then Lemma 2.6(1) yields that $|\pi(G)| = 2$ or $|\pi(G)| = 3$. But in the former case G is strongly supersoluble by Proposition 2.11, and so $|\pi(G)| = 3$ and every 3-maximal subgroup of G is either modular or S-quasinormal in G. Claim (1) and Lemma 2.6 imply that $G = R \rtimes S$, where S is a Miller–Moreno group. Moreover, since $|\pi(S)| = 2$ and S is strongly supersoluble, S is not nilpotent and so $S = Q \rtimes T$, where $|Q| = q$, $|T| = t$ and $C_S(Q) = Q$ for some distinct primes q and t by [13, I, Proposition 1.9]. Hence R is a 2-maximal subgroup of G, and so every maximal subgroup of R is either modular or S-quasinormal G. Therefore G is supersoluble by Lemma 2.10(2).

(4) G is strongly supersoluble.

From Claims (1) and (3) we get that for some maximal subgroup M of G we have $G = R \rtimes M = C_G(R) \rtimes M$ and $|R| = p$, and so M is cyclic. Since G is not strongly supersoluble, for some prime q dividing $|M|$ and for the Sylow q-subgroup Q of M we have $|Q| > q$. Assume first that $RQ \neq G$, and let $RQ \leq V$, where V is a maximal subgroup of G. Then V is strongly supersoluble by Claim (2). Hence $C_Q(R) \neq 1$, contrary to $R = C_G(R)$. Hence $RQ = G$ and so $|\pi(G)| = 2$. Therefore G is strongly supersoluble by Proposition 2.11; a contradiction. Thus we have (4).

(5) G induces on its every non-Frattini chief factor H/K an automorphism group $G/C_G(H/K)$ of order $p_1 \dots p_m$ where $m \leq n$ and $p_1, \dots, p_m$ are distinct primes.

If G is nearly nilpotent, it is clear. Suppose now that G is not nearly nilpotent. Let M be a maximal subgroup of G such that $K \leq M$ and $MH = G$. Then $G/M_G \simeq (H/K) \rtimes (G/C_G(H/K))$ by Lemma 2.7. If $M_G \neq 1$, the choice of G implies that $m \leq n$. Suppose now that $M_G = 1$, and so $G = H \rtimes M$, where $|H|$ is a prime and $H = C_G(H)$. Then, by Claim (4), M is a cyclic group of order $p_1 \dots p_m$ for some distinct primes $p_1, \dots, p_m$. Assume that $n < m$. Then G has an n-maximal subgroup T such that $T \leq M$ and $|T|$ is not a prime. But since G is not nearly nilpotent, this is impossible by Lemma 2.10(1). This contradiction completes the proof of the result.

3. Proof of Theorem B

Lemma 3.1 [17, p. 359]. *Given an ordering ϕ of the set of all primes, the class of all ϕ-dispersive groups is a saturated formation.*

Proposition 3.2. *Suppose that every 3-maximal subgroup of G is either S-quasinormal or modular in G. If G is not supersoluble, then either G is a group of order pq^2 for some distinct primes p and q, or $G = Q \rtimes P$, where $Q = C_G(Q)$ is a quaternion group of order 8 and $|P| = 3$.*

Proof. Assume the proposition false and let G be a counterexample of minimal order. Lemmas 2.2(4), 2.3(1), and Proposition 2.11 imply that every maximal subgroup of G is strongly supersoluble. Hence G is soluble by Lemma 2.6(1), and so $|\pi(G)| = 2$ by Theorem 2.12.

Since G is not supersoluble, G is a $\mathfrak{U}$-critical group. Let $D = G^{\mathfrak{U}}$ be the supersoluble residual of G. Lemma 2.6 implies that (a) D is a Sylow p-subgroup of G for some prime p, and if Q is a Sylow q-subgroup of G, where $q \neq p$, then $DQ = G$ and $Q/Q \cap \Phi(G)$ is either a cyclic prime power order group or a Miller–Moreno group; (b) $D/\Phi(D)$ is a noncyclic chief factor of G and if D is nonabelian, then the center, commutator subgroup, and Frattini subgroup of D coincide with one another; (c) if $p > 2$, then D is of exponent p, for $p = 2$ the exponent of D is at most 4. From (b) it follows that $Q^G = G$.

We first show that $|\Phi(D)| \leq p$. Indeed, assume that $|\Phi(D)| > p$, and let M be a maximal subgroup of G with $G = DM$ and $Q \leq M$. Then M is supersoluble, and so G has a 3-maximal subgroup T such that $Q \leq T$. Then $T^G = G$. If T is S-quasinormal in G, then G/T_G is nilpotent by Lemma 2.3(3). Hence QT_G/T_G is normal in G/T_G, which implies that $QT_G = G \leq M$. This contradiction shows that T is modular in G. Therefore G/T_G is a P-group by Lemma 2.1(ii). But then from the G-isomorphism

$$DT_G/T_G\Phi(D) \simeq D/D \cap T_G\Phi(D) = D/\Phi(D)(D \cap T_G) = D/\Phi(D)$$

we get that $D/\Phi(D)$ is cyclic. This contradiction shows that $|\Phi(D)| \leq p$.

We show now that $|Q| = q$. Assume that $|Q| > q$. Let M be a maximal subgroup of G with $|G : M| = q$. Then M is supersoluble, and so G has a 3-maximal subgroup T such that $|G : T| = pq^2$. Then $D \leq T^G$ and also we have $T_G \cap D \leq \Phi(D)$ and $T_G \leq \Phi(D)Q$. Moreover, T is not S-quasinormal in G since Q is a Sylow q-subgroup of G and $|T \cap \Phi(D)| > p$. Hence $G/T_G = (T^G/T_G) \times (K/T_G)$ where T^G/T_G is a nonabelian P-group of order prime to $|K/T_G|$ by Lemma 2.1(ii), But, clearly, q divides $|(G/T_G) : (T^G/T_G)|$, so $Q \leq K$ and hence

$$QT_G/T_G \leq C_{G/T_G}(T^G/T_G) \leq C_{G/T_G}(DT_G/T_G\Phi(D)),$$

where $DT_G/T_G\Phi(D)$ is G-isomorphic to $D/\Phi(D)$. Therefore $|D/\Phi(D)| = p$; a contradiction. Hence $|Q| = q$.

Note that if $|D/\Phi(D)| > p^2$ and T is a subgroup of D with $|D : T| = p^2$, then T is a 3-maximal subgroup of G. But T is neither S-quasinormal nor modular in G; a contradiction. Hence $|D/\Phi(D)| = p^2$.

Finally, assume that $\Phi(D) \neq 1$, and so $|D| = p^3$. Assume also that $p \neq 2$. Note first that since $|Q| = q$, D is a maximal subgroup of G. On the other hand, from (b) and (c) we get that some subgroup T of D of order p does not lie in $\Phi(D)$. It is clear that T is a 3-maximal subgroup of G and T is not S-quasinormal in G. Hence T is modular in G, and so $T^G = D$ is a nonabelian P-group by Lemma 2.1(ii). This contradiction shows that $p = 2$. Hence $q = 3$, since $G/C_G(D/\Phi(D)) \simeq Q$ and $|D/\Phi(D)| = 4$.

The proposition is proved.

Lemma 3.3. *Suppose that G is soluble and every n-maximal subgroup of G is either modular or S-quasinormal in G. If $n \leq |\pi(G)| + 1$, then G is ϕ-dispersive for some ordering ϕ of $\mathbb{P}$.*

Proof. Suppose the lemma false and let G be a counterexample of minimal order. Let N be a minimal normal subgroup of G and let P be a Sylow p-subgroup of G where p divides $|N|$. Then $N \leq P$.

(1) $C_G(N) = N$ and G/N is strongly supersoluble. Hence $N < P$.

Lemma 2.4 implies that the hypothesis holds for G/N. So the choice of G implies that G/N is ϕ-dispersive for some ordering ϕ of $\mathbb{P}$, and so $N < P$. Therefore the choice of G and Lemma 3.1 imply that $N \nleq \Phi(G)$. Thus, for some maximal subgroup M of G we have $G = N \rtimes M$. Then $\pi(M) = \pi(G)$, and so $G/N \simeq M$ is strongly supersoluble by Theorem 2.12. Therefore N is a unique minimal normal subgroup of G by Lemma 2.5. Hence $C_G(N) = N$.

(2) $|\pi(G)| > 2$.

Indeed, assume that $\pi(G) = \{p, q\}$, and let Q be a Sylow q-subgroup of G. Since G/N is Ore dispersive by Claim (1) and P is not normal in G, NQ/N is a normal Sylow subgroup of G/N, and so for some normal subgroup V of G we have $N \leq V$ and $|G : V| = p$. Then $\pi(V) = \pi(G)$. Hence V is strongly supersoluble by Theorem 2.12. It follows that for the largest prime $r \in \pi(V)$ a Sylow r-subgroup R of V is characteristic in V and so R is normal in G. Hence $r = p$ is the largest prime in $\pi(G)$. Since M is also Ore dispersive, a Sylow p-subgroup M_p of M is normal in M, and so normal in G since $N_G(M_p) \nleq M$. But then NM_p is a normal Sylow subgroup of G. This contradiction shows that $|\pi(G)| > 2$.

Take a prime divisor q of the order of G distinct from p. Take a Hall q'-subgroup E of G, and let $E \leq W$ where W is a maximal subgroup of G. Then $N \leq E$ and since G is soluble, Lemmas 2.2(4) and 2.3(1) imply that the hypothesis holds for W. Consequently, the choice of G implies that for some prime t dividing $|E|$ a Sylow t-subgroup Q of E is normal in E. Furthermore, since $C_G(N) = N$ we have $N \leq Q$. Hence, Q is a Sylow p-subgroup of E. It is clear also that Q is a Sylow p-subgroup of G and $(|G : N_G(Q)|, r) = 1$ for every prime $r \neq q$. Since $|\pi(G)| > 2$, it follows that Q is normal in G, and so $N = Q = P$. This contradiction completes the proof of the lemma.

In fact, Theorem B is a special case of the following

Theorem 3.4. *Suppose that G is soluble and every n-maximal subgroup of G is either modular or S-quasinormal in G. If $n \leq |\pi(G)| + 1$, then $G^{\mathfrak{U}_s}$ is a nilpotent Hall subgroup of G.*

Proof. Suppose the theorem false and let G be a counterexample of minimal order. Then G is not strongly supersoluble, and so $D = G^{\mathfrak{U}_s} \neq 1$. By Lemma 3.3, G has a normal Sylow p-subgroup P for some prime p dividing $|G|$.

(1) The conclusion of the theorem holds for every quotient $G/R \neq G/1$. (This is immediate from Lemma 2.4.)

(2) D is nilpotent.

Assume this false. Then, since $G^{\mathfrak{U}_s} \leq G'$; therefore, G is not supersoluble.

Let R be a minimal normal subgroup of G. By Claim (1) and [19, 2.2.8], $(G/R)^{\mathfrak{U}_s} = DR/D \simeq D/D \cap R$ is nilpotent. If G has a minimal normal subgroup $N \neq R$, then $D/D \cap (R \cap N)$ is nilpotent. Hence R is a unique minimal normal subgroup of G and $R \nleq \Phi(G)$ by [17, A, 13.2]. Therefore $R = C_G(R)$ by [17, A, 15.6], and $G = R \rtimes M$ for some maximal subgroup M of G with $M_G = 1$. Then $R = P$ is a Sylow p-subgroup of G by [17, A, 13.8]. It is clear that M is not supersoluble, and so $|R| > p$ since otherwise $M \simeq G/R = G/C_G(R)$ is cyclic.

Now, let T be a maximal subgroup of M. Then RT is a maximal subgroup of G and $|\pi(RT)| = |\pi(G)|$ or $|\pi(RT)| = |\pi(G)| - 1$. Hence, by Lemmas 2.2(4) and 2.3(1), RM satisfies the same assumptions as G, with $n - 1$ replacing n. The choice of G implies that $(RT)^{\mathfrak{U}_s} \leq F(RT) = R$. Therefore $T \simeq T/(T \cap (RT)^{\mathfrak{U}_s}) \simeq (RT)^{\mathfrak{U}_s}T/(RT)^{\mathfrak{U}_s}$ is strongly supersoluble. Hence M is a $\mathfrak{U}$-critical group.

By Lemma 2.6(1), $1 < |\pi(M)| \leq 3$. First assume that $|\pi(M)| = 2$. Then $n = 4$ by Theorem 2.12 since M is not supersoluble. Hence, every 3-maximal subgroup of M is either modular or S-quasinormal in G. Proposition 3.2 implies that M either is a nonsupersoluble group of order qr^2 for some distinct primes q and r or $M = Q \rtimes L$, where $Q = C_M(Q)$ is a quaternion group of order 8 and $|L| = 3$. Then R is a 3-maximal subgroup of G. Thus every maximal subgroup of R is either modular or S-quasinormal in G and so $|R| = p$ by Lemma 2.10(2); a contradiction. Thus $|\pi(M)| = 3$, so $n = 5$ and hence every 4-maximal subgroup of M is either modular or S-quasinormal in G. Let $|M| = q^a r^b t^c$, where p, r, and t are primes. If $a+b+c > 4$, then some member T of a composition series of M is a nonidentity 4-maximal subgroup of M since G is soluble, which is impossible by Lemma 2.10. Hence $a+b+c = 4$ since M is not supersoluble. Therefore, by Lemma 2.6, $M = Q \rtimes (L \rtimes T)$, where $|Q| = q^2$, $|L| = r$, and $|T| = t$. Then R is a 4-maximal subgroup of G, and so every maximal subgroup of R is either modular or S-quasinormal in G, which is impossible by Lemma 2.10(2). This contradiction completes the proof of Claim (2).

(3) D is a Hall subgroup of G.

Suppose this false. Then G is not strongly supersoluble. Let P be a Sylow p-subgroup of D such that $1 < P < G_p$, where $G_p \in \mathrm{Syl}_p(G)$.

(a) $D = P$ is a minimal normal subgroup of G.

Let R be a minimal normal subgroup of G lying in D. Then R is a q-group for some prime q. Moreover, $D/R = (G/R)^{\mathfrak{U}_s}$ is a Hall subgroup of G/R by Claim (1) and [19, 2.2.8]. Suppose that $PR/R \neq 1$. Then $PR/R \in \mathrm{Syl}_p(G/R)$. If $q \neq p$, then $P \in \mathrm{Syl}_p(G)$. This contradicts the fact that $P < G_p$. Hence $q = p$, and so $R \leq P$ and therefore $P/R \in \mathrm{Syl}_p(G/R)$ and we again get that $P \in \mathrm{Syl}_p(G)$. This contradiction shows that $PR/R = 1$, which implies that $R = P$ is the unique minimal normal subgroup of G contained in D. Since D is nilpotent by Claim (2), a p'-complement E of D is characteristic in D and so normal in G. Hence $E = 1$, which implies that $R = D = P$.

(b) $D \nleq \Phi(G)$. Hence for some maximal subgroup M of G we have $G = D \rtimes M$. (This follows from Lemma 2.5 since G is not strongly supersoluble.)

(c) If G has a minimal normal subgroup $L \neq D$, then $G_p = D \times L$. So $O_{p'}(G) = 1$.

Indeed, $DL/L \simeq D$ is a Hall subgroup of G/L by Claim (1). Hence $G_pL/L = RL/L$, and so $G_p = D \times (L \cap G_p)$. But $D < G_p$, and so $(L \cap G_p)$ is a nontrivial subgroup of L. Since G is soluble, L is a p-group and so $G_p = D \times L$. Thus $O_{p'}(G) = 1$.

(d) $\Phi(G_p) = 1$.

Suppose that $\Phi = \Phi(G_p) \neq 1$. Then, since G_p is normal in G by Claim (c), Φ is normal in G and so we can take a minimal normal subgroup L of G contained in Φ. But then $G_p = D \times L = D$ by Claim (c); a contradiction. Hence we have (d).

Final contradiction for (3). Claim (d) implies that G_p is an elementary abelian normal subgroup of G. By Maschke's Theorem $G_p = N_1 \times N_2$ is the direct product of some minimal normal subgroups of G. Claim (a) implies that $N_1 < G_p$. Let $M = N_2E$, where E is a complement to G_p in G. Then M is a maximal subgroup of G and $\pi(M) = \pi(G)$. On the other hand, every $(n-1)$-maximal subgroup of M is either modular or S-quasinormal in M by hypothesis and Lemmas 2.2(4) and 2.3(1). Thus $M \simeq G/N_1$ is strongly supersoluble by Theorem 2.12. Similarly we get that G/N_2 is strongly supersoluble. Hence $G \simeq G/N_1 \cap N_2$ is strongly supersoluble by Lemma 2.5. This contradiction shows that $D = G^{\mathfrak{U}_s}$ is a Hall subgroup of G.

The proof of the theorem is complete.

4. Final Remarks

1. Some preliminary results are of significance in its own right because they generalize some available results.

From Proposition 2.11 we get the following

Corollary 4.1 (Schmidt [1]). *If every 2-maximal subgroup M of G is modular, then G is nearly nilpotent.*

Corollary 4.2. *If every 2-maximal subgroup of G is S-quasinormal in G, then G is nearly nilpotent.*

Corollary 4.3 (Agrawal [20]). *If every 2-maximal subgroup of G is S-quasinormal in G, then G is supersoluble.*

From Proposition 3.2 we get the following known result.

Corollary 4.4 (Schmidt [1]). *If every 3-maximal subgroup M of G is modular in G and G is not supersoluble, then either G is a group of order pq^2 for some distinct primes p and q or $G = Q \rtimes P$, where $Q = C_G(Q)$ is a quaternion group of order 8 and $|P| = 3$.*

2. Note in closing that the restrictions on $|\pi(G)|$ in Theorems A and B cannot be weakened. Indeed, for Theorem A this follows from the example of the alternating group A_4 of degree 4. For Theorem B this follows from the example of $A_4 \times C_2$, where C_2 is a group of order 2.

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