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Similarity solutions of time-dependent relativistic radiation-hydrodynamical plane-parallel flows

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Abstract

Similarity solutions are examined for the frequency-integrated relativistic radiation-hydrodynamical flows, which are described by the comoving quantities. The flows are vertical plane-parallel time-dependent ones with a gray opacity coefficient. For adequate boundary conditions, the flows are accelerated in a somewhat homologous manner, but terminate at some singular locus, which originates from the pathological behavior in relativistic radiation moment equations truncated in finite orders.

Key words: accretion, accretion disks — astrophysical jets — radiative transfer — relativity

1 Introduction

Various astrophysical flows show time-dependent behaviors; nova outbursts, supernova explosions, gamma-ray bursts, accretion on to a gravitating body, winds from luminous accretion disks, and gravitational contractions of interstellar gas. These outflows and inflows are often relativistic, and sometimes radiation-dominated. Today, such relativistic radiation-hydrodynamical time-dependent flows are usually examined using relativistic radiation-hydrodynamical numerical simulations. In some limited situations, on the other hand, the self-similar treatment has been a strong tool for time-dependent flows (Sedov 1959).

Similarity solutions are sought and found for many astrophysical time-dependent phenomenae, including supernova explosions and gravitational collapses. In order to obtain similarity solutions, the numbers of physical constants are restricted, and the similarity behavior is often constrained, depending on the physical ingredients. For example, in the case of a point explosion with constant energy E into the surrounding medium with uniform density ρ, the radius r varies with time t as $r \propto (E/\rho)^{1/5} t^{2/5}$ (Sedov 1959). In the case of a self-similar flow around a central object of mass M, the reference radius must vary as $r \propto (GM)^{1/3} t^{2/3}$ (e.g., Sakashita 1974; Sakashita & Yokosawa 1974; Cheng 1977; Fukue 1984).

Because of this similarity constraint, the self-similar treatment in radiation hydrodynamical flows is rather restrictive, since the system involves the coupling constant, opacity, between radiation and matter. A few studies on this have been published (see Falize et al. 2011, and references therein). Furthermore, the self-similar treatment in relativistic flows is also restrictive, since the system involves the speed of light. Similarity solutions for relativistic blast waves, however, have been well examined (Blandford & McKee 1976; Sari 1997; Nakayama & Shigeyama 2005; van Eerten 2014). Hence, the self-similar treatment in *relativistic radiation hydrodynamical flows* is even more restrictive. As a result, similarity solutions in the relativistic radiation hydrodynamical flows are not known so well.

Among a few current of the studies, that by Lucy (2005) found and examined the similarity solutions in detail for radiation fields in time-dependent relativistic spherical flows under the assumptions of homologous changes, where the flow velocity v is assumed to vary as $v = r/t$. Under this homologous assumption, they found exact analytical solutions for time-dependent extinction coefficients.

Under these current situations, in this study we seek the similarity solutions, without assuming the homologous flow, for relativistic radiation-hydrodynamical flows in the plane-parallel geometry.

In the next section we describe the basic equations described by the comoving quantities. In section 3 we first obtain the homologous solutions, and we examine more general similarity solutions in section 4. The final section is devoted to concluding remarks. We further show the related steady case in the appendix to clarify the singular nature of the relativistic moment equations.

2 Comoving frame equations

In this study we consider a time-dependent relativistic radiation hydrodynamical flow in the plane-parallel geometry. We assume the following.

(i) The outflow is one-dimensional in the vertical (z) direction.
(ii) The gas pressure is ignored, and the flow is radiation-dominated.
(iii) The gravitational force and other forces are also ignored.
(iv) The radiative quantities measured in the comoving frame are adopted.
(v) The frequency-dependence is not included, and we use the gray approximation.
(vi) The radiation field in the flow is isotropic, and we adopt the Eddington approximation.

Under these assumptions, the basic equations in the vertical direction become as follows.

The comoving frame transfer equation and frequency-integrated moment equations are given in, e.g., Mihalas (1980) and Mihalas and Mihalas (1984). Following Mihalas (1980), the radiative quantities (J_0, H_0, K_0), density ρ_0, and opacities (κ_0, σ_0) are expressed in the comoving frame, while space–time coordinates (z, t) and flow velocity v ($= \beta c$) are measured in the rest frame.

The continuity equation is

$$\frac{\partial}{\partial t}(\gamma\rho_0) + \frac{\partial}{\partial z}(\gamma\rho_0\beta c) = 0, \tag{1}$$

where ρ_0 is the gas density, β ($= v/c$) the flow velocity normalized by the speed of light, and γ ($= 1/\sqrt{1 - v^2/c^2}$) the Lorentz factor. The equation of motion is

$$c^2\frac{\gamma^4}{c}\frac{\partial\beta}{\partial t} + c^2\gamma^4\beta\frac{\partial\beta}{\partial z} = \frac{\kappa_0 + \sigma_0}{c}4\pi H_0, \tag{2}$$

where κ_0 and σ_0 are the absorption and scattering opacities, respectively, and H_0 the mean radiative flux. As already stated, the gas pressure and gravity are ignored. Since we ignore the gas pressure, the energy balance equation is

$$0 = q^+ - 4\pi\rho_0\left(\frac{j_*}{4\pi} - \kappa_0 J_0\right), \tag{3}$$

where q^+ is the internal heating, j_* the mass emissivity, and J_0 the mean intensity.

The zeroth moment equation for radiation is

$$\begin{aligned}&\frac{\gamma}{c}\frac{\partial J_0}{\partial t} + \gamma\beta\frac{\partial J_0}{\partial z} + \frac{\gamma\beta}{c}\frac{\partial H_0}{\partial t} + \gamma\frac{\partial H_0}{\partial z}\\&+\gamma^3[2H_0 + \beta(J_0 + K_0)]\frac{1}{c}\frac{\partial\beta}{\partial t} + \gamma^3(2\beta H_0 + J_0 + K_0)\frac{\partial\beta}{\partial z}\\&= \rho_0\left(\frac{j_*}{4\pi} - \kappa_0 J_0\right) = \frac{q^+}{4\pi},\end{aligned} \tag{4}$$

where equation (3) is used. On the other hand, the first moment equation is

$$\begin{aligned}&\frac{\gamma}{c}\frac{\partial H_0}{\partial t} + \gamma\beta\frac{\partial H_0}{\partial z} + \frac{\gamma\beta}{c}\frac{\partial K_0}{\partial t} + \gamma\frac{\partial K_0}{\partial z}\\&+\gamma^3(2\beta H_0 + J_0 + K_0)\frac{1}{c}\frac{\partial\beta}{\partial t} + \gamma^3[2H_0 + \beta(J_0 + K_0)]\frac{\partial\beta}{\partial z}\\&= -\rho_0(\kappa_0 + \sigma_0)H_0,\end{aligned} \tag{5}$$

where K_0 is the second moment, and related to the mean intensity by the Eddington approximation,

$$K_0 = fJ_0 = \frac{1}{3}J_0. \tag{6}$$

3 Homologous solutions

In this section, we first seek the similarity solutions for the plane-parallel time-dependent radiation-hydrodynamical flows under the assumption of a homologous expansion, as Lucy (2005) did for the spherical case.

Under the homologous assumption, we assume that the velocity field changes as

$$\beta = \frac{z}{ct}, \tag{7}$$

and consider the radiation field equations (4) and (5). Using this homologous velocity as a similarity coordinate, we can transform moment equations (4) and (5) as

$$\begin{aligned}&\frac{\gamma}{c}\frac{dJ_0}{dt} + \frac{\gamma\beta}{c}\frac{dH_0}{dt} + \frac{1}{\gamma ct}\frac{\partial H_0}{\partial\beta} + \frac{\gamma}{ct}(1+f)J_0\\&= -\rho_0(\kappa_0 + \sigma_0)(J_0 - S_0)\end{aligned} \tag{8}$$

$$\frac{\gamma}{c}\frac{dH_0}{dt} + f\frac{\gamma\beta}{c}\frac{dJ_0}{dt} + f\frac{1}{\gamma ct}\frac{\partial J_0}{\partial \beta} + \frac{\gamma}{ct}2H_0$$
$$= -\rho_0(\kappa_0 + \sigma_0)H_0, \tag{9}$$

where $dt\,(= \partial/\partial t + v\partial/\partial z)$ is the Lagrangian time derivative, S_0 the source function, and the Eddington approximation is used. Furthermore, if we seek the similarity solutions in the form of

$$J_0 = (t/t_1)^{-p} J_1(\beta), \tag{10}$$

$$H_0 = (t/t_1)^{-p} H_1(\beta), \tag{11}$$

$$S_0 = (t/t_1)^{-p} S_1(\beta), \tag{12}$$

$$\chi_0 = \rho_0(\kappa_0 + \sigma_0)(t/t_1)^{-1}\chi_1(\beta), \tag{13}$$

then equations (8) and (9) are transformed into

$$\frac{dH_1}{d\beta} - p\gamma^2\beta H_1 + (1 + f - p)\gamma^2 J_1 = -\gamma ct_1\chi_1(J_1 - S_1), \tag{14}$$

$$f\frac{dJ_1}{d\beta} - fp\gamma^2\beta J_1 + (1 - p)\gamma^2 H_1 = -\gamma ct_1\chi_1 H_1, \tag{15}$$

and further rearranged as

$$\gamma^p\frac{d}{d\beta}\left(\gamma^{-p}H_1\right) + (1 + f - p)\gamma^2 J_1 = -\gamma\tau_1(J_1 - S_1), \tag{16}$$

$$f\gamma^p\frac{d}{d\beta}\left(\gamma^{-p}J_1\right) + (1 - p)\gamma^2 H_1 = -\gamma\tau_1 H_1, \tag{17}$$

where

$$\tau_1(\beta) = ct_1\chi_1(\beta) \tag{18}$$

is a typical optical depth.

In what follows, we assume the radiative equilibrium, $J_1 = S_1$, and the typical optical depth τ_1 is constant.

Similar to the spherical case of Lucy (2005), we can obtain an analytical solution in the case of $p = 1 + f = 4/3$, although the analytical solution of the spherical flow is obtained in the case of $p = 4$. That is, in this case we can easily integrate equation (16) to yield

$$H_1(\beta) = H_1(0)\gamma^p, \tag{19}$$

and further integrate equation (17) as

$$fJ_1(\beta) = fJ_1(0)\gamma^p - (1 - p)H_1(0)\gamma^p\frac{1}{2}\ln\frac{1+\beta}{1-\beta}$$
$$-\tau_1 H_1(0)\gamma^p\sin^{-1}\beta. \tag{20}$$

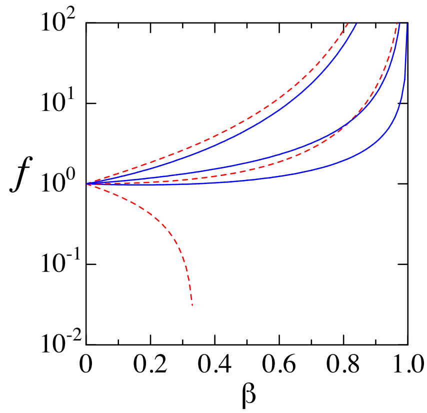

Fig. 1. Typical examples of homologous solutions under the radiative equilibrium condition. The dashed curves denote the mean intensity $J_1(\beta)$, while the solid ones represent the mean flux $H_1(\beta)$. The parameters are $\tau_1 = 1$, and $p = 1, 2, 3$, from bottom to top, for each quantity. (Color online)

Typical examples of other cases are shown in figure 1. In figure 1 the mean intensity $J_1(\beta)$ is shown by the dashed curves, while the mean flux $H_1(\beta)$ is denoted by the solid ones. The parameters are $\tau_1 = 1$ and $p = 1, 2, 3$, from bottom to top, for each quantity.

As seen in figure 1, except for limited cases of small p, the radiation field diverges as β approaches unity. This is interpreted by the relativistic effect. Similar to the radiative quantities in the rest frame, those in the comoving frame also concentrate towards the forward direction due to the relativistic aberration. In addition, they are enhanced due to the relativistic Doppler effect. As a result, the radiative quantities in the comoving frame become large as the flow speed increases, and ultimately diverge at $\beta = 1$.

It should be emphasized that the homologous assumption (7) is not coherent with equation of motion (2). That is, this homologous velocity field (7) gives $H_0 = 0$. This inconsistent situation is same in the spherical case of Lucy (2005). In other words, the homologous assumption abandons solving the flow dynamics, but concentrates the similarity solutions of the radiation field. Thus, in the next section we consider both the flow dynamics and radiation field, and seek the similarity solutions.

4 Non-homologous solutions

Now, we loosen the homologous assumption, and seek more general similarity solutions.

4.1 Similarity transformations

In this study, we assume the simple time-dependency, and we shall introduce the similarity coordinate in the form:

$$\zeta \equiv \frac{z}{ct} = \frac{z}{\tilde{t}}, \tag{21}$$

where $\tilde{t}$ ($\equiv ct$) is the rescaled time, and define the similarity variables by

$$\beta = \beta(\zeta) \tag{22}$$

$$\gamma\rho_0(\kappa_0 + \sigma_0) \equiv \tilde{t}^{-1} D(\zeta), \tag{23}$$

$$\frac{q^+}{4\pi}\frac{\kappa_0 + \sigma_0}{c^3} \equiv \tilde{t}^{-2} Q(\zeta) = \tilde{t}^{-2} Q_*, \tag{24}$$

$$\frac{\kappa_0 + \sigma_0}{c^3} J_0 \equiv \tilde{t}^{-1} J(\zeta), \tag{25}$$

$$\frac{\kappa_0 + \sigma_0}{c^3} H_0 \equiv \tilde{t}^{-1} H(\zeta), \tag{26}$$

$$\frac{\kappa_0 + \sigma_0}{c^3} K_0 \equiv \tilde{t}^{-1} K(\zeta), \tag{27}$$

where Q_* is assumed to be constant.

Using these similarity transformations, the continuity equation (1) and equation of motion (2) are transformed, respectively, as

$$D|\beta - \zeta| = D_*, \tag{28}$$

$$(\beta - \zeta)\frac{d\beta}{d\zeta} = \frac{4\pi}{\gamma^3} H, \tag{29}$$

where D_* is a constant.

The moment equations (4) and (5) are transformed as

$$(\beta - \zeta)\frac{dJ}{d\zeta} - J + (1 - \beta\zeta)\frac{dH}{d\zeta} - \beta H + \gamma^2[2(\beta - \zeta)H + (1 - \beta\zeta)(J + K)]\frac{d\beta}{d\zeta} = \frac{1}{\gamma} Q_*, \tag{30}$$

$$(\beta - \zeta)\frac{dH}{d\zeta} - H + (1 - \beta\zeta)\frac{dK}{d\zeta} - \beta K + \gamma^2[2(1 - \beta\zeta)H + (\beta - \zeta)(J + K)]\frac{d\beta}{d\zeta} = -\frac{1}{\gamma^2} DH, \tag{31}$$

where $K = fJ$. Moreover, these transformed moment equations are rearranged as

$$\begin{aligned}
&[f(1 - \beta\zeta)^2 - (\beta - \zeta)^2]\frac{dJ}{d\zeta} \\
&= [-(\beta - \zeta) + f\beta(1 - \beta\zeta)]J + \frac{1}{\gamma^2} H \\
&\quad - 2(1 - \zeta^2)\frac{4\pi}{\gamma^3(\beta - \zeta)} H^2 - \frac{\beta - \zeta}{\gamma} Q_* - \frac{1 - \beta\zeta}{\gamma^2} DH,
\end{aligned} \tag{32}$$

$$\begin{aligned}
&[f(1 - \beta\zeta)^2 - (\beta - \zeta)^2]\frac{dH}{d\zeta} \\
&= [-(\beta - \zeta) + f\beta(1 - \beta\zeta)]H + \frac{f}{\gamma^2} J \\
&\quad + 2(1 - f)(1 - \beta\zeta)\frac{4\pi}{\gamma} H^2 + (1 + f)[(\beta - \zeta)^2 \\
&\quad - f(1 - \beta\zeta)^2]\frac{4\pi}{\gamma(\beta - \zeta)} JH \\
&\quad \times f\frac{1 - \beta\zeta}{\gamma} Q_* + \frac{\beta - \zeta}{\gamma^2} DH.
\end{aligned} \tag{33}$$

As is easily seen, these equations have singular points, when the term on the left-hand side,

$$\mathcal{D} = f(1 - \beta\zeta)^2 - (\beta - \zeta)^2, \tag{34}$$

vanishes. This singularity will be discussed later.

4.2 Non-relativistic limit

Before calculating the similarity solutions, we briefly examine the non-relativistic limit, which is used as initial conditions.

In the non-relativistic limit, $\beta \ll 1$, $\zeta \ll 1$, and only the linear terms are retained. Then, besides (28), equations (29), (32) and (33) are approximated, respectively, as

$$(\beta - \zeta)\frac{d\beta}{d\zeta} = 4\pi H, \tag{35}$$

$$f\frac{dJ}{d\zeta} = -DH, \tag{36}$$

$$\frac{dH}{d\zeta} = J - (1 + f)\frac{4\pi}{\beta - \zeta} JH + Q_*. \tag{37}$$

Hence, if we assume the linear forms of

$$\beta = b\zeta, \tag{38}$$

$$H = h\zeta, \tag{39}$$

$$J = j_* + j\zeta, \tag{40}$$

then we can determine the relations among the coefficients (b, h, j_*, j) as

$$4\pi h = b(b - 1), \tag{41}$$

$$fj = -\frac{D_*}{4\pi} b, \tag{42}$$

$$j_* = \frac{Q_* - [b(b - 1)/4\pi]}{(1 + f)b - 1}. \tag{43}$$

These are the linear non-relativistic solutions, and used as initial solutions in the next subsection.

It should be noted that the remaining free parameters are D_*, Q_*, and b. Futhermore, in order for the physical solutions to exist, the following conditions should be fulfilled:

$$b > 1, \tag{44}$$

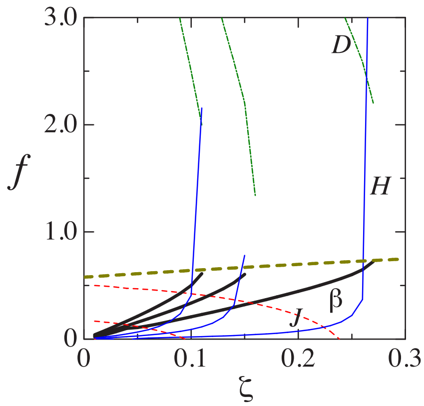

Fig. 2. Typical similarity solutions for the plane-parallel time-dependent radiation-hydrodynamical flows. The thick solid curves show the flow velocity β, the thin dashed ones denote the mean intensity J, the thin solid ones represent the mean flux H, the dash–dotted ones show the density D, and a thick dashed one is a critical locus (see the text). The parameters are $D_* = 1$, $Q_* = 1$ and $b = 2, 3, 4$ from right to left. (Color online)

$$Q_* > \frac{b(b-1)}{4\pi}. \tag{45}$$

4.3 Similarity solutions

Now, we shall solve the basic equations to obtain the typical solutions for the plane-parallel time-dependent radiation-hydrodynamical flows under the non-relativistic initial conditions. Typical solutions are shown in figure 2.

In figure 2 the flow velocity β is shown by the thick solid curves, the mean intensity J is denoted by the thin dashed ones, the mean flux H is depicted by the thin solid ones, and the density D is represented by the dash–dotted ones. The thick dashed curve is a critical locus, discussed later. The parameters are $D_* = 1$, $Q_* = 1$, and $b = 2, 3, 4$ from right to left.

As is seen in figure 2, similar to the homologous case, roughly speaking, the flow velocity increases linearly, but it deviates from the linear line, and terminates at some points. It seems that the behavior of the radiative flux is also qualitatively similar to the homologous case. However, it diverges before the flow speed reaches the speed of light. Hence, the apparent similarity of the divergence is pseudo; i.e., the reason of divergence is not the relativistic effect, but there is alternative reason.

Here, we remember that the basic equations have singularity at the point where the denominator (34) vanishes. From the condition of $\mathcal{D} = 0$, we have

$$\beta_c = \frac{\sqrt{f} + \zeta}{1 + \sqrt{f}\zeta}, \tag{46}$$

where we omit the minus solution. This relation between β and ζ, satisfying the singularity condition, *critical locus*, is shown by a thick dashed curve in figure 2. As is seen in figure 2, the flow velocity terminates at this critical locus, and the radiative flux diverges at the same position. Namely, the origin of the termination and divergence of solutions is attributable to the singular nature of the basic equations.

Actually, in the steady case, the similar singularity in relativistic radiation moment equations has been pointed out and examined (Turolla & Nobili 1988; Nobili et al. 1991; Turolla et al. 1995; Dullemond 1999). Namely, the moment equations for radiation transfer in relativistically moving steady flows generally have singular (critical) points. As a result, solutions behave pathologically in a relativistic regime.

The appearance of singularities is supposed to be related to the approximation of the full transfer equations with a finite number of moments (Dullemond 1999). For example, in one-dimensional relativistic flows, where the moment equations are truncated at the second order, using the Eddington approximation ($f = 1/3$), the singularity appears when the flow velocity v becomes $\pm c/\sqrt{3}$ (Turolla & Nobili 1988; Turolla et al. 1995). Hence, under the traditional Eddington approximation, we cannot obtain solutions accelerated beyond $c/\sqrt{3}$, although there exists a decelerating solution (Fukue 2005). In this steady case, when the flow speed v is equal to $c/\sqrt{3}$, the singularity occurs. In the present self-similar case, on the other hand, the singularity occurs at

$$\beta - \zeta = \sqrt{f}(1 - \beta\zeta), \tag{47}$$

where the left-hand side is the flow speed relative to the frame speed, and the right-hand side is $\sqrt{f}$ corrected by the non-linear advective term.

The invalidity of the Eddington approximation in such a relativistic flow can be understood as follows. In adopting the Eddington approximation, we assume that within the photon mean-free path the radiation field is *isotropic* in the comoving frame. However, in the relativistic regime, where the velocity gradient becomes large and there exist the Doppler and aberration effects of photons, the isotropy of the radiation field may break down even in the comoving frame (In this sense, this singularity is also the *relativistic* effect). Indeed, if we set the Eddington factor f to be unity, as is seen from equation (46), the critical velocity becomes $\beta_c = 1$, and the singularity is pushed away to the locus of the speed of light.

The steady case corresponding to the present flow is briefly shown in the appendix.

5 Concluding remarks

In this paper we examined similarity solutions for the relativistic radiation-hydrodynamical plane-parallel flows under the gray and Eddington assumptions. For adequate boundary conditions, the flows are accelerated in a somewhat homologous manner, but terminate at some singular locus, which originates from the pathological behavior in relativistic radiation moment equations truncated in finite orders.

For simplicity, we have dropped the gas pressure and gravity. If the gas pressure and gravity is included, we may construct the transonic similarity solutions in the relativistic radiation-hydrodynamical regime, similar to the non-relativistic hydrodynamical case (Cheng 1977; Fukue 1984).

Furthermore, in this study we used the similarity variable in the form of $\zeta = z/ct$. As a result, the solution becomes somewhat homologous-like. In general, we can adopt a similarity variable of $\zeta \propto z/t^{\delta}$, which may extend the similarity solutions of the present type.

Similar to the steady case, the singular point and pathological behavior in the relativistic moment equations under the Eddington approximation are a yet-unresolved problem in this field, and are left as future work.

Appendix. Steady case

The steady case using the rest-frame quantities under the Eddington approximation was discussed in several studies (e.g., Fukue 2005, 2006; Fukue & Akizuki 2006, 2007). In order to complement these studies and to clarify the existence of the singularity, we briefly show basic equations in the steady case, corresponding to the present problem, using the comoving-frame quantities.

Correspoinding to basic equations (1), (2), (4), and (5), basic equations in the steady case are as follows:

$$\frac{d}{dz}(\gamma\rho_0\beta c) = 0, \tag{48}$$

$$c^2\gamma^4\beta\frac{d\beta}{dz} = \frac{\kappa_0+\sigma_0}{c}4\pi H_0, \tag{49}$$

$$\gamma\frac{dH_0}{dz} + \gamma\beta\frac{dJ_0}{dz} + \gamma^3(2\beta H_0 + J_0 + K_0)\frac{d\beta}{dz} = \frac{q^+}{4\pi}, \tag{50}$$

$$\gamma\frac{dK_0}{dz} + \gamma\beta\frac{dH_0}{dz} + \gamma^3[2H_0 + \beta(J_0+K_0)]\frac{d\beta}{dz} = -\rho_0(\kappa_0+\sigma_0)H_0, \tag{51}$$

where $K_0 = fJ_0$ under the Eddington approximation.

Integrating the continuity equation (48) as

$$\rho_0\gamma\beta c = \dot{J}, \tag{52}$$

where $\dot{J}$ is the constant mass-loss rate per unit area, and introducing the optical depth by

$$d\tau \equiv -(\kappa_0+\sigma_0)\rho_0 dz, \tag{53}$$

and further rearranging moment equations (50) and (51), we finally have the following equations:

$$c^2\dot{J}\frac{d\beta}{d\tau} = -\frac{1}{\gamma^2}4\pi H_0, \tag{54}$$

$$(f-\beta^2)\frac{dJ_0}{d\tau} + 2H_0\frac{d\beta}{d\tau} = \frac{q^+}{4\pi(\kappa_0+\sigma_0)}\frac{\beta^2 c}{\dot{J}} + \frac{1}{\gamma}H_0, \tag{55}$$

$$(f-\beta^2)\frac{dH_0}{d\tau} + \gamma^2[-2(1-f)\beta H_0 + (1+f)(f-\beta^2)J_0]\frac{d\beta}{d\tau} = -\frac{q^+}{4\pi(\kappa_0+\sigma_0)}\frac{f\beta c}{\dot{J}} - \frac{\beta}{\gamma}H_0. \tag{56}$$

Thus, it is clearly seen that basic equations have singularity at $f-\beta^2 = 0$. Steady solutions can be found in previous studies.

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