Proc. Indian Acad. Sci. (Math. Sci.) (2018) 128:8
https://doi.org/10.1007/s12044-018-0382-7

Frobenius splitting of projective toric bundles

HE XIN

No. 19, Keji Road, New Songshan District, Jinzhou, LiaoNing Province,
People's Republic of China
E-mail: xinhe@bhu.edu.cn

MS received 21 May 2015; revised 18 December 2015; accepted 22 December 2015; published online 19 March 2018

Abstract. We prove that the projectivization of the tangent bundle of a nonsingular toric variety is Frobenius split.

Keywords. Frobenius splitting; toric bundles.

Mathematics Subject Classification. 14M25, 14G17.

1. Introduction

A toric variety is a normal algebraic variety endowed with an action of a torus T such that over an open subset of this variety the action is free and transitive. Given a toric variety X, a toric bundle on X is a vector bundle on X endowed with an action of T compatible with that on X.

Now we are given a complete toric variety X defined over an algebraically closed field k of characteristic $p > 0$. In [7, Quesiton 7.6], the authors ask when the projectivization of a toric bundle on X is Frobenius split. The answer is always affirmative when the toric bundle is split, i.e. a direct sum of line bundles, since in this case the projectivization is a toric variety [3, Proposition 7.3.3]. However, in general, very little is known. On the other hand, in [9] the authors proved that the cotangent bundle of a flag variety is Frobenius split. In the same vein, Lauritzen raised a question in [11] whether the cotangent bundle of a smooth toric variety is Frobenius split.

The aim of this paper is to prove the following:

Theorem 1. *Let X be a smooth toric variety and $\mathcal{T}_X$ be its tangent bundle, then $\mathbb{P}(\mathcal{T}_X)$ (following [6 Definition, p.162], we use $\mathbb{P}(\mathcal{E})$ to denote the associated* projective space bundle, i.e. $\mathrm{Proj}_X \mathrm{Sym}\, \mathcal{E}$ *for a locally free sheaf $\mathcal{E}$ on a scheme X) is Frobenius split.*

By [2, Lemma 1.1.11], our main result implies that the cotangent bundle of a smooth toric variety is Frobenius split. It should be noted that the converse of the lemma *loc. cit.* is also true, see [1, Proposition 8.1]. Besides, our main result is also obtained in [1] by using a different method.

In [1, Question 8.7], the authors ask whether the cotangent bundle is always Frobenius split for a smooth F-split variety. We answer this question negatively by giving an example in characteristic 2 at the end of this paper.

If there is no further specification, we will work throughout over an algebraically closed field k of characteristic $p > 0$.

2. Preliminaries

The objective of this section is twofold. The first is to review some facts on toric bundles and the second is to review a criterion for the Frobenius splitting of a smooth complete variety. The references for these two parts are [8] and [2] respectively.

2.1 *Toric bundles*

Here we will recall some facts on toric bundles, among which the *Klyachko data* of a toric bundle will play a key role later in the proof of our theorem.

2.1.1. *Toric varieties.* A toric variety is a normal algebraic variety on which there is an action of a torus such that over an open subset of the variety this action is free and transitive. Any affine toric variety can be constructed as follows [4]. Let N be a free abelian group of rank n, $\sigma \subseteq N \otimes_{\mathbb{Z}} \mathbb{R}$ be a strongly convex rational polyhedral cone, $\sigma^{\vee}$ be the set of vectors in $N^{\vee} \otimes_{\mathbb{Z}} \mathbb{R}$ which take nonnegative values on σ. Then $S_\sigma = \sigma^{\vee} \cap N^{\vee}$ is commutative semigroup and $U_\sigma = \operatorname{Spec} k[S_\sigma]$ is an affine toric variety. For a general toric variety X of dimension n, there exists a fan Σ in $N \otimes_{\mathbb{Z}} \mathbb{R}$ from which one can obtain the toric variety $X = X(\Sigma)$ by firstly constructing for each cone $\sigma \in \Sigma$ the associated affine toric varieties U_σ and then gluing $\{U_\sigma\}_{\sigma\in\Sigma}$.

If we denote by T the torus acting on the toric variety X, then the abelian group $N^{\vee}$ can be chosen to be the character group $\hat{T} = \operatorname{Hom}(T, \mathbb{G}_{\mathbf{m}})$ and N is given by $\hat{T}^0 = \operatorname{Hom}(\hat{T}, \mathbb{Z})$. In the sequel, the operation of the character group $\hat{T}$ will be written multiplicatively and the trivial character, i.e. the unit in the character group will be denoted by 0.

For each fan Σ in $N \otimes \mathbb{R}$, we use $|\Sigma|$ to denote set of ray generators of Σ.

A toric bundle on X is defined to be a vector bundle endowed with an action of T which is compatible with the action of T on X [8, 1.2]. In the sequel, by a toric bundle we will mean the associated locally free sheaf rather than the total space.

2.1.2. *Toric bundles over affine toric varieties.* The structure of toric bundles on affine toric varieties are easier to describe than the general case. For convenience, we first record the following result due to Klyachko.

PROPOSITION 2.1 [8, Proposition 2.1.1]

(1) *Let U_σ be an affine toric variety and $\mathcal{E}$ be a toric bundle on U_σ, then*

$$\mathcal{E} \cong \bigoplus_{\chi_i \in \hat{T}, 1 \le i \le n} \chi_i \mathcal{O}_{U_\sigma}. \tag{2.1}$$

The underlying line bundle of $\chi_i \mathcal{O}_{U_\sigma}$ is $\mathcal{O}_{U_\sigma}$ and if we denote by e_σ^i the unit of $\chi_i \mathcal{O}_{U_\sigma}$ the action of $t \in T$ on $\chi_i \mathcal{O}_{U_\sigma}$ is given by

$$t.e_\sigma^i = \chi_i(t) e_\sigma^i.$$

(2) *The toric bundle structure of $\mathcal{E}$ is uniquely determined by the induced representation of T_σ on $\mathcal{E}_x$, where x is a point of the unique closed orbit of U_σ and $T_\sigma \subseteq T$ is the stabilizer subgroup of x.*

The characters $\{\chi_i\}_{1\leq i\leq n}$ appearing on the right side of (2.1) might not be uniquely determined. For instance, any two toric line bundles on a torus are isomorphic.

The inverse images in $\mathcal{E}$ of the units $\{e^i_\sigma\}_{1\leq i\leq n}$ under the isomorphism (2.1) constitute a basis and will still be denoted by $\{e^i_\sigma\}_{1\leq i\leq n}$. This basis will be called an *eigen-basis* of $\mathcal{E}$ and the characters $\{\chi_i\}_{1\leq i\leq n}$ will be called *eigen-characters* of $\mathcal{E}$ associated to the eigen-basis $\{e^i_\sigma\}_{1\leq i\leq n}$.

2.1.3. *Klyachko data of a toric bundle.* Let $X = X(\Sigma)$ be a toric variety and $\mathcal{E}$ be a toric bundle of rank r on X. The *Klyachko data* associated to $\mathcal{E}$ is a vector space of dimension r together with a family of compatible filtrations indexed by the ray generators of Σ.

Now we recall how to obtain the Klyachko data associated to a toric bundle $\mathcal{E}$ on a toric variety $X = X(\Sigma)$. Firstly we assume X is affine. Let x be a point in the unique closed orbit of X. Then we have an induced representation of T_σ on $\mathcal{E}_x$, where $T_\sigma \subseteq T$ is the stabilizer subgroup of x. By Proposition 2.1, $\mathcal{E}$ is trivial as a vector bundle and we take $\{e^i\}_{1\leq i\leq r}$ to be its eigen-basis. Let x_0 be a point in the open orbit of X and e^i_x (resp. $e^i_{x_0}$) be the image of e^i in the fiber $\mathcal{E}_x$ (resp. $\mathcal{E}_{x_0}$). Then we can identify the vector spaces $\mathcal{E}_x$ with $\mathcal{E}_{x_0}$ by sending e^i_x to $e^i_{x_0}$, $1 \leq i \leq r$. In particular, if we denote by E the vector space $\mathcal{E}_{x_0}$, then we have an induced representation of T_σ on E. We extend this representation to a representation of T on E and let E_χ be the χ-isotypical component of this representation. Then for each $\alpha \in |\Sigma|$ and integer i, we define a subspace $E^\alpha(i)$ of E as follows

$$E^\alpha(i) = \bigoplus_{\langle\chi,\alpha\rangle\geq i} E_\chi. \tag{2.2}$$

One sees easily that $\{E^\alpha(i)\}$ is a decreasing filtration on E.

For a toric bundle $\mathcal{E}$ on a general toric variety X, one can choose a family of open affine sub toric varieties $\{U_\sigma\}$ covering X. Then for each cone σ and $\alpha \in |\sigma|$ we can define as in (2.2) a filtration of the vector space $E = \mathcal{E}_{x_0}$, where x_0 is a chosen point of the open orbit of X. Moreover, one can show ([8, Corollary 2.2.5]) the filtration $\{E^\alpha(i)\}$ obtained in this way is independent of the choice of σ that contains α.

The *Klyachko data* associated to the toric bundle $\mathcal{E}$ on $X = X(\Sigma)$ is nothing but the family of filtrations $\{E^\alpha(i)\}$ on the vector space E indexed by the ray generators of Σ. Conversely, given a family of indexed filtrations $\{E^\alpha(i)\}_{\alpha\in|\Sigma|}$ satisfying suitable compatible conditions one can construct a toric bundle on $X = X(\Sigma)$. To sum up, we have the following theorem.

Theorem 2 **[8, Theorem 2.2.1]**. *Giving a toric bundle of rank r on a toric variety $X(\Sigma)$ is equivalent to giving a vector space E of dimension r together with a family of compatible filtrations $\{E^\alpha(i)\}_{\alpha\in|\Sigma|}$ of E indexed by the ray generators of Σ satisfying the following condition.*

For each cone $\sigma \in \Sigma$, there exists a decomposition

$$E \cong \bigoplus_{\chi\in\widehat{T}_\sigma} E^{[\sigma]}(\chi), \tag{2.3}$$

where $\hat{T}_\sigma$ is the group of characters of the stabilizer subgroup $T_\sigma \subseteq T$ of any point in the unique closed orbit of U_σ, such that

$$E^\alpha(i) \cong \bigoplus_{\langle \chi, \alpha \rangle \geq i} E^{[\sigma]}(\chi)$$

for any $\alpha \in |\sigma|$ and $i \in \mathbb{Z}$.

2.1.4. *Global sections of a toric bundle and their local restrictions.* Given a toric bundle $\mathcal{E}$ on a smooth toric variety $X = X(\Sigma)$, there is an induced action of T on $H^0(X, \mathcal{E})$. The χ-isotypical component $H^0(X, \mathcal{E})_\chi$ of $H^0(X, \mathcal{E})$ can be described in terms of the Klyachko data of $\mathcal{E}$.

PROPOSITION 2.2 [8, Corollary 4.1.3(i)]

Let $X(\Sigma)$ be a smooth toric variety and $\mathcal{E}$ be a toric bundle on X, then

$$H^0(X, \mathcal{E})_\chi = \bigcap_{\alpha \in |\Sigma|} E^\alpha(\langle \chi, \alpha \rangle). \tag{2.4}$$

Remark 2.3. From the proposition above, one observes that the dimension of $H^0(X, \mathcal{E})_\chi$ is bounded by the rank of $\mathcal{E}$. On the other hand, there might exist two different characters χ_1 and χ_2 and sections $s_i \in H^0(X, \mathcal{E})_{\chi_i}$, $i = 1, 2$ such that s_1 and s_2 correspond to the same vector in E.

By the proposition above one reads readily the χ-isotypical component of the space of global sections of a toric bundle $\mathcal{E}$ on a smooth toric variety $X = X(\Sigma)$ from the associated Klyachko data. Now given a vector in E that corresponds to a global section s in $H^0(X, \mathcal{E})_\chi$ and a cone $\sigma \in \Sigma$, we consider the image of s under the restriction map

$$\rho : H^0(X, \mathcal{E}) \to \Gamma(U_\sigma, \mathcal{E}). \tag{2.5}$$

Firstly it is easy to see that the image of s under the restriction map (2.5) falls in the χ-isotypical component of $\Gamma(U_\sigma, \mathcal{E})$, i.e. for all $t \in T$, we have

$$t.\rho(s) = \chi(t)\rho(s).$$

Next we investigate the local sections of $\mathcal{E}$ appearing in the χ-isotypical component of $\Gamma(U_\sigma, \mathcal{E})$. By Proposition 2.1, we have

$$\mathcal{E}|_{U_\sigma} \cong \bigoplus_{1 \leq i \leq r} \chi_\sigma^i \mathcal{O}_{U_\sigma}.$$

We take $\{e_\sigma^i\}$ to be an eigen-basis of $\mathcal{E}|_{U_\sigma}$ whose associated eigen-characters are $\{\chi_\sigma^i\}$. Since U_σ is smooth, we can find $\alpha_{m+1}, \ldots, \alpha_n \in N = \hat{T}^0$ such that $|\sigma| \cup \{\alpha_{m+1}, \ldots, \alpha_n\}$ constitute a basis of N. Let $u_j \in \hat{T}$ be the characters defined by $\langle u_i, \alpha_j \rangle = \delta_{ij}$, $1 \leq i, j \leq n$, then

$$U_\sigma \cong k\,[u_1, \ldots, u_m, u_{m+1}^{\pm 1} \ldots, u_n^{\pm 1}]$$

and an element of $\Gamma(U_\sigma, \mathcal{E})$ can be written as

$$s = \sum_{1\le i\le r} a_i e^i_\sigma, \quad a_i \in k[u_1, \ldots, u_m, u^{\pm 1}_{m+1}, \ldots, u^{\pm 1}_n]. \tag{2.6}$$

By a suitable choice of the multiplicative coordinates of the torus T, the actions of $t = (t_1, \ldots, t_n) \in T$ on u_i, $1 \le i \le n$, and u_j^{-1}, $m+1 \le j \le n$ are given by

$$t.u_i = t_i^{-1} u_i, \quad t.u_j^{-1} = t_j u_j^{-1}.$$

Then, the action of T on a monomial $b = u_1^{i_1} \ldots u_n^{i_n}$, $i_1, \ldots, i_m \ge 0$, $i_{m+1}, \ldots, i_n \in \mathbb{Z}$ is given by

$$t.b = \chi_b(t) b, \quad \chi_b(t) = \prod_{1\le j\le n} t_j^{-i_j}. \tag{2.7}$$

PROPOSITION 2.4

Let $\{e^i_\sigma\}_{1\le i\le r}$ be an eigen-basis of $\mathcal{E}|_{U_\sigma}$ and $\{\chi^i_\sigma\}_{1\le i\le r}$ be the corresponding eigen-characters. Then $s = \sum_{1\le i\le r} a_i e^i_\sigma$ falls in the χ-isotypical component iff $a_i = c_i \prod_{1\le j\le n} u_j^{\langle \chi^i_\sigma, \alpha_j\rangle - \langle \chi, \alpha_j\rangle}$, where $c_i \in k$ and $1 \le i \le r$.

Proof. Let $a_i = \sum_h c_{ih} b_{ih}$, where $c_{ih} \in k$, $b_{ih} \in k[u_1, \ldots, u_m, u^{\pm 1}_{m+1}, \ldots, u^{\pm 1}_n]$ is a monomial. Then by (2.7) the action of $t \in T$ on v is given by

$$t.s = \sum_i \sum_h c_{ih} \chi_{b_{ih}}(t) b_{ih} \chi^i_\sigma(t) e^i_\sigma. \tag{2.8}$$

If s is in the χ-isotypical component of $\Gamma(U_\sigma, \mathcal{E})$, then we have

$$\sum_i \sum_h c_{ih} \chi_{b_{ih}}(t) b_{ih} \chi^i_\sigma(t) e^i_\sigma = \sum_i \sum_h c_{ih} \chi(t) b_{ih} e^i_\sigma.$$

By comparing the two sides of the above equality, one sees easily a_i is a monomial for $1 \le i \le r$. Moreover, by (2.7) we have $\chi_{a_i} = \frac{\chi}{\chi^i_\sigma}$ and $a_i = c_i \prod_{1\le j\le n} u_j^{\langle \chi^i_\sigma, \alpha_j\rangle - \langle \chi, \alpha_j\rangle}$ with $c_i \in k$. Conversely, if a_i can be written in the form $c_i \prod_{1\le j\le n} u_j^{\langle \chi^i_\sigma, \alpha_j\rangle - \langle \chi, \alpha_j\rangle}$, one checks easily $s = \sum_i a_i e^i_\sigma$ falls in the χ-isotypical component of $\Gamma(\mathcal{E}, U_\sigma)$. $\square$

As mentioned in Remark 2.3, for a vector $v \in E$ the characters $\chi \in \hat{T}$ allowing the existence of a global section $s \in H^0(X, \mathcal{E})_\chi$ corresponding to v (Proposition 2.2) might not be unique. Later we will consider the images of such sections under the restriction map (2.5). For that purpose, we choose an eigen-basis $\{e^i_\sigma\}_{1\le i\le r}$ of $\mathcal{E}|_{U_\sigma}$. Moreover, we will take the point x_0 from which we obtain the Klyachko data of $\mathcal{E}$ to be the distinguished point in the open orbit, i.e. the point with the defining ideal $(u_1 - 1, u_2 - 1, \ldots, u_n - 1)$ in $k[U_\sigma]$. Let $e^i_{x_0}$ be the image of e^i_σ under the restriction map $\mathcal{E}|_{U_\sigma} \to \mathcal{E} \otimes k(x_0)$. Then by the choice of x_0 and Proposition 2.4, we get the following.

COROLLARY 2.5

Let $s \in H^0(X, \mathcal{E})_\chi$ *be a global section corresponding to* $v = \sum_i c_i e^i_{x_0}$, $c_i \in k$ *and* x_0 *be the distinguished point in the open orbit. Then the image of* s *under the restriction map* (2.5) *is* $\sum_{1 \le i \le r} a_i e^i_\sigma$, *where* $a_i = c_i \prod_{1 \le j \le n} u_j^{\langle \chi^i_\sigma, \alpha_j \rangle - \langle \chi, \alpha_j \rangle}$.

2.1.5. *Examples*

Example 2.6 [8, §2.3, *Example* 5]. The cotangent and tangent bundle of a smooth toric variety.

The below two propositions first appear in [8, §2.3, Example 5] without proof.

PROPOSITION 2.7

Given a smooth toric variety $X = X(\Sigma)$ *of dimension* n, *the cotangent bundle* Ω^1_X *is a toric bundle on* X. *The associated Klyachko data is given by*

$$E^\alpha(i) = \begin{cases} \Omega = \hat{T} \otimes k, & \text{if } i \le -1; \\ \{\omega \in \Omega \mid \langle \omega, \alpha \rangle = 0\}, & \text{if } i = 0; \\ 0, & \text{if } i > 0. \end{cases} \tag{2.9}$$

Proof. It suffices to prove (2.9) for affine smooth toric varieties. In this case, X is isomorphic to $\mathbb{A}^d \times \mathbb{G}_{\mathbf{m}}^{n-d}$ for some $0 \le d \le n$. Let $\{\alpha_i\}_{1 \le i \le n}$ be a basis of the dual lattice $\hat{T}^0$ of the character lattice $\hat{T}$, then X can be realized as the affine toric variety associated to the cone Σ generated by $\{\alpha_i\}_{1 \le i \le d}$ if $d \ge 1$ or the origin if $d = 0$.

Now we take a basis $\{u_i\}_{1 \le i \le n}$ of the character lattice $\hat{T}$ such that $\langle u_i, \alpha_j \rangle = \delta_{ij}$, $1 \le i, j \le n$. Then the open immersion $T \hookrightarrow X$ is given by

$$k[u_1, \ldots, u_d, u_{d+1}^{\pm 1}, \ldots, u_n^{\pm 1}] \hookrightarrow k[u_1^{\pm 1}, \ldots, u_n^{\pm 1}]$$

and the action of an element $t = (t_1, \ldots, t_n)$ on $\mathcal{O}_X$ is given by

$$t.u_i = t_i^{-1} u_i, \quad 1 \le i \le n. \tag{2.10}$$

Thus the action of t on du_i is given by

$$t.du_i = t_i^{-1} du_i, \quad 1 \le i \le n. \tag{2.11}$$

Then $\{du_i\}_{1 \le i \le n}$ consist of an eigen-basis of Ω^1_X and $\{\chi_i\}_{1 \le i \le n}$ defined by $\chi_i(t) = t_i^{-1}$ for $t = (t_1, \ldots, t_n) \in T$ and the associated eigen-characters.

Now let α be a ray generator of Σ. Then $\alpha = \alpha_l$ for some $1 \le l \le d$. By Theorem 2, we have

$$E^{\alpha_l}(j) = \bigoplus_{\langle \chi_i, \alpha_l \rangle \ge j} k du_i.$$

By identifying the k-vector spaces with basis $\{du_i\}_{1\le i\le n}$ and $\{u_i\}_{1\le i\le n}$, we obtain the conclusion of the proposition. $\square$

The tangent bundle $\mathcal{T}_X$ is also a toric bundle on X and over any open affine sub toric variety of X the representation of the torus associated to $\mathcal{T}_X$ (see Proposition 2.1(2)) is the dual of the representation associated to Ω^1_X. Thus we have the following

PROPOSITION 2.8

The Klyachko data associated to the tangent bundle $\mathcal{T}_X$ of a smooth toric variety is given by

$$E^\alpha(i) = \begin{cases} \mathscr{T} = \hat{T}^0 \otimes k, & \text{if } i \le 0;\\ k\alpha, & \text{if } i = 1;\\ 0, & \text{if } i > 1. \end{cases} \tag{2.12}$$

Proof. It suffices to prove the case when X is an affine toric variety. Following the notations in the proof of Proposition 2.7, we have an eigen-basis $\{\frac{\partial}{\partial u_i}\}_{1\le i\le n}$ of $\mathcal{T}_X$. Moreover, the action of an element $t = (t_1, \ldots, t_n)$ of T on the eigen-basis is given by

$$t.\frac{\partial}{\partial u_i} = t_i\frac{\partial}{\partial u_i}, \quad 1 \le i \le n.$$

Similar as in the proof of Proposition 2.7, one gets the Klyachko data associated to $\mathcal{T}_X$ is (2.12). $\square$

By applying Proposition 2.2 to the tangent bundle, we get the following.

COROLLARY 2.9

Let X be a smooth toric variety and $0 \in \hat{T}$ be the trivial character. Then the restriction map $H^0(X, \mathcal{T}_X)_0 \to H^0(T, \mathcal{T}_T)_0$ is bijective.

The corollary above says nothing but the dimension of global T-invariant tangent vector fields of a smooth toric variety is equal to the dimension of the toric variety.

Example 2.10. The tensor product, symmetric product and wedge product of toric bundles.

Let X be a toric variety and $\mathcal{E}$ be a toric bundle on X whose Klyachko data is given by $\{E^\alpha(i)\}_{\alpha\in|\Sigma|}$, $i \in \mathbb{Z}$. We define the following integer-valued function on E for each $\alpha \in |\Sigma|$:

$$\varphi^\alpha_{\mathcal{E}}: \quad E \to \mathbb{Z}, \quad e \mapsto \max_i \{i \in \mathbb{Z} \mid e \in E^\alpha(i)\}.$$

Let $\mathcal{E}$ and $\mathcal{F}$ be two toric bundles on X, whose Klyachko data are given by $\{E^\alpha(i)\}_{\alpha\in|\Sigma|}$ and $\{F^\alpha(i)\}_{\alpha\in|\Sigma|}$. Then $\mathcal{E}\otimes\mathcal{F}$ is also a toric bundle and the Klyachko data can be read from those of $\mathcal{E}$ and $\mathcal{F}$ as follows: For each $\alpha \in |\Sigma|$, $(E\otimes F)^\alpha(i)$ is the subspace spanned by the set of vectors

$$\{e\otimes f \mid e \in E, f \in F;\ \varphi^\alpha_{\mathcal{E}}(e) + \varphi^\alpha_{\mathcal{F}}(f) \ge i\}.$$

Similarly, for each integer $m \geq 1$ and $n \leq r$, where r is the rank of $\mathcal{E}$, the symmetric product $S^m\mathcal{E}$ and wedge product $\wedge^n\mathcal{E}$ are also toric bundles. The associated Klyachko data of these toric bundles are the families of filtrations on the vector spaces $S^m E$ and $\wedge^n E$ described as follows:

$$(S^m E)^\alpha(i) = \left\{ \sum \prod_{1\leq j\leq m} e_j \mid e_j \in E, 1 \leq j \leq m; \sum_{1\leq j\leq m} \varphi^\alpha_{\mathcal{E}}(e_j) \geq i \right\},$$

$$(\wedge^n E)^\alpha(i) = \left\{ \sum e_1 \wedge \cdots \wedge e_j \wedge \cdots \wedge e_n \mid e_j \in E, 1 \leq j \leq n; \right.$$

$$\left. \sum_{1\leq j\leq m} \varphi^\alpha_{\mathcal{E}}(e_j) \geq i \right\}.$$

Example 2.11. The determinant of a toric bundle.

Let $\mathcal{E}$ be a toric bundle on a toric variety X whose Klyachko data is given by $\{E^\alpha(i)\}_{\alpha\in|\Sigma|}$, $i \in \mathbb{Z}$ as in Theorem 2. Now for each $\alpha \in |\Sigma|$, we introduce a finite subset of integers $I^\alpha(\mathcal{E}) \subset \mathbb{Z}$, which is defined by

$$i \in I^\alpha(\mathcal{E}) \Leftrightarrow E^\alpha(i+1) \subsetneq E^\alpha(i).$$

Then the Klyachko data associated to the line bundle $\det \mathcal{E}$ is given by the integer-valued function [8, §2.3, Example 1]

$$\rho \mapsto \sum_{i\in I^\alpha(\mathcal{E})} d_i i \ , \tag{2.13}$$

where d_i is the dimension of the vector space $E^\alpha(i)/E^\alpha(i+1)$.

Example 2.12. The pullback of a toric bundle by a toric morphism.

Let X_1, X_2 be two toric varieties and $f : X_1 \to X_2$ be a toric morphism, i.e. a morphism compatible with the actions of T_i on X_i, $i = 1, 2$. Then one checks easily for any toric bundle $\mathcal{E}$ on X_2 the pullback $\mathcal{F} = f^*\mathcal{E}$ is a toric bundle on X_1.

Next we consider the Klyachko data of $\mathcal{F}$. By Theorem 2, it suffices to investigate the special case when both X_1 and X_2 are affine. In this case we assume X_i is isomorphic to U_{σ_i}, where σ_i is a strongly convex rational polyhedra cone in $N_i = \widehat{T}^0_i \otimes \mathbb{R}$, $i = 1, 2$. Since f is a toric morphism it induces a homomorphism $T_1 \to T_2$ and the following homomorphisms of lattices

$$\varphi : N_1 \to N_2, \quad \varphi^\vee : N_2^\vee \to N_1^\vee.$$

By abuse of notation, we still denote by φ the induced morphism $N_1 \otimes_{\mathbb{Z}} \mathbb{R} \to N_2 \otimes_{\mathbb{Z}} \mathbb{R}$. Then the image of σ_1 under φ falls in σ_2 since f is a toric morphism.

Suppose the toric bundle $\mathcal{E}$ over X_2 has the following decomposition:

$$\mathcal{E} \cong \bigoplus_{1\leq j\leq r} \chi_j \mathcal{O}_{X_2}$$

with eigen-basis $\{e^j_{\sigma_2}\}_{1\le j\le r}$ and eigen-characters $\{\chi_j\}_{1\le j\le r}$. Then $\mathcal{F} = f^*\mathcal{E}$ over X_1 decomposes as

$$\mathcal{F} = f^*\mathcal{E} \cong \bigoplus_{1\le j\le r} \varphi^\vee(\chi_j)\mathcal{O}_{X_1}.$$

Let x_0 be a point in the open orbit of X_1, $F = \mathcal{F} \otimes k(x_0)$ and e^j, $1 \le j \le n$ be the image of $e^j_{\sigma_2}$ under the restriction map $\mathcal{E} \to \mathcal{E} \otimes k(f(x_0))$. Then by Theorem 2 for each $\beta \in |\sigma_1|$ and $i \in \mathbb{Z}$, we have

$$F^\beta(i) = \bigoplus_{\langle\varphi^\vee(\chi_j),\beta\rangle\ge i} k(1\otimes e^j) = \bigoplus_{\langle\chi_j,\varphi(\beta)\rangle\ge i} k(1\otimes e^j) \tag{2.14}$$

since $\langle\varphi^\vee(\chi_j), \beta\rangle = \langle\chi_j, \varphi(\beta)\rangle$. In particular, if $\varphi(\beta)$ is a ray generator of σ_2 and the Klyachko data of $\mathcal{E}$ is given by $\{E^\alpha(i)\}_{\alpha\in|\sigma_2|}$, we have $F^\beta(i) = f^*(E^{\varphi(\beta)}(i))$.

Following the notations of Example 2.12, if we assume further X_1 and X_2 are both smooth, then we have an induced morphism $\mathcal{T}_{X_1} \to f^*\mathcal{T}_{X_2}$. Since f is toric, this morphism is T_1-equivariant. Next we will compute the corresponding morphism between the Klyachko data of $\mathcal{T}_{X_1}$ and $f^*\mathcal{T}_{X_2}$. By Proposition 2.8, this is indeed a morphism from $N_1 \otimes k$ to $N_2 \otimes k$. Moreover, it is the dual of the morphism from ${N_2}^\vee \otimes k$ to ${N_1}^\vee \otimes k$ which corresponds to the morphism $f^*\Omega^1_{X_2} \to \Omega^1_{X_1}$.One can check easily over the open orbit that the latter morphism is nothing but the one induced from $\varphi^\vee : {N_2}^\vee \to {N_1}^\vee$ hence the morphism $N_1 \otimes k \to N_2 \otimes k$ is the one induced from $\varphi : N_1 \to N_2$, which defines the toric morphism $X_1 \to X_2$. To sum up, we get the following.

PROPOSITION 2.13

Let $X_1 \to X_2$ be a toric morphism of smooth toric varieties, then the induced morphisms $f^*\Omega^1_{X_2} \to \Omega^1_{X_1}$ and $\mathcal{T}_{X_1} \to f^*\mathcal{T}_{X_2}$ are morphisms of toric bundles and the corresponding morphisms on Klyachko data are those induced from $\varphi^\vee : {N_2}^\vee \to {N_1}^\vee$ and $\varphi : N_1 \to N_2$, respectively.

2.2 *A criterion for Frobenius splitting of smooth complete varieties*

Given an algebraic variety X over k, it is said to be *Frobenius split* or *F-split* if the injective morphism

$$\mathcal{O}_X \to F_{X*}\mathcal{O}_X \tag{2.15}$$

defined by sending a section to its p-th power splits as a homomorphism of $\mathcal{O}_X$-modules.

It is easy to see that to prove an algebraic variety X is F-split can be reduced to find a global section φ of $\mathscr{H}om(F_*\mathcal{O}_X, \mathcal{O}_X)$ such that $\varphi(1) = 1$. Based on this fact, we will give a more explicit description of $\mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X)$.

By definition of the upper shriek functor [6, Chapter 2, exe. 6.10], we have the following isomorphism

$$\mathscr{H}om(F_{X*}\mathcal{O}_X, \mathcal{O}_X) \cong F_{X*}\mathscr{H}om(\mathcal{O}_X, F^!\mathcal{O}_X). \tag{2.16}$$

Then by [5], we have an isomorphism

$$F^!\mathcal{O}_X \cong \mathscr{H}om(F^*\omega_X, \omega_X) \cong \omega_X^{1-p}. \tag{2.17}$$

Now by combining (2.16) and (2.17), we have the following isomorphism of sheaves of $\mathcal{O}_X$-modules,

$$F_{X*}\omega_X^{1-p} \cong \mathscr{H}om(F_{X*}\mathcal{O}_X, \mathcal{O}_X), \tag{2.18}$$

whence there is an induced isomorphism on spaces of global sections.

Next we recall a more direct way to derive (2.18). First, note that we have the following isomorphism:

$$\epsilon : F_{X*}\omega_X^{1-p} \cong F_{X*}\mathscr{H}om(F_X^*\omega_X, \omega_X) \cong \mathscr{H}om(\omega_X, F_{X*}\omega_X). \tag{2.19}$$

We will define an isomorphism from $\mathscr{H}om(\omega_X, F_{X*}\omega_X)$ to $\mathscr{H}om(F_{X*}\mathcal{O}_X, \mathcal{O}_X)$ and our new description of (2.18) will be the composite of this isomorphism with (2.19). For this purpose, we need to use the *trace map*

$$\tau : F_{X*}\omega_X \to \omega_X.$$

Recall that τ is the composite of the projection $F_{X*}\omega_X \to \mathcal{H}^n(F_{X*}\Omega^\bullet_{X/k})$ with the Cartier isomorphism

$$\mathcal{H}^n(F_{X*}\Omega^\bullet_{X/k}) \xrightarrow{\approx} \omega_X,$$

where $n = \dim X$. The trace map can be written explicitly in terms of local coordinates as follows.

Lemma 1 [2, *Proposition* 1.3.6]. *Let X be a nonsingular variety of dimension n and $x_1, \ldots, x_n$ be a set of local coordinates at a point x of X. Then the trace map τ is given by*

$$\tau(f dx_1 \wedge \cdots \wedge dx_n) = Tr(f) dx_1 \wedge \cdots \wedge dx_n,$$

where $f = \sum_{\mathbf{i}} f_{\mathbf{i}} x^{\mathbf{i}} \in \mathcal{O}_{X,x} \subset k[[x_1, \ldots, x_n]]$ and

$$Tr(f) := \sum_{\mathbf{i}} f_{\mathbf{i}}^{\frac{1}{p}} x^{\mathbf{j}}$$

with the summation taken over all multi-index $\mathbf{i}$ such that $\mathbf{i} = \mathbf{p} - \mathbf{1} + p\mathbf{j}$ for some $\mathbf{j} \in \mathbb{N}^n$.

By introducing the trace map τ, one can prove the isomorphism (2.18) is nothing but the composite of (2.19) and a morphism ι given by the following

Lemma 2 [2, *Proposition* 1.3.7]. *The morphism*

$$\iota : \mathscr{H}om(\omega_X, F_{X*}\omega_X) \to \mathscr{H}om(F_{X*}\mathcal{O}_X, \mathcal{O}_X) \tag{2.20}$$

defined by $\iota(\psi)(f)\omega = \tau(f\psi(\omega))$, where f is a local section of $\mathcal{O}_X$ and ω is a local generator of ω_X, is an isomorphism.

Now by replacing the isomorphism (2.18) with the composite $\iota\epsilon$, we can give the following criterion for Frobenius splitting.

PROPOSITION 2.14

Let X be a smooth variety over k. Then X is Frobenius split iff there exists $\varphi \in H^0(X, \omega_X^{1-p})$ such that $\iota(\epsilon(\varphi))(1) = 1$.

For simplicity, from now on we will just say $\varphi \in H^0(X, \omega_X^{1-p})$ defines a Frobenius splitting of X if it satisfies $\iota(\epsilon(\varphi))(1) = 1$.

We conclude this section with the following lemma, which will be used later.

Lemma 3. *Let X be a smooth algebraic variety, $\mathcal{E}$ be a vector bundle of rank r on X and $\pi : \mathbb{P}(\mathcal{E}) \to X$ be the projection. Then for any integer $n \geq 0$, we have the following isomorphism*:

$$\pi_*\omega_{\mathbb{P}(\mathcal{E})}^{-n} \cong S^{rn}\mathcal{E} \otimes (\det \mathcal{E})^{-n} \otimes \omega_X^{-n}. \tag{2.21}$$

Proof. Firstly, we have the following relative Euler exact sequence over $\mathbb{P}(\mathcal{E})$,

$$0 \to \Omega^1_{\mathbb{P}(\mathcal{E})/X} \to \pi^*\mathcal{E} \otimes \mathscr{O}_{\mathbb{P}(\mathcal{E})}(-1) \to \mathscr{O}_{\mathbb{P}(\mathcal{E})} \to 0.$$

By taking determinant, we get

$$\omega_{\mathbb{P}(\mathcal{E})/X} \cong \det(\pi^*\mathcal{E}) \otimes \mathscr{O}_{\mathbb{P}(\mathcal{E})}(-r) \cong \pi^* \det \mathcal{E} \otimes \mathscr{O}_{\mathbb{P}(\mathcal{E})}(-r).$$

Therefore, $\omega_{\mathbb{P}(\mathcal{E})} \cong \omega_{\mathbb{P}(\mathcal{E})/X} \otimes \pi^*\omega_X \cong \pi^*(\det \mathcal{E} \otimes \omega_X) \otimes \mathscr{O}_{\mathbb{P}(\mathcal{E})}(-r)$ and $\omega_{\mathbb{P}(\mathcal{E})}^{-n} \cong \pi^*((\det \mathcal{E})^{-n} \otimes \omega_X^{-n}) \otimes \mathscr{O}_{\mathbb{P}(\mathcal{E})}(nr)$. Then by applying the projection formula one gets the isomorphism (2.21). □

3. Frobenius splitting of projectivization of toric bundles

By Proposition 2.14, to prove a smooth algebraic variety Y is F-split, it suffices to find an element of $H^0(Y, \omega_Y^{1-p})$ whose image under $\iota\epsilon$ sends 1 to 1. We will see later (3.4) that for $Y = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a toric bundle on a toric variety X, the vector space $H^0(Y, \omega_Y^{1-p})$ is endowed with an action of the torus. In this case, we obtain a new criterion (Proposition 3.3) for the Frobenius splitting of Y, by which we prove our main theorem.

3.1 *Frobenius splitting of a toric variety*

Next we investigate the Frobenius splitting of a toric variety in detail. By the main result of [10], all smooth toric varieties are F-split. Then Proposition 2.14 implies that for any smooth complete toric variety X there exists a global section of ω_X^{1-p} defining a Frobenius splitting of X. On the other hand, we have the following decomposition:

$$H^0(X, \omega_X^{1-p}) \cong \bigoplus_{\chi \in \widehat{T}} H^0(X, \omega_X^{1-p})_\chi, \tag{3.1}$$

where $H^0(X, \omega_X^{1-p})_\chi$ is the χ-isotypical component of $H^0(X, \omega_X^{1-p})$. The global section of ω_X^{1-p} defining the Frobenius splitting of X can be chosen as described in the following.

PROPOSITION 3.1

Let X be a smooth toric variety and T be its open orbit, then there is a character $\chi \in \hat{T}$ and a nonzero element of $H^0(X, \omega_X^{1-p})_\chi$ defining a Frobenius splitting of X. Moreover, trivial character is the only character satisfying this property.

Proof. Firstly, by [2, Exe. 1.3.6], there exists a unique T-equivariant Frobenius splitting of X. Therefore, there is an unique character $\chi \in \hat{T}$ such that this Frobenius splitting is defined by an element of $H^0(X, \omega_X^{1-p})_\chi$ (Proposition 2.14). Next we prove the second claim. Since the image of $H^0(X, \omega_X^{1-p})_\chi$ under the restriction morphism

$$H^0(X, \omega_X^{1-p}) \to \Gamma(T, \omega_X^{1-p})$$

falls in the χ-isotypical component $\Gamma(T, \omega_X^{1-p})_\chi$, it suffices to prove the claim in the special case when $X = T$. Let $\{\alpha_i\}_{1 \le i \le n}$ be a basis of the lattice $\hat{T}^0$ and $u_j \in \hat{T}$ be the characters defined by $\langle u_j, \alpha_i \rangle = \delta_{ij}$, $1 \le i, j \le n$. Then an eigen-basis of ω_T^{1-p} is given by

$$\left(\frac{\partial}{\partial u_1} \wedge \cdots \wedge \frac{\partial}{\partial u_n}\right)^{\otimes(p-1)}.$$

Moreover, by Proposition 2.7 and Example 2.10, the corresponding eigen-character χ_0 is defined by $\langle \chi_0, \alpha_i \rangle = p - 1$, $1 \le i \le n$. Now by Proposition 2.4, an element of $H^0(T, \omega_T^{1-p})_\chi$ is of the following form:

$$c \prod_i u_i^{p-1-\langle \chi, \alpha_i \rangle} \left(\frac{\partial}{\partial u_1} \wedge \cdots \wedge \frac{\partial}{\partial u_n}\right)^{\otimes(p-1)}, \quad c \in k.$$

By Lemma 1 and Lemma 2, the above section defines a Frobenius splitting of T iff $c = 1$ and $\langle \chi, \alpha_i \rangle = 0$ for all $1 \le i \le n$, i.e. χ is the trivial character as desired. □

3.2 *Proof of the main theorem*

Before proving our main result, we will investigate Frobenius splittings of the projectivization of a toric bundle in a little more general setting.

Let X be a smooth algebraic variety, $\mathcal{E}$ be a vector bundle on X of rank r, $Y = \mathbb{P}(\mathcal{E})$ and $f : Y \to X$ be the projection. Then we have an isomorphism $\mathcal{O}_X \cong f_*\mathcal{O}_Y$, which induces a morphism

$$\mathrm{Hom}(F_{Y*}\mathcal{O}_Y, \mathcal{O}_Y) \to \mathrm{Hom}(F_{X*}\mathcal{O}_X, \mathcal{O}_X), \quad \varphi \mapsto f_*\varphi. \tag{3.2}$$

One sees easily that φ defines a Frobenius splitting of Y iff $f_*\varphi$ defines a Frobenius splitting of X. By applying the isomorphism (2.18) to both sides of (3.2), we get the following morphism:

$$\eta : H^0(Y, \omega_Y^{1-p}) \to H^0(X, \omega_X^{1-p}). \tag{3.3}$$

On the other hand, by Lemma 3, we have the following isomorphism:

$$\beta : H^0(Y, \omega_Y^{1-p}) \cong H^0(X, S^{r(p-1)}\mathcal{E} \otimes (\det \mathcal{E})^{1-p} \otimes \omega_X^{1-p}). \tag{3.4}$$

If X is a toric variety and $\mathcal{E}$ is a toric bundle on X, then $S^{r(p-1)}\mathcal{E} \otimes (\det \mathcal{E})^{1-p} \otimes \omega_X^{1-p}$ is also a toric bundle on X. Hence there is an induced action of T on $H^0(Y, \omega_Y^{1-p})$. Next we will prove the following.

PROPOSITION 3.2

The morphism $\eta\beta^{-1}$ is T-equivariant.

Proof. It suffices to show for any $\chi \in \hat{T}$, the image of the χ-isotypical component $H^0(Y, \omega_Y^{1-p})_\chi$ under η falls in $H^0(X, \omega_X^{1-p})_\chi$ and this can be checked locally. Thus we may just assume $X = T$. Since any two toric line bundles on T are isomorphic as toric bundles we can assume $\mathcal{E}$ is a direct sum of toric line bundles with trivial eigen-characters. Hence $\det \mathcal{E}$ is a trivial toric line bundle. Let $\{e_i\}_{1\le i\le r}$ be an eigen-basis of $\mathcal{E}$. Then

$$e^{\mathbf{I}} = e_1^{k_1} \cdots e_r^{k_r},$$

indexed by $\mathbf{I} \in S = \{(k_1, \ldots, k_r) \mid k_1, \ldots, k_r \ge 0,\ k_1 + \cdots + k_r = r(p-1)\}$ form an eigen-basis of $S^{r(p-1)}\mathcal{E}$.

Let $\{\alpha_i\}_{1\le i\le n}$ be a basis of the lattice $\hat{T}^0 u_j$, $1 \le j \le n$ and χ_0 be the characters defined as follows:

$$\langle u_j, \alpha_i\rangle = \delta_{ij}, \quad \langle \chi_0, \alpha_i\rangle = p-1, \quad 1 \le i, j \le n.$$

Then by Proposition 2.7, the toric line bundle ω_T^{1-p} is isomorphic to $\chi_0\mathcal{O}_T$ and the eigen-basis is

$$w_0 = \left(\frac{\partial}{\partial u_j} \wedge \cdots \wedge \frac{\partial}{\partial u_n}\right)^{\otimes(p-1)}.$$

Then the toric bundle $S^{r(p-1)}\mathcal{E} \otimes \omega_T^{1-p}$ is isomorphic to

$$\bigoplus_{\mathbf{I}\in S} \chi_0\mathcal{O}_T,$$

where the corresponding eigen-basis is $\{e^{\mathbf{I}} \otimes w_0\}_{\mathbf{I}\in S}$. Then a section s in the χ-isotypical component of $H^0(S^{r(p-1)}\mathcal{E} \otimes \omega_T^{1-p})$ can be written as

$$s = \sum_{\mathbf{I}\in S} a_{\mathbf{I}} e^{\mathbf{I}} \otimes w_0,$$

with $a_{\mathbf{I}} \in \mathcal{O}_T$ are as described in Proposition 2.4.

Next we consider $\beta^{-1}(s)$. Let x be a closed point of T and V be the fiber $\mathcal{E}_x$ and $e_{i,x}$ be the image of e_i in $\mathcal{E}_x$. Then we have $Y \cong \mathbb{P}(V) \times T$. Let $p_1 : Y \to \mathbb{P}(V)$, $p_2 : Y \to T$ be the two projections. Then we get

$$\omega_Y^{1-p} \cong p_1^* \omega_{\mathbb{P}(V)}^{1-p} \otimes p_2^* \omega_T^{1-p}$$

and

$$H^0(\omega_Y^{1-p}) \cong H^0(\omega_{\mathbb{P}(V)}^{1-p}) \otimes H^0(\omega_T^{1-p}).$$

Moreover, recall that we have the following isomorphism:

$$\gamma : H^0(\omega_{\mathbb{P}(V)}^{1-p}) \cong H^0(\mathcal{O}_{\mathbb{P}(V)}(r(p-1))).$$

Then we get

$$\beta^{-1}(e^{\mathbf{I}} \otimes a_{\mathbf{I}} w_0) = p_1^*(\gamma^{-1}(e_x^{\mathbf{I}})) \otimes p_2^*(a_{\mathbf{I}} w_0),$$

where $e_x^{\mathbf{I}} = \prod_{1 \le i \le r} e_{i,x}^{k_i}$ for $\mathbf{I} = (k_1, \ldots, k_r)$. By definition, the image of $\beta^{-1}(e^{\mathbf{I}} \otimes a_{\mathbf{I}} w_0)$ under η is nonzero only if $e_x^{\mathbf{I}}$ defines a Frobenius splitting of $\mathbb{P}(V)$, which by [2, Exe 1.3.(1)] is true only for $\mathbf{I} = \mathbf{p-1} = (p-1, p-1, \ldots, p-1)$. Thus we can assume $s = a_{\mathbf{p-1}} e^{\mathbf{p-1}} \otimes w_0$.

Next we prove for $s = a_{\mathbf{p-1}} e^{\mathbf{p-1}} \otimes w_0$ contained in the χ-isotypical component of $H^0(S^{r(p-1)} \otimes \omega_X^{1-p})$, $\eta\beta^{-1}(s)$ falls in the χ-isotypical component of $H^0(\omega_X^{1-p})$. By the above argument, one sees easily $\eta\beta^{-1}(s) = a_{\mathbf{p-1}} w_0$. By our assumption on $\mathcal{E}$, the eigen-character corresponding to $e^{\mathbf{p-1}}$ is the trivial character. Then by Proposition 2.4,

$$a_{\mathbf{p-1}} = c \prod_{1 \le j \le n} u_j^{p-1-\langle \chi, \alpha_j \rangle}$$

for some $c \in k$. Consequently, $a_{\mathbf{p-1}} w_0$ falls in the χ-isotypical component of $H^0(\omega_X^{1-p})$ as desired. □

Now by combining Propositions 3.2 and 3.1, one gets the following

PROPOSITION 3.3

Let $\mathcal{E}$ be a toric bundle on a smooth complete toric variety X. Then $Y = \mathbb{P}(\mathcal{E})$ is F-split iff there exists $w \in H^0(Y, \omega_Y^{1-p})_0$, where $0 \in \hat{T}$ is the trivial character, such that $\eta(w)$ (see (3.3)) is nonzero.

Proof of Theorem 1. By Proposition 3.3, to prove the theorem, it suffices to find that $w \in H^0(\mathbb{P}(\mathcal{T}_X), \omega_{\mathbb{P}(\mathcal{T}_X)}^{1-p})_0$ such that $\eta(w)$ (see (3.3)) is nonzero. Now we consider the following diagram:

$$\begin{array}{ccccc} H^0(X, S^{n(p-1)}\mathcal{T}_X)_0 & \xleftarrow{\beta_X} & H^0(\mathbb{P}(\mathcal{T}_X), \omega_{\mathbb{P}(\mathcal{T}_X)}^{p-1})_0 & \xrightarrow{\eta_X} & H^0(X, \omega_X^{p-1})_0 = k \\ \downarrow & & \downarrow & & \| \\ H^0(T, S^{n(p-1)}\mathcal{T}_T)_0 & \xleftarrow{\beta_T} & H^0(\mathbb{P}(\mathcal{T}_T), \omega_{\mathbb{P}(\mathcal{T}_T)}^{p-1})_0 & \xrightarrow{\eta_T} & H^0(T, \omega_T^{p-1})_0 = k \end{array}$$

in which the horizontal arrows are defined as (3.3) and (3.4) and the vertical arrows are restrictions to the open orbit T and the open subset $\mathbb{P}(\mathcal{T}_T)$ of $\mathbb{P}(\mathcal{T}_X)$. Note that by Corollary 2.9, we have $H^0(X, \mathcal{T}_X)_0 \cong H^0(T, \mathcal{T}_T)_0$. Let f_i, $1 \le i \le n$ be n linearly independent elements of $H^0(T, \mathcal{T}_T)_0$. One sees easily that $f_1^{p-1} \cdots f_n^{p-1}$ defines a nonzero element of $H^0(T, S^{n(p-1)}\mathcal{T}_T)_0$ and $H^0(X, S^{n(p-1)}\mathcal{T}_X)_0$. Moreover, $\eta_X \beta_X^{-1}(f_1^{p-1} \cdots f_n^{p-1})$ is nonzero iff $\eta_T \beta_T^{-1}(f_1^{p-1} \cdots f_n^{p-1})$ is. Therefore, we are reduced to the case when X is a torus. If we follow the notations of the proof of Proposition 3.2 and take $f_i = \frac{\partial}{\partial u_i}$, $1 \le i \le n$, then $\eta_T \beta_T^{-1}(f_1^{p-1} \cdots f_n^{p-1})$ is nothing but $u_1^{p-1} \cdots u_n^{p-1}(\frac{\partial}{\partial u_1} \wedge \cdots \wedge \frac{\partial}{\partial u_n})^{\otimes(p-1)}$, hence the theorem is proved. $\square$

Remark 3.4. The theorem above provides a $(p-1)$-th power Frobenius splitting ([2, Exe. 1.3(2)]) for the cotangent bundle of a toric variety. Moreover, like toric varieties, this Frobenius splitting is indeed defined over $\mathbb{Z}$. In other words, we can find an element s in $H^0(\omega_{\mathbb{P}(\mathcal{T}_X)}^{-1})$ for a toric variety X defined over $\mathbb{Z}$ such that the image of s^{p-1} induces a Frobenius splitting of the reduction of $\mathbb{P}(\mathcal{T}_X)$ modulo p for each prime p.

3.3 *An F-split variety whose cotangent bundle is not F-split*

In [1, Question 8.7] the authors asked whether the cotangent bundle of an F-split variety is always F-split. The following proposition combined with [1, Lemma 8.9] provides a negative answer in characteristic 2. It is not known whether this proposition is true in larger characteristics.

PROPOSITION 3.5

Let S be a del Pezzo surface of degree 6 over an algebraically closed filed k of characteristic 2, P be a point of S and $Bl_P S$ be the blow-up of S at P. Then the cotangent bundle of $Bl_P S$ is Frobenius split iff P is a fixed point of S under the torus action.

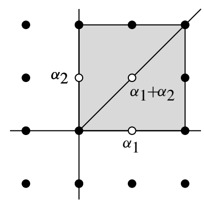
First we prove a lemma, which will be used in the proof of the proposition.

Lemma 4. *Let P be a closed point of $\mathbb{A}^2 \cong \operatorname{Spec} k[x, y]$, $I_P = (x-a, y-b)$ be the defining ideal of P in $\mathcal{O}_{\mathbb{A}^2}$, $X = Bl_P\mathbb{A}^2$ be the blow-up of $\mathbb{A}^2$ at P and $\pi : X \to \mathbb{A}^2$ be the projection. Then $\pi_* S^2\mathcal{T}_X$ regarded as a submodule of $S^2\mathcal{T}_{\mathbb{A}^2}$ is generated by the following elements*:

(1) $(x-a)^2(\frac{\partial}{\partial x})^2, (x-a)(y-b)(\frac{\partial}{\partial x})^2, (y-b)^2(\frac{\partial}{\partial x})^2$;
(2) $(x-a)^2(\frac{\partial}{\partial y})^2, (x-a)(y-b)(\frac{\partial}{\partial y})^2, (y-b)^2(\frac{\partial}{\partial y})^2$;
(3) $(x-a)^2\frac{\partial}{\partial x}\frac{\partial}{\partial y}, (x-a)(y-b)\frac{\partial}{\partial x}\frac{\partial}{\partial y}, (y-b)^2\frac{\partial}{\partial x}\frac{\partial}{\partial y}$;
(4) $(y-b)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + (x-a)(\frac{\partial}{\partial x})^2$;

(5) $(x-a)\frac{\partial}{\partial x}\frac{\partial}{\partial y}+(y-b)(\frac{\partial}{\partial y})^2$.

Proof. It is easy to see that one needs only to prove the lemma in the special case when $I_P=(x,y)$. In this case X is a toric variety endowed with an action by a two-dimensional torus T. As is shown in the figure below, the toric structure of X can be given by the fan Σ in $N\otimes\mathbb{R}=\hat{T}^0\otimes\mathbb{R}\cong\mathbb{R}^2$ (see Definition 2.1) with $|\Sigma|=\{\alpha_1,\alpha_2,\alpha_1+\alpha_2\}$.



Let σ be the cone with $|\sigma|=\{\alpha_1,\alpha_2\}$, then $U_\sigma\cong\mathbb{A}^2$. Note that the identity map of N induces a map of fans $\Sigma\to\sigma$, whose corresponding toric morphism is nothing but the projection $\pi:X\to\mathbb{A}^2$. Then by Proposition 2.13, π induces a morphism of toric bundles

$$\phi:\mathcal{T}_X\to\pi^*\mathcal{T}_{\mathbb{A}^2}.$$

One sees easily the morphism above is an isomorphism when restricted to the open subset $V=X\backslash\{C\}$, where C is the exceptional curve. Therefore the following induced morphism

$$\varphi:S^2\mathcal{T}_X\to S^2(\pi^*\mathcal{T}_{\mathbb{A}^2})\cong\pi^*S^2\mathcal{T}_{\mathbb{A}^2}$$

is injective. By projection formula, we have $H^0(\pi^*S^2\mathcal{T}_{\mathbb{A}^2})\cong H^0(S^2\mathcal{T}_{\mathbb{A}^2})$. Therefore, to prove the lemma it suffices to decide the subspace $\varphi(H^0(S^2\mathcal{T}_X))$ of $H^0(\pi^*S^2\mathcal{T}_{\mathbb{A}^2})$. Furthermore, since φ is equivariant, we are reduced to decide the subspace $\varphi(H^0(S^2\mathcal{T}_X)_\chi)$ of $H^0(\pi^*S^2\mathcal{T}_{\mathbb{A}^2})_\chi$ for each $\chi\in\hat{T}$.

Let E be the k-vector space spanned by α_1,α_2. Then by Proposition 2.8, the Klyachko data for $\mathcal{T}_X$ and $\mathcal{T}_{\mathbb{A}^2}$ can be realized as families of filtrations on E indexed by ray generators of $|\Sigma|$ and $|\sigma|$ respectively. Then by applying Examples 2.10 and 2.12, the Klyachko data of $S^2\mathcal{T}_X$ and $\pi^*S^2\mathcal{T}_{\mathbb{A}^2}$ can be realized as families of filtration on S^2E indexed by α_1,α_2 and $\alpha_1+\alpha_2$. Moreover, the morphisms of Klyachko data corresponding to ϕ and φ are the identity maps of E and S^2E respectively.

Now we will apply Proposition 2.2 to represent the global section by using Klyachko data. To eliminate ambiguity of notations, we use E_1 and E_2 to denote E equipped with indexed filtrations corresponding to the Klyachko data of $\mathcal{T}_X$ and $\pi^*\mathcal{T}_{\mathbb{A}^2}$ respectively. Then we get

$$H^0(S^2\mathcal{T}_X)_\chi=S^2E_1^{\alpha_1}(\langle\chi,\alpha_1\rangle)\cap S^2E_1^{\alpha_2}(\langle\chi,\alpha_2\rangle)\cap S^2E_1^{\alpha_1+\alpha_2}(\langle\chi,\alpha_1+\alpha_2\rangle) \tag{3.5}$$

and

$$H^0(\pi^*S^2\mathcal{T}_{\mathbb{A}^2})_\chi=H^0(S^2\mathcal{T}_{\mathbb{A}^2})_\chi=S^2E_2^{\alpha_1}(\langle\chi,\alpha_1\rangle)\cap S^2E_2^{\alpha_2}(\langle\chi,\alpha_2\rangle). \tag{3.6}$$

Since for any $\chi \in \hat{T}$, $H^0(\pi^* S^2 \mathcal{T}_{\mathbb{A}^2})_\chi$ can be regarded as a subspace of $S^2 E$, hence it can be written as

$$c_1\alpha_1^2 + c_2\alpha_1\alpha_2 + c_3\alpha_2^2$$

with $c_i \in k$, $i = 1, 2, 3$. Then $c_1\alpha_1^2 + c_2\alpha_1\alpha_2 + c_3\alpha_2^2$ is also the image of an element of $H^0(S^2\mathcal{T}_X)_\chi$ iff the multiplicity of the irreducible factor $\alpha_1 + \alpha_2$ in $c_1\alpha_1^2 + c_2\alpha_1\alpha_2 + c_3\alpha_2^2$ is at least $\langle \chi, \alpha_2 + \alpha_2 \rangle$. Let $i_1 = \langle \chi, \alpha_1 \rangle$, $i_2 = \langle \chi, \alpha_2 \rangle$, then it is easy to see we have only the following possible cases:

(1) $i_1 \leq 0, i_2 \leq 0$;
(2) $i_1 = 1, i_2 < 0$;
(3) $i_1 < 0, i_2 = 1$;
(4) $i_1 = 1, i_2 = 0$;
(5) $i_1 = 0, i_2 = 1$;
(6) $i_1 = 2, i_2 \leq -2$;
(7) $i_1 \leq -2, i_2 = 2$.

In the cases above, $H^0(S^2\mathcal{T}_X)_\chi$ regarded as a subspace of $H^0(S^2\mathcal{T}_{\mathbb{A}^2})_\chi$ are spanned by the following vectors respectively:

(1) $\alpha_1^2, \alpha_2^2, \alpha_1\alpha_2$;
(2) $\alpha_1^2, \alpha_1\alpha_2$;
(3) $\alpha_2^2, \alpha_1\alpha_2$;
(4) $\alpha_1(\alpha_1 + \alpha_2)$;
(5) $\alpha_2(\alpha_1 + \alpha_2)$;
(6) α_1^2;
(7) α_2^2.

On the other hand, an eigen-basis of $S^2\mathcal{T}_{\mathbb{A}^2}$ over $\mathbb{A}^2$ is given by $(\frac{\partial}{\partial x})^2, \frac{\partial}{\partial x}\frac{\partial}{\partial y}, (\frac{\partial}{\partial y})^2$, whose corresponding eigen-characters are x^2, xy, y^2 respectively. Moreover, let x_0 be the distinguished point of the open orbit. Then the images of this eigen-basis in the fiber $S^2\mathcal{T}_{\mathbb{A}^2} \otimes k(x_0)$ are $\alpha_1^2, \alpha_1\alpha_2, \alpha_2^2$ respectively. Therefore by applying Corollary 2.5 and Proposition 2.4, the vectors listed above correspond to the following sections of $H^0(S^2\mathcal{T}_{\mathbb{A}^2})$ respectively:

(1) $x^{2-i_1}y^{-i_2}(\frac{\partial}{\partial x})^2, x^{1-i_1}y^{1-i_2}\frac{\partial}{\partial x}\frac{\partial}{\partial y}, y^{2-i_2}(\frac{\partial}{\partial y})^2$;
(2) $xy^{-i_2}(\frac{\partial}{\partial x})^2, y^{1-i_2}\frac{\partial}{\partial x}\frac{\partial}{\partial y}$;
(3) $x^{-i_1}y(\frac{\partial}{\partial y})^2, x^{1-i_1}\frac{\partial}{\partial x}\frac{\partial}{\partial y}$;
(4) $y\frac{\partial}{\partial x}\frac{\partial}{\partial y} + x(\frac{\partial}{\partial x})^2$;
(5) $x\frac{\partial}{\partial x}\frac{\partial}{\partial y} + y(\frac{\partial}{\partial y})^2$;
(6) $y^{-i_2}(\frac{\partial}{\partial x})^2$;
(7) $x^{-i_1}(\frac{\partial}{\partial y})^2$.

Now by using the conditions on i_1, i_2, one checks easily that $\pi_* S^2\mathcal{T}_X$ regarded as a submodule of $S^2\mathcal{T}_{\mathbb{A}^2}$ is generated by the following sections:

(1) $x^2(\frac{\partial}{\partial x})^2, xy(\frac{\partial}{\partial x})^2, y^2(\frac{\partial}{\partial x})^2$;
(2) $x^2(\frac{\partial}{\partial y})^2, xy(\frac{\partial}{\partial y})^2, y^2(\frac{\partial}{\partial y})^2$;

(3) $x^2\frac{\partial}{\partial x}\frac{\partial}{\partial y}, xy\frac{\partial}{\partial x}\frac{\partial}{\partial y}, y^2\frac{\partial}{\partial x}\frac{\partial}{\partial y}$;
(4) $y\frac{\partial}{\partial x}\frac{\partial}{\partial y} + x(\frac{\partial}{\partial x})^2$;
(5) $x\frac{\partial}{\partial x}\frac{\partial}{\partial y} + y(\frac{\partial}{\partial y})^2$.

□

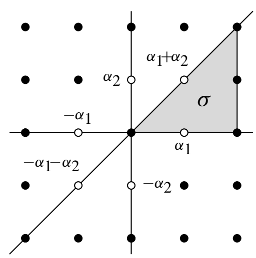
Proof of Proposition 3.5. Let $X = Bl_P S$ and $\pi : X \to S$ be the projection. If P is a fixed point of S under the torus action, then X is also a toric variety hence $\mathbb{P}(\mathcal{T}_X)$ is Frobenius split by Theorem 1.

Now suppose $\mathbb{P}(\mathcal{T}_X)$ is F-split, then by (3.4) there exists a section $s \in H^0(S^2\mathcal{T}_X)$ such that $\beta_X^{-1}(s)$ defines a Frobenius splitting of $\mathbb{P}(\mathcal{T}_X)$. By Lemma 4, we notice that $\pi_* S^2\mathcal{T}_X$ is contained in $S^2\mathcal{T}_S \otimes \mathcal{I}_P$. In particular, we have the following inclusion:

$$i : H^0(S^2\mathcal{T}_X) \hookrightarrow H^0(S^2\mathcal{T}_S).$$

On the other hand, let $U = X\backslash C$, where C is the exceptional curve. Then π defines an isomorphism $\pi^{-1}(U) \cong U$ and $\beta_S^{-1}(i(s))$ induces a Frobenius splitting of $\mathbb{P}(\mathcal{T}_U)$ hence also defines a Frobenius splitting of $\mathbb{P}(\mathcal{T}_S)$.

As is shown in the figure below, a toric structure of S can be given by a fan Σ in $\mathbb{R}^2$ with a set of ray generators $|\Sigma| = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, -\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2\}$.



We may just assume that P is a closed point of U_σ, where σ is the cone with ray generators α_1 and $\alpha_1 + \alpha_2$. Next we will prove that P must be the unique fixed point of U_σ under the torus action.

To simplify notations, we will just write s for $i(s)$ from now on, then s can be written as

$$s = \sum_{\chi \in \widehat{T}} s_\chi, \tag{3.7}$$

where $s_\chi \in H^0(S^2\mathcal{T}_S)_\chi$. Since S is complete, $H^0(S^2\mathcal{T}_S)_\chi \neq 0$ for finitely many $\chi \in \hat{T}$ hence the sum (3.7) has only finitely many terms. Note that by Proposition 3.3, $\beta_S^{-1}(s_0)$ defines a Frobenius splitting of $\mathbb{P}(\mathcal{T}_S)$ but s_0 might not be liftable to an element of $H^0(S^2\mathcal{T}_X)$.

We divide the remaining part of this proof into three steps.

Step 1. First we will decide all the characters $\chi \in \hat{T}$ such that $H^0(S^2\mathcal{T}_S)_\chi \neq 0$. By applying Propositions 2.2 and 2.8, an element of the χ-isotypical component $H^0(S^2\mathcal{T}_S)_\chi$ corresponds to a homogeneous polynomial in two variables of degree 2, say, $F(\alpha_1, \alpha_2)$. Given a ray generator α of $|\Sigma|$, it is obviously a linear combination of α_1 and α_2 with integer coefficients. If $\langle \chi, \alpha \rangle > 0$, in order that $H^0(S^2\mathcal{T}_S)_\chi \neq 0$, the multiplicity of the irreducible factor α in $F(\alpha_1, \alpha_2)$ is at least $\langle \chi, \alpha \rangle$. Under such restrictions one checks easily that χ must be defined by one of the following conditions:

(1) $\langle \chi, \alpha_1 \rangle = \langle \chi, \alpha_1 + \alpha_2 \rangle = -1$;
(2) $\langle \chi, \alpha_1 \rangle = \langle \chi, \alpha_1 + \alpha_2 \rangle = 1$;
(3) $\langle \chi, \alpha_1 \rangle = \langle \chi, \alpha_1 + \alpha_2 \rangle = 0$;
(4) $\langle \chi, \alpha_1 \rangle = 1, \langle \chi, \alpha_1 + \alpha_2 \rangle = 0$;
(5) $\langle \chi, \alpha_1 \rangle = 0, \langle \chi, \alpha_1 + \alpha_2 \rangle = 1$;
(6) $\langle \chi, \alpha_1 \rangle = 0, \langle \chi, \alpha_1 + \alpha_2 \rangle = -1$;
(7) $\langle \chi, \alpha_1 \rangle = -1, \langle \chi, \alpha_1 + \alpha_2 \rangle = 0$.

The i-th character appearing in the list above will be denoted by χ_i from now on. Let E be the k-vector space spanned by α_1, α_2. Then by Propositions 2.2, 2.8 and Example 2.10, the vector space $H^0(S^2\mathcal{T}_S)_\chi$ corresponds to the following subspaces of S^2E:

(1) $k\alpha_1(\alpha_1 + \alpha_2)$;
(2) $k\alpha_1(\alpha_1 + \alpha_2)$;
(3) S^2E;
(4) $k\alpha_1\alpha_2$;
(5) $k(\alpha_1 + \alpha_2)\alpha_2$;
(6) $k(\alpha_1 + \alpha_2)\alpha_2$;
(7) $k\alpha_1\alpha_2$.

Now we choose $x, y \in \hat{T}$ such that $\langle x, \alpha_1 \rangle = 1, \langle x, \alpha_1 + \alpha_2 \rangle = 0$ and $\langle y, \alpha_1 \rangle = 0, \langle y, \alpha_1 + \alpha_2 \rangle = 1$. Then $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ form an eigen-basis of $\mathcal{T}_{U_\sigma}$. Therefore by applying Corollary 2.5 and Proposition 2.4, the image of an element of $H^0(S^2\mathcal{T}_S)_{\chi_i}, 1 \leq i \leq 5$ under the restriction map (2.5) can be written in following forms:

(1) $c_1x^2y^2\frac{\partial}{\partial x}\frac{\partial}{\partial y}$;
(2) $c_2\frac{\partial}{\partial x}\frac{\partial}{\partial y}$;
(3) $c_3xy\frac{\partial}{\partial x}\frac{\partial}{\partial y} + c_4x^2(\frac{\partial}{\partial x})^2 + c_5y^2(\frac{\partial}{\partial y})^2$;
(4) $c_6(x(\frac{\partial}{\partial x})^2 + y\frac{\partial}{\partial x}\frac{\partial}{\partial y})$;
(5) $c_7(x\frac{\partial}{\partial x}\frac{\partial}{\partial y} + y(\frac{\partial}{\partial y})^2)$;
(6) $c_8(x^3(\frac{\partial}{\partial x})^2 + x^2y\frac{\partial}{\partial x}\frac{\partial}{\partial y})$;
(7) $c_8(y^3(\frac{\partial}{\partial y})^2 + xy^2\frac{\partial}{\partial x}\frac{\partial}{\partial y})$;

where $c_i \in k, 1 \leq i \leq 9$. We denote by s_{χ_i} the section appearing in the i-th item of the list above. Let $\rho : H^0(S^2\mathcal{T}_S) \to \Gamma(U_\sigma, S^2\mathcal{T}_S)$ be the restriction map, then for a global section s of $S^2\mathcal{T}_S$, we have

$$\rho(s) = \sum_{1 \leq i \leq 5} s_{\chi_i} \tag{3.8}$$

for some $c_i \in k, 1 \leq i \leq 7$. This can be also written as

$$\rho(s) = f_1(x, y)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + f_2(x, y)\left(\frac{\partial}{\partial x}\right)^2 + f_3(x, y)\left(\frac{\partial}{\partial y}\right)^2. \tag{3.9}$$

Note that the local sections s_{χ_i}, $1 \le i \le 5$ are extendable to global sections of $S^2\mathcal{T}_S$, but the three terms in the sum (3.9) are necessarily not.

Step 2. Next we will prove if P is not the fixed point of U_σ under the torus action. Then the section s defining the Frobenius splitting of $\mathbb{P}(\mathcal{T}_S)$ can be chosen such that the coefficients $f_2(x, y)$ and $f_3(x, y)$ in (3.9) are both 0.

First we will prove when we choose s such that $f_2(x, y) = f_3(x, y) = 0$. Since $\rho(s)$ can be represented as a sum in another way (3.8), one obtains

$$f_2(x, y) = c_8x^3 + c_4x^2 + c_6x, \quad f_3(x, y) = c_9y^3 + c_5y^2 + c_7y$$

in all cases. By Lemma 4, $\rho(s) \in S^2\mathcal{T}_S \otimes I_P$. In particular, let $(x-a, y-b)$ be the defining ideal of P, then $f_2(a, b) = f_3(a, b) = 0$, i.e. $c_8a^3+c_4a^2+c_6a = c_6y^3+c_5y^2+c_7y = 0$.

Now we consider the following two local sections:

$$\begin{aligned} s_1 &= c_6(y-b)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + (c_4x^2 + c_6x)\left(\frac{\partial}{\partial x}\right)^2 \\ &= -bc_6\frac{\partial}{\partial x}\frac{\partial}{\partial y} + c_4x^2\left(\frac{\partial}{\partial x}\right)^2 + c_6\left(y\frac{\partial}{\partial x}\frac{\partial}{\partial y} + x\left(\frac{\partial}{\partial x}\right)^2\right), \end{aligned} \tag{3.10}$$

$$\begin{aligned} s_2 &= c_7(x-a)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + (c_5y^2 + c_7y)\left(\frac{\partial}{\partial y}\right)^2 \\ &= -ac_7\frac{\partial}{\partial x}\frac{\partial}{\partial y} + c_5y^2\left(\frac{\partial}{\partial y}\right)^2 + c_7\left(x\frac{\partial}{\partial x}\frac{\partial}{\partial y} + y\left(\frac{\partial}{\partial y}\right)^2\right). \end{aligned} \tag{3.11}$$

One observes that s_1 and s_2 can be extended to global sections of $S^2\mathcal{T}_S$. On the other hand, since $c_4a^2 + c_6a = c_5b^2 + c_7b = 0$, the sections s_1 and s_2 can be written as

$$s_1 = c_6\left((y-b)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + (x-a)\left(\frac{\partial}{\partial x}\right)^2\right) + c_4(x-a)^2\left(\frac{\partial}{\partial x}\right)^2, \tag{3.12}$$

$$s_2 = c_7\left((x-a)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + (y-b)\left(\frac{\partial}{\partial y}\right)^2\right) + c_5(y-b)^2\left(\frac{\partial}{\partial y}\right)^2. \tag{3.13}$$

Then one sees by Lemma 4 that the above sections fall in $\Gamma(U_\sigma, \pi_*S^2\mathcal{T}_X)$. Furthermore, let $U = S\backslash\{P\}$, then π induces an isomorphism $\pi^{-1}(U) \to U$. Therefore, the sections s_1 and s_2 are indeed liftable to global sections of $S^2\mathcal{T}_X$. We still denote their global liftings by s_1 and s_2. Then one checks easily

$$\rho(s - s_1 - s_2) = f(x, y)\frac{\partial}{\partial x}\frac{\partial}{\partial y} \tag{3.14}$$

for some $f(x, y) \in k[x, y]$. Moreover, the section $\beta_S^{-1}(s-s_1-s_2)$ still defines a Frobenius splitting of $\mathbb{P}(\mathcal{T}_S)$ as $s_1 + s_2$ has no contribution to the coefficient of the element $xy\frac{\partial}{\partial x}\frac{\partial}{\partial y}$. Now we get a section satisfying the condition described at the beginning of this step.

Step 3. Next we prove a section s defining a Frobenius splitting of $\mathbb{P}(\mathcal{T}_X)$ such that

$$\rho(s) = f(x, y)\frac{\partial}{\partial x}\frac{\partial}{\partial y} \tag{3.15}$$

for some $f(x, y) \in k[x, y]$. Then P must be the unique fixed point of U_σ under the torus action. By comparing (3.15) and (3.8) one sees easily $f(x, y) = c_1x^2y^2 + c_2xy + c_3$.

Moreover, by Lemma 4, $f(x, y) \in I_P^2$, where $I_P = (x - a, y - b)$ is the defining ideal of P in $\mathcal{O}_{U_\sigma}$.

If P is not the fixed point of U_σ under the torus action, one of a, b is nonzero. Let $f(z) = c_1 z^2 + c_2 z + c_3 = c(z + w_1)(z + w_2)$, $c, w_1, w_2 \in k$. Then we claim in order that $f(xy) \in I_P^2$, $w_1 = w_2$. Indeed, neither $xy + w_1$ nor $xy + w_2$ falls in I_P^2 hence both fall in I_P. If $w_1 \neq w_2$, we will have nonzero intersection of I_P with k, a contradiction. Thus $f(z)$ has a multiple root and hence $c_2 = 0$ since $p = 2$. However, by Proposition 3.1, this implies that s does not define a Frobenius splitting of $\mathbb{P}(\mathcal{T}_S)$, which contradicts our assumption. $\square$

Acknowledgements

The author would like to thank the referee for pointing out numerous typos, inaccuracies and mistakes and at the same time providing many helpful suggestions on improving this work.

References

[1] Achinger P, Ilten N and Süß H, F-split and F-regular varieties with a diagonalizable group action, preprint arXiv:1503.03116 (2015) pp. 1–40

[2] Brion M and Kumar S, Frobenius splitting methods in geometry and representation theory, Progress in Mathematics, vol. 231 (2005) (Birkhäuser)

[3] Cox D A, Little J B and Schenck H K, Toric varieties (2011) (American Mathematical Society)

[4] Fulton W, Introduction to toric varieties 131 (1993) (Princeton University Press)

[5] Hartshorne R (1966) Residues and duality, Lecture Notes in Mathematics 20 (Berlin: Springer)

[6] Hartshorne R, Algebraic geometry, 52 (1977) (Springer)

[7] Hering M, Mustaţă M and Payne S, Positivity properties of toric vector bundles, *Annales de L'institut Fourier* **60(2)** (2010) 607–640

[8] Klyachko A, Equivariant vector bundles on toral varieties, *Math. USSR-Izv.* **35** (1990) 337–375

[9] Kumar S, Lauritzen N and Thomsen J F, Frobenius splitting of cotangent bundles of flag varieties, *Inventiones mathematicae* **136(3)** (1999) 603–621

[10] Thomsen J F (2000) Frobenius direct images of line bundles on toric varieties, *J. Algebra* **226**, 865–874

[11] Various authors, Open problems, Frobenius splitting in algebraic geometry, commutative algebra and representation theory, Conference at the University of Michigan, https://sites.google.com/site/frobeniussplitting/shedule (2010)

COMMUNICATING EDITOR: Nitin Nitsure