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# BETTI NUMBERS OF SMALL COVERS AND THEIR TWO-FOLD COVERINGS

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**Abstract:** We compute the Betti numbers of two-fold coverings of small covers with some special properties; in particular, we use the results of Davis and Januszkiewicz on the cohomology of small covers. It turned out that their proof contains some gap that we describe in detail and fill in.

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**1. Introduction.** This article deals with the problem of computation of the Betti numbers over the field $\mathbb{Z}_2$ of two-fold coverings of small covers. The problem has a general solution which reduces the computation to finding the dimension of certain special $\mathbb{Z}_2$-algebras; for instance, the Gysin exact sequence applies. However, it often turns out useless in practice, as in the case of graph-associahedra. We point out a wide class of two-fold coverings over small covers for which the problem of computation of the mod-2 Betti numbers can be solved explicitly. Namely, we introduce the concept of *section classes* in the first cohomology of a small cover.

The study of the topology of two-fold coverings over small covers of simple polytopes is motivated among others by connections with the classical problem on realization of cycles; cf. [1]. Recall that an oriented connected closed smooth manifold $M^n$ is said to be a *universal manifold for realization of cycles* (or simply a *URC-manifold*) if for any topological space $X$ and any homology class $z \in H_n(X;\mathbb{Z})$, there exist a finite-fold covering $\widehat{M} \to M^n$ and a continuous mapping $f : \widehat{M} \to X$ such that $f_*[\widehat{M}] = kz$ for a nonzero integer $k$. In [1], Gaifullin found many examples of URC-manifolds among the orientation two-fold coverings over small covers of simple polytopes, in particular, of the so-called graph-associahedra. Hence, it is natural to ask which of these manifolds is the "smallest" in any reasonable sense. For instance, we may ask which of these manifolds has the smallest mod-2 Betti numbers. Unfortunately, the author still cannot answer this question, since for most of small covers over graph-associahedra their orientation two-fold coverings correspond to nonsectional cohomology classes.

In the context of this general problem we define a special class of two-fold coverings of small covers containing some orientation two-fold coverings. In this class we calculate Betti numbers over $\mathbb{Z}_2$. This computation rests, in particular, on the classical results of Davis and Januszkiewicz on the cohomology of small covers. It turned out that their proof contains some gap that we describe in detail and fill in.

**2. Small covers.** The small covers of simple convex polytopes were originally introduced by Davis and Januszkiewicz in their pioneer work [2]. We recall the definition of small covers and their basic properties (see [2, 3]).

Let $P$ be an $n$-dimensional simple convex polytope with facets $F_1, \dots, F_m$. A map

$$\lambda : \{F_1, \dots, F_m\} \to \mathbb{Z}_2^n$$

is called *characteristic* if for each pairwise different facets $F_{i_1}, \dots, F_{i_n}$ intersecting in a vertex, the vectors $\lambda(F_{i_1}), \dots, \lambda(F_{i_n})$ form a basis for $\mathbb{Z}_2^n$. Given $x \in P$, we denote by $\mathrm{St}_\lambda(x)$ the linear span of the vectors $\lambda(F)$, where $F$ runs over all facets containing $x$. We denote by $\lambda_i(F)$ the $i$th coordinate of $\lambda(F)$.

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Definition. The *small cover* $M_{P,\lambda}$ of a simple polytope $P$ corresponding to a characteristic map $\lambda$ is the quotient space of $P \times \mathbb{Z}_2^n$ by the equivalence $\sim$ such that $(x,a) \sim (y,b)$ if and only if

$$x = y, \quad (a+b) \in \mathrm{St}_\lambda(x).$$

Denote by $[x,a]$ the coset of a point $(x,a) \in P \times \mathbb{Z}_2^n$. The small cover $M_{P,\lambda}$ is an $n$-dimensional smooth manifold with the natural $\mathbb{Z}_2^n$-action given by $b \cdot [x,a] = [x, a+b]$. The orbit space of this action is $P$, the projection map $p : M_{P,\lambda} \to P$ is given by $[x,a] \mapsto x$, and the stabilizer subgroup of $x \in P$ is $\mathrm{St}_\lambda(x)$.

**3. Davis–Januszkiewicz Theorems.** Recall that the $h$-numbers of a simple polytope $P$ are the numbers $h_i = h_i(P)$, $i = 1, \dots, n$, given by

$$h_0 t^n + h_1 t^{n-1} + \dots + h_n = (t-1)^n + f_0 (t-1)^{n-1} + \dots + f_{n-1},$$

where $f_i$ is the number of codimension $i+1$ faces of $P$.

**Theorem 1** [2, Theorem 3.1, Corollary 3.7]. *The mod-2 Betti numbers of $M_{P,\lambda}$ are equal to the $h$-numbers of $P$. The action of $\mathbb{Z}_2^n$ on $H^*(M_{P,\lambda}; \mathbb{Z}_2)$ induced by the natural action of $\mathbb{Z}_2^n$ on $M_{P,\lambda}$ is trivial.*

Theorem 1 plays a key role in the proof of the following classical theorem:

**Theorem 2** [2, Theorem 4.14]. *There is an isomorphism of graded rings*

$$H^*(M_{P,\lambda}; \mathbb{Z}_2) = \mathbb{Z}_2[v_1, v_2, \dots, v_m]/(\mathscr{I} + \mathscr{J}),$$

*where $\mathscr{I}$ is the Stanley–Reisner ideal of $P$ and $\mathscr{J}$ is the ideal generated by the linear forms $\sum_{k=1}^m \lambda_i(F_k) v_k$, $i = 1, \dots, n$. The generators $v_i$ are the cohomology classes Poincaré dual to the preimages $p^{-1}(F_i)$ of the facets $F_i$ of $P$, respectively.*

A key role in the proof of Theorem 1 due to Davis and Januszkiewicz is played by the following construction (cf. [2, Section 3]).

Let $\ell(x) = \langle x, l \rangle$ be a generic height function on $\mathbb{R}^n$, i.e., a height function corresponding to a vector $l$ that is not orthogonal to any edge of $P$. We will say that a set $A$ lies *below* a set $B$ if $\ell(x) \le \ell(y)$ for all $x \in A$ and $y \in B$. The *index* of a vertex $v$ of $P$ is the number of vertices adjacent to $v$ which lie below $v$.

Let $F_v$ be the union of the relative interiors of all faces of $P$ containing $v$ and lying below $v$. The closure of $F_v$ is a face of $P$, and $p^{-1}(F_v)$ is an open cell of dimension equal to the index of $v$. The number of $i$-dimensional cells $p^{-1}(F_v)$ is equal to the number of index $i$ vertices of $P$. It is well known that the latter number is independent of $\ell$ and equals $h_i(P)$; see [4].

Davis and Januszkiewicz claimed and used substantially in their proof of Theorem 1 that the cells $p^{-1}(F_v)$ form a cell structure on $M_{P,\lambda}$. Nevertheless, this is not always true.

**Gap.** *The boundary of a cell $p^{-1}(F_v)$ can be not contained in the union of cells $p^{-1}(F_w)$ of smaller dimensions. Hence, the cells $p^{-1}(F_v)$ do not form a CW decomposition of $M_{P,\lambda}$ in general.*

Example. Consider a pentagon $ABCDE$ with an arbitrary characteristic map $\lambda$. Assume that the height function $\ell$ is chosen so that $\ell(A) > \ell(B) > \ell(C) > \ell(D) > \ell(E)$. Then both endpoints of the one-dimensional cell $p^{-1}(F_B)$ coincide with the point $p^{-1}(C)$ contained in the relative interior of the one-dimensional cell $p^{-1}(F_C)$.

**4. Correction of the proof.** Let $s_i$ be the sequence of heights of vertices of $P$ numbered in the ascending order. Consider the filtration on $M_{P,\lambda}$ formed by the sets

$$G_p := \bigcup_{\ell(v) \le s_p} F_v,$$

and the corresponding spectral sequence in cohomology with the coefficient group $\mathbb{Z}_2$. The first page of this spectral sequence is

$$E_1^{p,q} = H^{p+q}(G_p, G_{p-1}; \mathbb{Z}_2).$$

Since the filtration $\{G_p\}$ is invariant under the action of $\mathbb{Z}_2^n$ on $M_{P,\lambda}$, we obtain a well-defined $\mathbb{Z}_2^n$-action on the spectral sequence $E^r_{p,q}$. Since $p^{-1}(F_v)$ is an open cell for each vertex $v$, the quotient space $G_p/G_{p-1}$ is homeomorphic to the sphere whose dimension equals the index of the vertex at height $s_p$. Hence every group $E_1^{p,q}$ is either trivial or isomorphic to $\mathbb{Z}_2$. Therefore, the action of $\mathbb{Z}_2^n$ on every $E_1^{p,q}$, hence on all $E_r^{p,q}$ for $r > 1$, is trivial. Thus, $\mathbb{Z}_2^n$ acts trivially on the mod-2 cohomology of $M_{P,\lambda}$. We have

$$\dim H^i(M_{P,\lambda}; \mathbb{Z}_2) = \sum_{p+q=i} \dim E_\infty^{p,q} \le \sum_{p+q=i} \dim E_1^{p,q} = h_i(P).$$

These estimates and the triviality of the $\mathbb{Z}_2$-action on $H^*(M_{P,\lambda}; \mathbb{Z}_2)$ are exactly those two facts that were deduced by Davis and Januszkiewicz from the wrong fact that the cells $p^{-1}(F_v)$ form a CW decomposition of $M_{P,\lambda}$. Since we gave another proof of these two facts, we can further proceed with the original proof of Theorem 1 and then Theorem 2 due to Davis and Januszkiewicz without any changes. In consequence, the estimates above turn out into the equalities $\dim H^i(M_{P,\lambda}; \mathbb{Z}_2) = h_i(P)$.

Remark. There are analogs of Theorems 1 and 2 for quasitoric manifolds which are also due to Davis and Januszkiewicz. There is a similar gap in their proofs, and this gap can be filled in similarly. However, unlike Theorems 1 and 2, their analogs for quasitoric manifolds admit a different proof that was obtained by Buchstaber and Panov; e.g., see [3].

An analog of the spectral sequence of the filtration $\{G_p\}$ can be constructed for every covering over $M_{P,\lambda}$. However, it will have many nontrivial differentials in general.

**5. Two-fold coverings over small covers: Section classes.** Let $S$ be a hyperplane section of $P$. Its preimage $p^{-1}(S)$ is a mod-2 singular cycle in $M_{P,\lambda}$; but if $S$ is not a facet of $P$, then this cycle is homologically trivial. On the other hand, if $p^{-1}(S)$ is disconnected then its connected component always represents a nontrivial homology class.

The preimage of $S$ may be nonconnected only if $S$ is a generic section; i.e., $S$ contains no vertex of $P$. Further we assume that either $S$ is a generic hyperplane section with disconnected preimage $p^{-1}(S)$ or $S$ is a facet of $P$. In the former case we denote by $M_{S,\lambda_S}$ a connected component of $p^{-1}(S)$, and in the latter case we put $M_{S,\lambda_S} = p^{-1}(S)$. It is easy to see that in both cases $M_{S,\lambda_S}$ is the small cover over $S$ corresponding to the characteristic function given by $\lambda_S(F \cap S) = \lambda(F)$ whenever $F \cap S$ is nonempty and does not coincide with $S$.

Definition. We say that $w \in H^1(M_{P,\lambda}; \mathbb{Z}_2)$ is a section class if there exists $S$ as above such that $w$ is the Poincaré dual of the homology class of $M_{S,\lambda_S}$.

By definition, the section classes corresponding to the facets of $P$ are the generators $v_i$. If $S$ is a generic hyperplane section with disconnected $p^{-1}(S)$; then $(D[M_{S,\lambda_S}])^2 = 0$, where $D$ stands for the Poincaré duality operator. Hence, nontriviality of $w^2$ is an obstruction for $w$ to be a section class corresponding to a generic hyperplane section.

It is interesting if the orientation two-fold covering over $M_{P,\lambda}$ can correspond to a section class. Let us consider a simple example.

Example. Suppose that $P$ admits a regular coloring of its facets in $n$ colors. Then $P$ admits a characteristic map $\lambda$ that sends every facet of the $i$th color to the $i$th basic vector $e_i \in \mathbb{Z}_2^n$. For instance, a permutohedron admits such coloring. Choose $a \in \langle e_2, \dots, e_n \rangle$ and a facet $G$ of the first color, and consider the map $\nu$ given by

$$\nu(F) = \begin{cases} \lambda(G) + a & \text{if } F = G, \\ \lambda(F) & \text{if } F \ne G. \end{cases}$$

We can easily check that $\nu$ is characteristic. Suppose now that $a$ has an odd number of nonzero coordinates. Then the first Stiefel–Whitney class of $M_{P,\nu}$ is the Poincaré dual of $p^{-1}(G)$. Consequently, the orientation two-fold covering over $M_{P,\nu}$ corresponds to a section class.

**6. The Betti numbers of two-fold coverings corresponding to section classes.** It is well known that the cosets of two-fold coverings over a topological space are in a one-to-one correspondence with the elements of its first cohomology group with the coefficient group $\mathbb{Z}_2$. We denote by $M_w$ the two-fold covering over $M_{P,\lambda}$ corresponding to $w \in H^1(M_{P,\lambda}; \mathbb{Z}_2)$.

We start with the following proposition showing that the problem of the computation of the Betti numbers of $M_w$ can be reduced to the case when $w$ is a sum of two section classes corresponding to facets. Thus it shows that the section classes are reasonable family of cohomology classes in the context of this problem.

**Proposition.** *The problem of computing the Betti numbers of $M_w$ can be reduced to the case of $w = D[p^{-1}(F)] + D[p^{-1}(G)]$, where $F$ and $G$ are facets of a polytope.*

Proof. Suppose that $w = \sum_{i \in I} D[p^{-1}(F_i)]$ for some set of indices $I$. It is easy to see that the following map on the set of facets of the prism $P \times [0,1]$ is characteristic:

$$\lambda_w(G) = \begin{cases} \lambda(F_i), & \text{if } G = F_i \times [0,1] \text{ and } i \notin I, \\ \lambda(F_i) + e_{n+1}, & \text{if } G = F_i \times [0,1] \text{ and } i \in I, \\ e_{n+1}, & \text{if } G = P \times \{0\} \text{ or } P \times \{1\}. \end{cases}$$

It is not hard to check that $M_{P\times[0,1],\lambda_w}$ coincides with the projectivization $P(\xi \oplus 1)$, where $\xi$ is the real linear bundle over $M_{P,\lambda}$ corresponding to $w$, and 1 denotes the trivial real linear bundle over $M_{P,\lambda}$. Moreover, if $\pi : M_{P\times[0,1],\lambda_w} \to M_{P,\lambda}$ is the projection map, then $\pi^*(w) = D[\tilde{p}^{-1}(P \times 0)] + D[\tilde{p}^{-1}(P \times 1)]$, where $\tilde{p} : M_{P\times[0,1],\lambda_w} \to P \times [0,1]$ is the projection of the small cover $M_{P\times[0,1],\lambda_w}$.

Finally, the two-fold covering over $M_{P\times[0,1],\lambda_w}$ corresponding to $\pi^*(w)$ is homeomorphic to $M_w \times \mathbb{S}^1$. Consequently, if we could compute the Betti numbers of this covering space, then we would immediately obtain the Betti numbers of the original two-fold covering $M_w$. $\square$

Let $h^*(-;\mathbb{Z}_2)$ be the dimension of $H^*(-;\mathbb{Z}_2)$. The main result of this section is as follows:

**Theorem 3.** *Suppose that $w \in H^1(M_{P,\lambda}; \mathbb{Z}_2)$ is a section class corresponding to $S$. Then*

$$h^j(M_w; \mathbb{Z}_2) = 2h_j(P) - h_{j-1}(S) - h_j(S),$$

*where $h_*(S)$ and $h_*(P)$ are the $h$-numbers of $S$ and $P$.*

The proof of this theorem reduces to the following lemma.

**Lemma.** *Let $X$ be a connected smooth closed manifold and let $\widetilde{X}$ be the two-fold covering of it corresponding to a nontrivial class $w \in H^1(X; \mathbb{Z}_2)$. Let $i : Y \hookrightarrow X$ be a connected smooth submanifold of $X$ such that $i_*[Y]$ is the Poincaré dual of $w$. Then*

$$h^j(\widetilde{X}; \mathbb{Z}_2) = 2h^j(X; \mathbb{Z}_2) - h^j(Y; \mathbb{Z}_2) - h^{j-1}(Y; \mathbb{Z}_2).$$

Proof. Recall the exact Gysin sequence:

$$\cdots \to H^j(X; \mathbb{Z}_2) \to H^j(\widetilde{X}; \mathbb{Z}_2) \to H^j(X; \mathbb{Z}_2) \xrightarrow{w \smile -} H^{j+1}(X; \mathbb{Z}_2) \to \cdots .$$

The exactness of this sequence implies easily that

$$h^{j+1}(\widetilde{X}; \mathbb{Z}_2) = h^{j+1}(X; \mathbb{Z}_2) - h^j(X; \mathbb{Z}_2) + k_j + k_{j+1},$$

where $k_j$ is the dimension of the kernel of $(w \smile -) : H^j(X; \mathbb{Z}_2) \to H^{j+1}(X; \mathbb{Z}_2)$. Thus, the proof reduces to computing the numbers $k_j$. The manifolds $X$ and $Y$ are closed and connected; hence, for every cohomology class $\sigma \in H^*(X; \mathbb{Z}_2)$, there is a homology class $z \in H_{n-1-*}(Y; \mathbb{Z}_2)$ such that

$$D_X(w \smile \sigma) = i_*(z), \quad D_Y(z) = i^*(\sigma),$$

where $D_X$ and $D_Y$ are the Poincaré duality operators for $X$ and $Y$, respectively. Since $i_*([Y]) \neq 0$, the map $i_*$ is injective, hence, $\operatorname{Ker}(w \smile -) = \operatorname{Ker} i^*$. Similarly, the dual map $i^*$ is surjective, therefore, $k_j = h^j(X;\mathbb{Z}_2) - h^j(Y;\mathbb{Z}_2)$. $\square$

Proof of Theorem 3. By definition, a section class cannot be trivial. Since the mod-2 Betti numbers of a small cover of a polyhedron coincide with the $h$-numbers of this polyhedron, the theorem follows from the above lemma. $\square$

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