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Qualitative behaviour of incompressible two-phase flows with phase transitions: The isothermal case

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Abstract. A thermodynamically consistent model for incompressible two-phase flows with phase transitions is considered mathematically. The model is based on first principles, i.e., balance of mass, momentum and energy. In the isothermal case, this problem is analysed to obtain local well-posedness, stability of non-degenerate equilibria, and global existence and convergence to equilibria of solutions which do not develop singularities.

Keywords. Two-phase incompressible flows; Navier–Stokes equations; free boundary problems; surface tension; isothermal; phase transitions; stability; convergence to equilibria.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^{3-}, $n \geq 2$. Ω contains two phases: at time t, phase k occupies subdomain $\Omega_k(t)$ of Ω $(k = 1, 2)$. Assume that $\partial\Omega_1(t) \cap \partial\Omega = \emptyset$; this means no boundary intersection to avoid the contact angle problem. The closed compact hyper-surface $\Gamma(t) := \partial\Omega_1(t) \subset \Omega$ forms the interface between the phases. $\Omega_2(t) = \Omega\backslash\overline{\Omega}_1(t)$ is the continuous phase which typically will be connected, whereas $\Omega_1(t)$ is the disperse phase which consists of m components.

In this paper, we consider incompressible flows. Let $\varrho_1, \varrho_2 > 0$ denote the constant densities, $\mu_1, \mu_2 > 0$ the constant viscosities, and $\sigma > 0$ the constant coefficient of surface tension. Let u_1, u_2 be the velocity fields, π_1, π_2 be the pressures which are unknown functions defined in Ω_1, Ω_2, respectively. In this paper, we assume that the temperature is constant, i.e we consider the isothermal case. ν_Γ designates the outer normal of Ω_1, V_Γ the normal velocity of $\Gamma(t)$, $H_\Gamma = H(\Gamma(t)) = -\mathrm{div}_\Gamma \nu_\Gamma$ the curvature of $\Gamma(t)$, and j_Γ the phase flux defined by

$$\mathrm{j}_\Gamma := \varrho(u \cdot \nu_\Gamma - V_\Gamma).$$

Note that j_Γ is well-defined due to the jump condition $[\![\varrho(u\cdot\nu_\Gamma - V_\Gamma)]\!] = 0$, which expresses conservation of mass across the interface. The quantity $[\![\phi]\!] = (\phi_2 - \phi_1)|_\Gamma$ denotes the jump of the variable ϕ across $\Gamma(t)$. Let $D(u) = (\nabla u + [\nabla u]^\mathsf{T})/2$ denote the rate of strain tensor, T the stress tensor, which is given by

$$T(u,\pi) = 2\mu D(u) - \pi I,$$

and ψ_1, ψ_2 be the given Helmholtz free energies, which are constants in the isothermal, incompressible case. Hereafter we drop the index i, as there is no danger of confusion; $\phi(t) = \phi_1(t)$ in $\Omega_1(t)$, $\phi(t) = \phi_2(t)$ in $\Omega_2(t)$, but we keep in mind that the unknown functions u, π, and constants μ, ϱ and ψ depend on the phases.

By *incompressible isothermal two-phase flow with phase transition*, we mean the following problem with sharp interface: Find a family of closed compact hypersurfaces $\{\Gamma(t)\}_{t\ge 0}$ contained in Ω and appropriately smooth functions $u : \mathbb{R}_+ \times \bar\Omega \to \mathbb{R}^n$, and $\pi : \mathbb{R}_+ \times \bar\Omega \to \mathbb{R}_+$ such that

$$\begin{cases} \varrho(\partial_t u + (u\cdot\nabla)u) - \operatorname{div} T = 0, \quad \text{in } \Omega\backslash\Gamma(t), \\ \operatorname{div} u = 0 \quad \text{in } \Omega\backslash\Gamma(t), \\ [\![u]\!] = [\![1/\varrho]\!]\mathrm{j}_\Gamma\nu_\Gamma \quad \text{on } \Gamma(t), \\ [\![1/\varrho]\!]\mathrm{j}_\Gamma^2\nu_\Gamma - [\![T\nu_\Gamma]\!] = \sigma H_\Gamma\nu_\Gamma \quad \text{on } \Gamma(t), \\ [\![\psi]\!] + [\![1/(2\varrho^2)]\!]{\mathrm{j}_\Gamma}^2 - [\![T\nu_\Gamma\cdot\nu_\Gamma/\varrho]\!] = 0 \quad \text{on } \Gamma(t), \\ V_\Gamma - u\cdot\nu_\Gamma + \mathrm{j}_\Gamma/\varrho = 0 \quad \text{on } \Gamma(t), \\ u = 0 \quad \text{on } \partial\Omega, \\ u(0) = u_0 \ \text{ in } \Omega\backslash\Gamma_0, \quad \Gamma(0) = \Gamma_0. \end{cases} \tag{1.1}$$

This model is explained in our previous paper [4] and in much more detail in [9, Chapter 1], and also in Prüss and Shimizu [6]. Note that j_Γ is a dummy variable as it can be eliminated from the system according to

$$\mathrm{j}_\Gamma = [\![u\cdot\nu_\Gamma]\!]/[\![1/\varrho]\!], \quad V_\Gamma = [\![\varrho u\cdot\nu_\Gamma]\!]/[\![\varrho]\!],$$

and replacing the third equation in (1.1) by $P_\Gamma[\![u]\!] = 0$, where $P_\Gamma = I - \nu_\Gamma\otimes\nu_\Gamma$. This works if $\varrho_1 \ne \varrho_2$, i.e. if $[\![\varrho]\!] \ne 0$. Throughout this paper, we assume this condition. In the physically uninteresting situation $[\![\varrho]\!] = 0$, the problem is not well-posed, as then V_Γ hence equivalently j_Γ are not uniquely defined by u and π, and in addition for a given interface the system for (u,π) has too many transmission conditions on the interface, i.e. it is over-determined. This shortcoming can be removed by either considering the non-isothermal case as in [4], see also [9], or by introducing *kinetic undercooling*. The latter means to replace the *Gibbs–Thomson law*, i.e. the fifth equation in (1.1) by

$$[\![\psi]\!] + [\![1/(2\varrho^2)]\!]{\mathrm{j}_\Gamma}^2 - [\![T\nu_\Gamma\cdot\nu_\Gamma/\varrho]\!] = -\gamma\mathrm{j}_\Gamma \quad \text{on } \Gamma(t),$$

with some constant $\gamma > 0$. Then there will be energy dissipation on the interface, in contrast to the case considered here. Moreover, the functional analytic setting will also be different, and therefore we concentrate here on the physically relevant case $[\![\varrho]\!] \ne 0$.

The density of the mass-specific available energy is defined by $\mathsf{e}_\mathsf{a} := \frac{1}{2}|u|^2 + \psi$, hence the total available energy of the system is given by

$$\mathsf{E}_\mathsf{a}(t) = \int_\Omega \varrho\mathsf{e}_\mathsf{a}(t,x)\,\mathrm{d}x + \sigma|\Gamma(t)|,$$

We know from Prüss and Shimizu [6] that along smooth solutions

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}_\mathsf{a}(t) = -\int_\Omega 2\mu|D(u)|^2\,\mathrm{d}x.$$

Therefore, if $\mu_1,\ \mu_2 > 0$, the total available energy E_a is a Lyapunov functional.

Now we look at the equilibria of (1.1). If $\frac{d}{dt}\mathsf{E}_a = 0$ in some interval (t_1, t_2), then $D(u) = 0$ on $\Omega\backslash\Gamma(t)$, hence $u = 0$ by [9, Lemma 1.2.1], because the third equation in (1.1) shows $[\![u \cdot \nu_\Gamma]\!]\nu_\Gamma = [\![1/\varrho]\!]j_\Gamma\nu_\Gamma = [\![u]\!]$, namely $P_\Gamma[\![u]\!] = 0$. This implies $\nabla\pi = 0$ in each component of the phases. Balance of normal stress and the Gibbs–Thomson relation, i.e. the fifth equation in (1.1) on $\Gamma(t)$ yield

$$[\![\pi/\varrho]\!] + [\![\psi]\!] = 0 \quad \text{and} \quad [\![\pi]\!] = \sigma H_\Gamma.$$

Therefore, by the first equation in (1.1), we know that the pressure is constant in the components of the phases and by the second equation, the curvature H_Γ is constant on the components of Γ. As ψ is constant even in the phases, we see that that the pressure is also constant in the phases. This implies that Ω_2 is *connected* and that Γ consists of finitely many disjoint spheres of *equal size*.

We explain now the main results of this paper. To fix the functional analytic setting, let X be a Banach space, J be a time interval and assume that $p \in (1,\infty)$ and $1/p < \mu \le 1$. We introduce weighted L_p-spaces

$$L_{p,\mu}(J;X) := \{u : J \to X : t^{1-\mu}u \in L_p(J;X)\},$$
$$H^1_{p,\mu}(J;X) := \{u : L_{p,\mu}(J;X) \cap H^1_1(J;X) : \frac{d}{dt}u \in L_{p,\mu}(J;X)\}.$$

More details on weighted L_p-spaces can be found e.g. in the monograph [9]. The basic result for local well-posedness of problem (1.1) in an L_p-setting is the following theorem.

Theorem 1.1. *Let* $p > n+2$, $\mu \in (1/2+(n+2)/2p, 1]$, $\sigma, \varrho_k, \mu_k > 0$, $\psi_k \in \mathbb{R}$, $k = 1,2$. *and suppose* $\varrho_1 \neq \varrho_2$. *Assume the* regularity conditions

$$u_0 \in W_p^{2\mu-2/p}(\Omega\backslash\Gamma_0)^n, \quad \Gamma_0 \in W_p^{2+\mu-2/p},$$

and the compatibility conditions

$$\operatorname{div} u_0 = 0 \quad \text{in } \Omega\backslash\Gamma_0, \quad u_0 = 0 \quad \text{on } \partial\Omega,$$
$$P_{\Gamma_0}[\![u_0]\!] = P_{\Gamma_0}[\![\mu(\nabla u_0 + [\nabla u_0]^{\mathsf{T}})\nu_{\Gamma_0}]\!] = 0 \quad \text{on } \Gamma_0.$$

Then there exists a unique $L_{p,\mu} - L_p$*-solution* (u,π,Γ) *of problem* (1.1) *on some possibly small but nontrivial time interval* $J = [0,\tau]$.

Remark 1.2.

(1) Here the notion $\Gamma_0 \in W_p^{2+\mu-2/p}$ means that Γ_0 is a C^2-manifold, such that its outer normal field ν_{Γ_0} is of class $W_p^{1+\mu-2/p}(\Gamma_0)$. Therefore the curvature tensor $L_{\Gamma_0} = -\nabla_{\Gamma_0}\nu_{\Gamma_0}$ of Γ_0 belongs to $W_p^{\mu-2/p}(\Gamma_0)$ which embeds into $C^{\alpha+1/p}(\Gamma_0)$, with $\alpha = \mu - (n+2)/p > 0$ since $p > n+2$ by assumption.
(2) For the same reason we also have $u_0 \in C^{1+\beta}(\bar{\Omega}_j(0))^n$, $j = 1,2$ and $V_0 \in C^{1+\beta}(\Gamma_0)$, with $\beta = 2\mu - 1 - (n+2)/p > 0$.
(3) The notion L_p-solution means that (u,π,Γ) is obtained as the push-forward of an L_p-solution $(\bar{u},\bar{\pi},h)$ of the transformed problem (2.1), which means that $(\bar{u},h)$ belongs to $\mathbb{E}_\mu(J) = \mathbb{E}_{u,\mu}(J) \times \mathbb{E}_{h,\mu}(J)$ with $J = [0,\tau]$ defined by

$$\mathbb{E}_{u,\mu}(J) = H^1_{p,\mu}(J;L_p(\Omega))^n \cap L_{p,\mu}(J;H^2_p(\Omega\backslash\Sigma))^n,$$

$$\mathbb{E}_{h,\mu}(J) = W^{2-1/2p}_{p,\mu}(J; L_p(\Sigma)) \cap H^1_{p,\mu}(J; W^{2-1/p}_p(\Sigma)) \\ \cap L_{p,\mu}(J; W^{3-1/p}_p(\Sigma)).$$

Recall that the closed C^2-hypersurfaces contained in Ω form a C^2-manifold, which is denoted by $\mathcal{MH}^2(\Omega)$. Define the state manifold $\mathcal{SM}$ as follows (here we set the time weight parameter $\mu = 1$):

$$\begin{aligned} \mathcal{SM} := \big\{&(u, \Gamma) \in L_p(\Omega)^n \times \mathcal{MH}^2 : u \in W^{2-2/p}_p(\Omega\backslash\Gamma)^n,\ \Gamma \in W^{3-2/p}_p, \\ &\operatorname{div} u = 0 \ \text{ in } \ \Omega\backslash\Gamma, \quad u = 0 \ \text{ on } \partial\Omega, \\ &[\![\mathcal{P}_\Gamma u]\!] = \mathcal{P}_\Gamma[\![\mu(\nabla u + [\nabla u]^{\mathsf{T}})]\!]\nu_\Gamma = 0 \quad \text{on } \Gamma\big\}; \end{aligned} \tag{1.2}$$

cf. [7, (6.1)] and also [9, Chapter 11]. Problem (1.1) induces a local semiflow on $\mathcal{SM}$.

PROPOSITION 1.3

Let $p > n + 2, \sigma, \varrho_k, \mu_k > 0, \psi_k \in \mathbb{R}, k = 1, 2,$ and suppose $\varrho_2 \neq \varrho_1$. Then problem (1.1) *generates a local semiflow on the state manifold $\mathcal{SM}$. Each solution (u, Γ) exists on a maximal time interval* $[0, t_+)$, *where $t_+ > 0$ depends on the initial value u_0 and Γ_0.*

We denote by

$$\begin{aligned} \mathcal{E} := \{(0, \Gamma_*) :\ \Gamma_* = &\bigcup_{1\le l\le m} S_{R_*}(x_l) \subset \Omega, \\ &x_1, \ldots x_m \in \mathbb{R}^n,\ |x_k - x_l| > 2R_*, k \neq l\} \end{aligned}$$

the set of *non-degenerate equilibria* i.e. the spheres satisfy

$$S_{R_*}(x_l) \cap S_{R_*}(x_k) = \emptyset\ (l \neq k), \quad S_{R_*}(x_l) \cap \partial\Omega = \emptyset.$$

Note that $\mathcal{E}$ forms a real analytic manifold of dimension $mn + 1$, where n dimensions come from the center and m from the number of components, 1 comes from the radius of the sphere R_* which is determined by $m(\omega_n/n)R^n_* = |\Omega_1(0)|$. Note that for $\varrho_1 \neq \varrho_2$, the volumes of the phases are conserved. Indeed,

$$\varrho_1|\Omega_1(t)| + \varrho_2|\Omega_2(t)| \equiv \varrho_1|\Omega_1(0)| + \varrho_2|\Omega_2(0)| =: c_0$$

which implies

$$[\![\varrho]\!]|\Omega_1(t)| = \varrho_2|\Omega| - c_0.$$

For the study of stability of non-degenerate equilibria, a major difficulty lies in the fact that the equilibria are not isolated in the state manifold, but form a finite-dimensional submanifold $\mathcal{E}$ of $\mathcal{SM}$. For the linearization of the transformed problem, this implies that the kernel of the linear operator L is nontrivial, i.e. the imaginary axis is not in the resolvent set of L, and so the standard principle of linearized stability is not applicable. Fortunately, 0 is the only eigenvalue of L on $i\mathbb{R}$ and it is nicely behaved: the kernel $\mathsf{N}(L)$ is isomorphic to the tangent space of $\mathcal{E}$ at this equilibrium, and 0 is semi-simple. Therefore, we may employ what is called the *generalized principle of linearized stability*, a method which is adapted to such a situation and has been worked out for quasilinear parabolic evolution equations in [10]. For a through discussion of this principle for two-phase problems like that considered here, we refer to Prüss and Simonett [9, Chapter 11].

Theorem 1.4. *Let $p > n+2, \sigma, \varrho_k, \mu_k > 0, \psi_k \in \mathbb{R}, k = 1, 2$ and suppose $\varrho_1 \neq \varrho_2$. Then in the topology of the state manifold $\mathcal{SM}$, we have*

(i) *$(0, \Gamma_*) \in \mathcal{E}$ is stable if and only if Γ_* is connected.*
(ii) *Any solution starting in a neighborhood of a stable equilibrium exists globally and converges to a probably different stable equilibrium in the topology of $\mathcal{SM}$.*
(iii) *Any solution starting and* staying in *a neighborhood of an unstable equilibrium exists globally and converges to a probably different unstable equilibrium in the topology of $\mathcal{SM}$.*

There is a large literature on incompressible Newtonian two-phase flows without phase transitions. On the other hand, only recently, phase transitions have been taken into account. For the more difficult non-isothermal case with different but constant densities, the system is *velocity-dominated*, here we refer to the papers [5,7,8,12,13]. We emphasize that the results in this paper are complementary to those in the isothermal case, but it is not possible to deduce them, as the functional settings and the Lyapunov functionals are different. The isothermal case appears to be a *singular limit* of the non-isothermal problem. In the non-isothermal case with equal densities, the system is *temperature-dominated*. For this, we refer to the papers [4,11]. Both cases are discussed in a much wider framework and in much greater detail in the monograph [9].

In the remainder of this paper, we give proofs based on the methods and results from the monograph [9]. In §2, we employ the direct mapping approach by using a Hanzawa transformation to transform the problem to a fixed domain. Employing a fixed point argument, the local well-posedness result Theorem 1.1 is obtained in §3. Section 4 is devoted to study the linearization of the problem at a non-degenerate equilibrium, this is the essential part of the proof of Theorem 1.4. Employing the *generalized principle of linearized stability* we derive the stability assertions Theorem 1.4 in §5. In §6, we show that a solution which does not develop singularities exist globally and converges to an equilibrium.

2. Transformation to a fixed domain

A basic idea is to transform the problem to a domain with a fixed interface Σ, where $\Gamma(t)$ is parametrized over Σ by means of a height function $h(t)$. For this we rely on the so-called *Hanzawa transform* which we will explain below. This transformation was introduced in the famous paper by Hanzawa [1] in connection with the classical Stefan problem. For the necessary geometric background, we refer to [9, Chapter 2].

Recall that the *second order bundle* of Γ is given by

$$\mathcal{N}^2\Gamma := \{(p, \nu_\Gamma(p), \nabla_\Gamma \nu_\Gamma(p)) : \ p \in \Gamma\}.$$

The hypersurface Γ can be approximated by a real analytic hypersurafce Σ, in the sense that the Hausdorff distance of the second order normal bundles is as small as we please. More precisely, given $\eta > 0$, there exists a real analytic hypersurface Σ such that $d_H(\mathcal{N}^2\Sigma, \mathcal{N}^2\Gamma) \leq \eta$. If $\eta > 0$ is small enough, then Σ bounds a domain Ω_1^Σ with $\overline{\Omega_1^\Sigma} \subset \Omega$ and then we set $\Omega_2^\Sigma = \Omega \backslash \overline{\Omega_1^\Sigma} \subset \Omega$. The hyper-surface Σ admits a tubular neighbourhood, which means that there is $a_0 > 0$ such that the map

$$\begin{aligned} &\Lambda : \Sigma \times (-a_0, a_0) \to \mathbb{R}^n, \\ &\Lambda(p, r) := p + r\nu_\Sigma(p) \end{aligned}$$

is a diffeomorphism from $\Sigma \times (-a_0, a_0)$ onto $\mathrm{im}(\Lambda)$, the image of Λ. The inverse

$$\Lambda^{-1} : \mathrm{im}(\Lambda) \to \Sigma \times (-a_0, a_0)$$

of this map is conveniently decomposed as

$$\Lambda^{-1}(x) = (\Pi_\Sigma(x), d_\Sigma(x)), \quad x \in \mathrm{im}(\Lambda).$$

Here $\Pi_\Sigma(x)$ means the metric projection of x onto Σ and $d_\Sigma(x)$ the signed distance from x to Σ; so $|d_\Sigma(x)| = \mathrm{dist}(x, \Sigma)$ and $d_\Sigma(x) < 0$ if and only if $x \in \Omega_1^\Sigma$. In particular, we have $\mathrm{im}(\Lambda) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, \Sigma) < a_0\}$. The maximal number a_0 is given by the radius $r_\Sigma > 0$, defined as the largest number r such that the exterior and interior ball conditions for Σ in Ω holds. In the following, we choose

$$a_0 = r_\Sigma/2 \quad \text{and} \quad a = a_0/3.$$

The derivatives of $\Pi_\Sigma(x)$ and $d_\Sigma(x)$ are given by

$$\nabla d_\Sigma(x) = \nu_\Sigma(\Pi_\Sigma(x)), \quad \partial \Pi_\Sigma(x) = M_0(d_\Sigma(x)) P_\Sigma(\Pi_\Sigma(x)),$$

where $P_\Sigma(p) = I - \nu_\Sigma(p) \otimes \nu_\Sigma(p)$ denotes the orthogonal projection onto the tangent space $T_p\Sigma$ of Σ at $p \in \Sigma$, and $M_0(r) = (I - rL_\Sigma)^{-1}$, with L_Σ the Weingarten tensor. Then

$$|M_0(r)| \leq 1/(1 - r|L_\Sigma|) \leq 3 \quad \text{for all } |r| \leq 2r_\Sigma/3.$$

If $\mathrm{dist}(\Gamma, \Sigma)$ is small enough, we may use the map Λ to parametrize the unknown free boundary $\Gamma(t)$ over Σ by means of a *height function* $h(t)$ via

$$\Gamma(t) = \{p + h(t, p)\nu_\Sigma(p) : p \in \Sigma\}, \quad t \geq 0,$$

for small $t \geq 0$, at least. Extend this diffeomorphism to all of $\bar{\Omega}$ by means of

$$\Xi_h(t, x) = x + \chi(d_\Sigma(x)/a) h(t, \Pi_\Sigma(x)) \nu_\Sigma(\Pi_\Sigma(x)) =: x + \xi_h(t, x).$$

Here χ denotes a suitable cut-off function. More precisely, let $\chi \in \mathcal{D}(\mathbb{R}), 0 \leq \chi \leq 1, \chi(r) = 1$ for $|r| < 1$ and $\chi(r) = 0$ for $|r| > 2$. We may choose χ in such a way that $1 < |\chi'|_\infty \leq 3$. Note that $\Xi_h(t, x) = x$ for $|d_\Sigma(x)| > 2a$, and

$$\Pi_\Sigma(\Xi_h(t, x)) = \Pi_\Sigma(x), \quad |d_\Sigma(x)| < a,$$

as well as

$$d_\Sigma(\Xi_h(t, x)) = d_\Sigma(x) + \chi(d_\Sigma(x)/a) h(t, \Pi_\Sigma(x)), \quad |d_\Sigma(x)| < 2a.$$

This yields

$$\Xi_h^{-1}(t, x) = x - h(t, \Pi_\Sigma(x)) \nu_\Sigma(\Pi_\Sigma(x)) \quad \text{for } |d_\Sigma(x)| < a.$$

Now we define the transformed quantities

$$\bar{u}(t, x) = u(t, \Xi_h(t, x)), \quad \bar{\pi}(t, x) = \pi(t, \Xi_h(t, x)), \quad t > 0, \ x \in \Omega\backslash\Sigma,$$

the *pull backs* of u and π. This way we have transformed the time varying regions $\Omega\backslash\Gamma(t)$ to the fixed domain $\Omega\backslash\Sigma$. This transformation leads to the following quasi-linear problem, dropping the bars and collecting its principal linear part on the left hand side.

$$\begin{aligned}
&\varrho\partial_t u - \mu\Delta u + \nabla\pi = F_u(u, \pi, h) \quad \text{in } \Omega\backslash\Sigma,\\
&\operatorname{div} u = G_d(u, h) \quad \text{in } \Omega\backslash\Sigma,\\
&P_\Sigma [\![u]\!] + c(t, x)\nabla_\Sigma h = G_m(u, h) \quad \text{on } \Sigma,\\
&-P_\Sigma [\![\mu(\nabla u + [\nabla u]^{\mathsf{T}})\nu_\Sigma]\!] = G_{u\tau}(u, h) \quad \text{on } \Sigma,\\
&-[\![\mu(\nabla u + [\nabla u]^{\mathsf{T}})\nu_\Sigma]\!] \cdot \nu_\Sigma + [\![\pi]\!] - \sigma\Delta_\Sigma h = G_{u\nu}(u, h) \quad \text{on } \Sigma,\\
&-[\![(\mu/\varrho)(\nabla u + [\nabla u]^{\mathsf{T}})\nu_\Sigma]\!] \cdot \nu_\Sigma + [\![\pi/\varrho]\!] = G_s(u, h) \quad \text{on } \Sigma,\\
&[\![\varrho]\!]\partial_t h - [\![\varrho u \cdot \nu_\Sigma]\!] + b(t, x) \cdot \nabla_\Sigma h = F_h(u, h) \quad \text{on } \Sigma,\\
&u = 0 \quad \text{on } \partial\Omega,\\
&u(0) = u_0 \quad \text{in } \Omega\backslash\Sigma, h(0) = h_0 \quad \text{in } \Sigma.
\end{aligned} \tag{2.1}$$

The nonlinearities are given by

$$\begin{aligned}
F_u(u,\pi,h) &= M_1(h)\nabla\pi - \varrho(u\cdot(I-M_1(h)) - R(h)\cdot)\nabla u \\
&\quad - \mu(M_2(h):\nabla^2)u - \mu(M_3(h)\cdot\nabla)u + \mu M_4(h):\nabla u, \\
G_d(u,h) &= M_1(h):\nabla u, \\
G_m(u,h) &= [\![u\cdot\nu_\Sigma]\!]M_0(h) - e^{\Delta_\Sigma t}[\![u_0\cdot\nu_\Sigma]\!]\nabla_\Sigma h, \\
G_{u\tau}(u,h) &= -P_\Sigma[\![\mu(\nabla u + [\nabla u]^{\mathsf T})M_0(h)\nabla_\Sigma h]\!] \\
&\quad - P_\Sigma[\![\mu(M_1(h)\nabla u + [M_1(h)\nabla u]^{\mathsf T})(\nu_\Sigma - M_0(h)\nabla_\Sigma h)]\!] \\
&\quad + [\![\mu((I-M_1)\nabla u + [(I-M_1)\nabla u]^T) \\
&\qquad (\nu_\Sigma - M_0\nabla_\Sigma h)\cdot\nu_\Sigma]\!]M_0(h)\nabla_\Sigma h, \\
G_{u\nu}(u,h) &= -[\![\mu(\nabla u + [\nabla u]^{\mathsf T})M_0(h)\nabla_\Sigma h\cdot\nu_\Sigma]\!] \\
&\quad - [\![\mu(M_1(h)\nabla u + [M_1(h)\nabla u]^{\mathsf T})(\nu_\Sigma - M_0(h)\nabla_\Sigma h)\cdot\nu_\Sigma]\!] \\
&\quad + \sigma(H_\Gamma(h) - \Delta_\Sigma h) - [\![u\cdot\nu_\Gamma]\!]/[\![1/\varrho]\!], \\
G_s(u,h) &= -[\![\psi]\!] - [\![1/2\varrho^2]\!]j^2 \\
&\quad + 2[\![(\mu/\varrho)\partial_\nu u]\!] - [\![(\mu/\varrho)(M_1(h)\nabla u + [M_1(h)\nabla u]^{\mathsf T})\nu_\Gamma\cdot\nu_\Gamma]\!], \\
F_h(u,h) &= (b(t,x) - [\![\varrho M_0(h)u]\!])\nabla_\Sigma h, \\
\mathrm{j}_\Gamma &= [\![u\cdot\nu_\Sigma]\!]/\beta(h)[\![1/\varrho]\!], \quad \nu_\Gamma = \beta(h)(\nu_\Sigma - M_0(h)\nabla_\Sigma h).
\end{aligned}$$

Here we employed the abbreviations

$$\begin{aligned}
&M_1(h) = [D\xi_h]^{\mathsf T}[I + D\xi_h]^{-\mathsf T}, \quad M_2(h) = M_1^{\mathsf T}(h) - M_1(h)M_1^{\mathsf T}(h), \\
&M_3(h) = (I - M_1(h))\operatorname{div} M_2(h), \\
&M_4(h) = ((I - M_1(h))\nabla)M_1(h) - [((I - M_1(h))\nabla)M_1(h)]^{\mathsf T}
\end{aligned}$$

and $c(t,x) = e^{\Delta_\Sigma t}[\![u_0\cdot\nu_\Sigma]\!]$, $b(t,x) = e^{\Delta_\Sigma t}[\![\varrho u_0]\!]$ are artificially added in order to deal with large initial data u_0 for local well-posedness.

3. Local well-posedness

The proof of Theorem 1.1 is based on maximal L_p-regularity of the following principal part of the linearized problem

$$\begin{aligned}
&\varrho\partial_t u - \mu\Delta u + \nabla\pi = \varrho f_u \quad \text{in } \Omega\backslash\Sigma, \\
&\operatorname{div} u = g_d \quad \text{in } \Omega\backslash\Sigma, \\
&P_\Sigma[\![u]\!] + c(t,x)\nabla_\Sigma h = g_m \quad \text{on } \Sigma, \\
&-2[\![\mu D(u)\nu_\Sigma]\!] + [\![\pi]\!]\nu_\Sigma - \sigma\Delta_\Sigma h\nu_\Sigma = g_u \quad \text{on } \Sigma, \\
&-2[\![\mu D(u)\nu_\Sigma\cdot\nu_\Sigma/\varrho]\!] + [\![\pi/\varrho]\!] = g_s \quad \text{on } \Sigma, \\
&u = 0 \quad \text{on } \partial\Omega \\
&[\![\varrho]\!]\partial_t h - [\![\varrho u\cdot\nu_\Sigma]\!] + b(t,x)\cdot\nabla_\Sigma h = [\![\varrho]\!]f_h \quad \text{on } \Sigma, \\
&u(0) = u_0 \quad \text{in } \Omega\backslash\Sigma, \qquad h(0) = h_0 \quad \text{on } \Sigma.
\end{aligned} \tag{3.1}$$

For this problem we have maximal regularity result in the L_p-setting, which is a special case of [9, Theorem 8.4.1].

Theorem 3.1. *Let $p > n+2$, $\mu \in (1/2+(n+2)/2p, 1]$, $\sigma, \varrho_k, \mu_k > 0$, $k = 1, 2$, $\varrho_2 \neq \varrho_1$ and*

$$(b, c) \in [W^{1-1/2p}_{p,\mu}(J; L_p(\Sigma)) \cap L_{p,\mu}(J; W^{2-1/p}_p(\Sigma))]^{n+1}$$

with $J = [0, \tau]$. Then (3.1) admits a unique solution (u, π, h) with regularity

$$\begin{aligned}
&u \in H^1_{p,\mu}(J; L_p(\Omega))^n \cap L_{p,\mu}(J; H^2_p(\Omega\backslash\Sigma))^n,\\
&[\![u \cdot \nu_\Sigma]\!] \in H^1_{p,\mu}(J; \dot{W}^{-1/p}_p(\Sigma)), \quad \pi \in L_{p,\mu}(J; \dot{H}^1_p(\Omega\backslash\Sigma)),\\
&\pi_k := \pi_{|\partial\Omega_k} \in W^{1/2-1/2p}_{p,\mu}(J; L_p(\Sigma)) \cap L_{p,\mu}(J; W^{1-1/p}_p(\Sigma)),\ k = 1, 2,\\
&h \in W^{2-1/2p}_{p,\mu}(J; L_p(\Sigma)) \cap H^1_{p,\mu}(J; W^{2-1/p}_p(\Sigma)) \cap L_{p,\mu}(J; W^{3-1/p}_p(\Sigma))
\end{aligned} \tag{3.2}$$

if and only if the data $f_u, g_d, g_m, g_u, g_s, f_h, u_0, h_0$ satisfy the following regularity and compatibility conditions:

(*a*) $f_u \in L_{p,\mu}(J; L_p(\Omega, \mathbb{R}^{n+1}))$,
(*b*) $g_d \in H^1_{p,\mu}(J; \dot{H}^{-1}_p(\Omega)) \cap L_{p,\mu}(J; H^1_p(\Omega\backslash\Sigma))$,
(*c*) $(g_u, g_s) \in W^{1/2-1/2p}_{p,\mu}(J; L_p(\Sigma, \mathbb{R}^{n+1})) \cap L_{p,\mu}(J; W^{1-1/p}_p(\Sigma, \mathbb{R}^{n+1}))$,
(*d*) $(g_m, f_h) \in W^{1-1/2p}_{p,\mu}(J; L_p(\Sigma, \mathbb{R}^{n+1})) \cap L_{p,\mu}(J; W^{2-1/p}_p(\Sigma, \mathbb{R}^{n+1}))$,
(*e*) $u_0 \in W^{2\mu-2/p}_p(\Omega\backslash\Sigma, \mathbb{R}^n)$, $h_0 \in W^{2+\mu-2/p}_p(\Sigma)$,
(*f*) $div\, u_0 = g_d(0)$ *in* $\Omega\backslash\Sigma$,
(*g*) $P_\Sigma[\![u_0]\!] + c(0, \cdot)\nabla_\Sigma h_0 = g_m(0)$ *on* Σ *if* $1 + \mu > 3/p$,
(*h*) $-P_\Sigma[\![\mu_0(\cdot)(\nabla u_0 + [\nabla u_0]^{\mathsf{T}})]\!] = P_\Sigma g_u(0)$ *on* Σ *if* $\mu > 3/p$.

The solution map $[(f_u, g_d, g_m, g_u, g_s, f_h, u_0, h_0) \mapsto (u, \pi, h)]$ is continuous between the corresponding spaces.

Based on this maximal regularity result, the proof of Theorem 1.1 then follows by the contraction mapping principle as in [7]; we refer also to [9, Chapter 9].

4. Linear stability of equilibria

The analysis in this section follows the arguments of [9, Chapter 10]. See also [7] for an earlier analysis.

(1) We obtain the following fully linearized problem at a non-degenerate equilibrium $e_* := (0, \Gamma_*) \in \mathcal{E}$ with reference hyper-surface $\Sigma = \Gamma_*$.

$$\begin{aligned}
&\varrho\partial_t u - \mu\Delta u + \nabla\pi = \varrho f_u \quad \text{in } \Omega\backslash\Sigma,\\
&\operatorname{div} u = g_d \quad \text{in } \Omega\backslash\Sigma,\\
&P_\Sigma[\![u]\!] = g_m \quad \text{on } \Sigma,\\
&-[\![T(u, \pi)\nu_\Sigma]\!] + \sigma\mathcal{A}_\Sigma h\nu_\Sigma = g_u \quad \text{on } \Sigma,\\
&-[\![T(u, \pi)\nu_\Sigma \cdot \nu_\Sigma/\varrho]\!] = g_s \quad \text{on } \Sigma,\\
&u = 0 \quad \text{on } \partial\Omega,\\
&\partial_t h - [\![\varrho u \cdot \nu_\Sigma]\!]/[\![\varrho]\!] = f_h \quad \text{on } \Sigma,\\
&u(0) = u_0 \quad \text{in } \Omega\backslash\Sigma, \qquad h(0) = h_0 \quad \text{on } \Sigma.
\end{aligned} \tag{4.1}$$

where $\mathcal{A}_\Sigma = -H'(0) = -(n-1)/R_*^2 - \Delta_\Sigma$. The time-trace space $\mathbb{E}_\gamma$ of $\mathbb{E}(J)$ is given by

$$(u_0, h_0) \in \mathbb{E}_\gamma = W_p^{2-2/p}((\Omega\backslash\Sigma)^n \times W_p^{3-2/p}(\Sigma),$$

and the space of the right-hand sides is

$$(f_u, g_d, (g_u, g_s), (g_m, f_h)) \in \mathbb{F}(J) := \mathbb{F}_u(J) \times \mathbb{F}_d(J) \times \mathbb{G}_u(J)^{n+1} \times \mathbb{G}_h(J)^n,$$

where

$$\begin{aligned}
\mathbb{F}_u(J) &= L_p(J \times \Omega)^{n+1},\\
\mathbb{F}_d(J) &= H_p^1(J; \dot{H}_p^{-1}(\Omega)) \cap L_p(J; H_p^1(\Omega)),\\
\mathbb{G}_u(J) &= W_p^{1/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{1-1/p}(\Sigma)),\\
\mathbb{G}_h(J) &= W_p^{1-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{2-1/p}(\Sigma)).
\end{aligned}$$

By localization and coordinate transformations it follows from the maximal regularity result in [5] that the linear operator defined by the left-hand side of (4.1) is an isomorphism from $\mathbb{E}(J)$ into $\mathbb{F}(J) \times \mathbb{E}_\gamma$. If the time derivatives ∂_t are replaced by $\partial_t + \omega, \omega > 0$ sufficiently large, then this result is also true for $J = \mathbb{R}_+$. As a base space for the underlying analytic semigroup, we use

$$X_0 = L_{p,\sigma}(\Omega)^n \times W_p^{2-1/p}(\Sigma),$$

where the subscript σ means solenoidal, and define the operator L by

$$L(u, h) = \big(-(\mu/\varrho)\Delta u + \nabla\pi/\varrho, -[\![\varrho u \cdot \nu_\Sigma]\!]/[\![\varrho]\!]\big)$$

with

$$\begin{aligned}
D(L) = \{(u, h) \in H_p^2(\Omega\backslash\Sigma)^n \times W_p^{3-1/p}(\Sigma) \cap X_0 :&\\
\operatorname{div} u = 0 \ \text{ in } \ \Omega\backslash\Sigma, \quad u = 0 \ \text{ on } \ \partial\Omega,&\\
P_\Sigma[\![u]\!] = P_\Sigma[\![\mu D(u)\nu_\Sigma]\!] = 0 \ \text{ on } \ \Sigma\}.&
\end{aligned}$$

The phase flux j_Σ is given by $\mathrm{j}_\Sigma = [\![u \cdot \nu_\Sigma]\!]/[\![\varrho^{-1}]\!]$, and π is determined as the solution of the weak transmission problem

$$\begin{aligned}
(\nabla\pi|\nabla\phi/\varrho)_2 &= ((\mu/\varrho)\Delta u|\nabla\phi)_2, \quad \phi \in \dot{H}_{p'}^1(\Omega), \ \phi = 0 \text{ on } \Sigma,\\
[\![\pi]\!] &= -\sigma\mathcal{A}_\Sigma h + 2[\![\mu(D(u)\nu_\Sigma|\nu_\Sigma)]\!], \quad \text{on } \Sigma,\\
[\![\pi/\varrho]\!] &= 2[\![(\mu/\varrho)(D(u)\nu_\Sigma|\nu_\Sigma)]\!] \quad \text{on } \Sigma.
\end{aligned}$$

Let us introduce solution operators T_j, $j \in \{1, 2, 3\}$ as follows:

$$\begin{aligned}
\frac{1}{\varrho}\nabla\pi = {} & T_1((\mu/\varrho)\Delta u) + T_2(-\sigma\mathcal{A}_\Sigma h + 2[\![\mu(D(u)\nu_\Sigma|\nu_\Sigma)]\!])\\
& + T_3(2[\![(\mu/\varrho)(D(u)\nu_\Sigma|\nu_\Sigma)]\!]).
\end{aligned}$$

We refer to [2], and for a more detailed exposition, see [9, Chapter 6.4] for the analysis of such transmission problems.

Then the linearized problem can be rewritten as an abstract evolution problem in X_0,

$$\dot{z} + Lz = f, \quad t > 0, \quad z(0) = z_0, \tag{4.2}$$

where $z = (u, h)$, $f = (f_u, f_h)$, $z_0 = (u_0, h_0)$, provided $g_d = g_m = g_u = g_s = 0$. The linearized problem has maximal L_p-regularity, hence (4.2) has this property as well. Therefore, by a result due to Hieber and Prüss, $-L$ generates an analytic C_0-semigroup in X_0; cf. [3, Proposition 1.1]. Since the embedding $X_1 \hookrightarrow X_0$ is compact, the semigroup e^{-Lt} as well as the resolvent $(\lambda + L)^{-1}$ of $-L$ are compact, too. Therefore the spectrum $\sigma(L)$ of L consists of countably many eigenvalues of finite algebraic multiplicity, and it is independent of p.

(2) We study the eigenvalues of $-L$. Suppose that λ with $\mathrm{Re}\,\lambda \geq 0$ is an eigenvalue of $-L$. This means

$$\begin{aligned}
&\lambda\varrho u - \mu\Delta u + \nabla\pi = 0 \quad \text{in } \Omega\backslash\Sigma,\\
&\mathrm{div}\, u = 0 \quad \text{in } \Omega\backslash\Sigma,\\
&P_\Sigma [\![u]\!] = 0 \quad \text{on } \Sigma,\\
&-[\![T(u,\pi)\nu_\Sigma]\!] + \sigma\mathcal{A}_\Sigma h\nu_\Sigma = 0 \quad \text{on } \Sigma,\\
&-[\![T(u,\pi)\nu_\Sigma\cdot\nu_\Sigma/\varrho]\!] = 0 \quad \text{on } \Sigma,\\
&u = 0 \quad \text{on } \partial\Omega,\\
&\lambda[\![\varrho]\!]h - [\![\varrho u\cdot\nu_\Sigma]\!] = 0 \quad \text{on } \Sigma.
\end{aligned} \tag{4.3}$$

Observe that on Σ we may write

$$u_k = P_\Sigma u_k + \lambda h\nu_\Sigma + j_\Sigma\nu_\Sigma/\varrho_k, \quad k = 1, 2.$$

By this identity, taking the inner product of the problem for u with u and integrating by parts, we get

$$\begin{aligned}
0 &= \lambda|\varrho^{1/2}u|_2^2 - (\mathrm{div}\, T(u,\pi)|u)_2\\
&= \lambda|\varrho^{1/2}u|_2^2 + 2|\mu^{1/2}D(u)|_2^2\\
&\quad + ([\![T(u,\pi)\nu_\Sigma]\!]|P_\Sigma u_k + \lambda h\nu_\Sigma)_\Sigma + ([\![T(u,\pi)\nu_\Sigma\cdot\nu_\Sigma/\varrho]\!]|j_\Sigma)_\Sigma\\
&= \lambda|\varrho^{1/2}u|_2^2 + 2|\mu^{1/2}D(u)|_2^2 + \sigma\bar{\lambda}(\mathcal{A}_\Sigma h|h)_\Sigma.
\end{aligned}$$

Taking real parts yields

$$0 = \mathrm{Re}\,\lambda|\varrho^{1/2}u|_2^2 + 2|\mu^{1/2}D(u)|_2^2 + \sigma\mathrm{Re}\,\lambda(\mathcal{A}_\Sigma h|h)_\Sigma. \tag{4.4}$$

Taking imaginary parts yields

$$0 = \mathrm{Im}\,\lambda|\varrho^{1/2}u|_2^2 - \sigma\mathrm{Im}\,\lambda(\mathcal{A}_\Sigma h|h)_\Sigma,$$

hence if $\mathrm{Im}\,\lambda \neq 0$, then

$$\sigma(\mathcal{A}_\Sigma h|h)_\Sigma = |\varrho^{1/2}u|_2^2.$$

Inserting this identity into (4.4) leads to

$$0 = 2\mathrm{Re}\,\lambda|\varrho^{1/2}u|_2^2 + 2|\mu^{1/2}D(u)|_2^2. \tag{4.5}$$

This shows that when $[\![\varrho]\!] \neq 0$, if λ is an eigenvalue of $-L$ with $\mathrm{Re}\,\lambda \geq 0$ then λ is real. In fact, identity (4.5) implies $D(u) = 0$, then $u = 0$ by [9, Lemma 1.2.1], which is a variant of Korn's inequality, the no-slip condition on $\partial\Omega$ and $P_\Sigma[\![u]\!] = 0$ on Σ. Substituting $u = 0$ in the last equation in (4.3), we obtain $\lambda[\![\varrho]\!]h = 0$. If λ satisfies $\mathrm{Re}\,\lambda \geq 0$ and $\mathrm{Im}\,\lambda \neq 0$, then $[\![\varrho]\!]h = 0$, hence $h = 0$. This shows that λ with $\mathrm{Re}\,\lambda \geq 0$ and $\mathrm{Im}\,\lambda \neq 0$ are not eigenvalues.

The identity

$$\lambda \int_\Sigma h \mathrm{d}\Sigma = \int_\Sigma (u_k \cdot \nu_\Sigma - j/\varrho_k)\mathrm{d}\Sigma = -\varrho_k^{-1} \int_\Sigma j \mathrm{d}\Sigma,$$

shows that the mean values $\bar{h}$ of h and $\bar{j}$ of j vanish in case $\lambda \neq 0$ since the densities are non-equal. If Σ is connected, then $\mathcal{A}_\Sigma$ is positive semi-definite on functions with mean zero. Therefore by (4.4) we see that $\lambda > 0$ are not in the spectrum of $-L$, provided Σ is connected.

Hereafter we use the notation

$$L_{2,0}(\Sigma) = \{h \in L_2(\Sigma) \, : \, \int_\Sigma h \mathrm{d}\Sigma = 0\}.$$

(3) Next we consider the asymmetric Stokes problem

$$\begin{aligned}
&\varrho\lambda u - \mu\Delta u + \nabla\pi = 0 \quad \text{in } \Omega\backslash\Sigma,\\
&\operatorname{div} u = 0 \quad \text{in } \Omega\backslash\Sigma,\\
&P_\Sigma[\![u]\!] = P_\Sigma[\![T(u,\pi)\nu_\Sigma]\!] = 0 \quad \text{on } \Sigma,\\
&-[\![T(u,\pi)\nu_\Sigma \cdot \nu_\Sigma]\!] = g \quad \text{on } \Sigma,\\
&-[\![T(u,\pi)\nu_\Sigma \cdot \nu_\Sigma/\varrho]\!] = 0 \quad \text{on } \Sigma,\\
&u = 0 \quad \text{on } \partial\Omega
\end{aligned} \tag{4.6}$$

to obtain as output

$$[\![\varrho u \cdot \nu_\Sigma]\!]/[\![\varrho]\!] = S_\lambda g.$$

For this problem we have the following result which is a special case of [9, Proposition 10.7.1].

PROPOSITION 4.1

The operator S_λ for the Stokes problem (4.6) *admits a bounded extension to $L_{2,0}(\Sigma)$ for $\lambda \geq 0$ and has the following properties*:

(*i*) *If u denotes the solution of* (4.6), *then*

$$(S_\lambda g|g)_{L_2(\Sigma)^2} = \lambda \int_\Omega \varrho|u|^2 \mathrm{d}x + 2\int_\Omega \mu|D(u)|_2^2 \mathrm{d}x,$$
$$\lambda \geq 0,\ g \in L_{2,0}(\Sigma) \cap H_2^{1/2}(\Sigma).$$

(*ii*) *$S_\lambda \in \mathcal{B}(L_{2,0}(\Sigma))$ is self-adjoint, positive semidefinite and compact.*

(*iii*) *$|S_\lambda|_{\mathcal{B}(L_{2,0}(\Sigma), H_2^1(\Sigma))} \leq C$ uniformly for $\lambda \geq 0$.*

(*vi*) *$S_\lambda : L_{2,0}(\Sigma) \to H_2^1(\Sigma) \cap L_{2,0}(\Sigma)$ is isomorphism for each $\lambda \geq 0$.*

(4) Now suppose that $\lambda > 0$ is an eigenvalue of $-L$. We set $g = -\sigma\mathcal{A}_\Sigma h$ to obtain $\lambda h = S_\lambda(-\sigma\mathcal{A}_\Sigma h)$, namely

$$\lambda T_\lambda h + \sigma\mathcal{A}_\Sigma h = 0 \tag{4.7}$$

with $T_\lambda = S_\lambda^{-1}$. $\lambda > 0$ is an eigenvalue of $-L$ if and only if the problem (4.7) admits a nontrivial solution, i.e. if and only if 0 is an eigenvalue for $B_\lambda := \lambda T_\lambda + \sigma\mathcal{A}_\Sigma$. Here the domain of B_λ is that of $\mathcal{A}_\Sigma$, T_λ is a relatively compact perturbation of $\mathcal{A}_\Sigma$.

We consider this problem in $L_{2,0}(\Sigma)$. Then $\mathcal{A}_\Sigma$ is self-adjoint and

$$\sigma(\mathcal{A}_\Sigma h|h)_\Sigma \geq -\frac{\sigma(n-1)}{R^2}|h|_\Sigma^2.$$

If $\tau > 0$ is an eigenvalue of T_λ, then

$$\tau^{-1}h = T_\lambda^{-1}h = S_\lambda h,$$

hence we get

$$\tau^{-1}|h|_\Sigma \le C|h|_\Sigma,$$

since by Propositions 4.1, S_λ is bounded in $L_{2,0}(\Sigma)$, uniformly for every $\lambda \ge 0$. Therefore $\tau = \tau(\lambda) \ge c_0 > 0$ for every λ, and so

$$(B_\lambda h|h)_\Sigma = \lambda(T_\lambda h|h)_\Sigma + \sigma(\mathcal{A}_\Sigma h|h)_\Sigma \ge \left(c_0\lambda - \frac{\sigma(n-1)}{R^2}\right)|h|_\Sigma^2.$$

This proves that B_λ is positive definite, hence (4.7) has no nontrivial solution for large λ. But for small $\lambda > 0$ we have with $h = \sum_{k=1}^m \chi_{\Sigma_k} h_k$, where χ_A denotes the characteristic function of A, $h_k =$ constant on Σ_k, $\sum_{k=1}^m h_k = 0$,

$$\sigma(\mathcal{A}_\Sigma h|h)_\Sigma = -\frac{\sigma(n-1)}{R^2}\omega_n R^{n-1}\sum_{k=1}^m h_k^2 < 0.$$

For $h \in H_2^{3/2}(\Sigma) \cap L_{2,0}(\Sigma)$, we have $T_\lambda h = S_\lambda^{-1}h \to T_0 h$ in $L_{2,0}(\Sigma)$ as $\lambda \to 0$. This shows that B_λ is not positive semi-definite for small λ. Therefore, by a continuity argument, B_λ has a nontrivial kernel for some $\lambda_0 > 0$, which implies that $-L$ has at least one positive eigenvalue.

Even more is true. We have seen that B_λ is positive definite for large λ and $B_0 = \sigma\mathcal{A}_\Sigma$ has $-\sigma(n-1)/R_*^2$ as an eigenvalue of multiplicity $m-1$ in $L_{2,0}(\Sigma)$. Therefore, as λ increases to infinity, $m-1$ eigenvalues of B_λ must cross through zero, this way inducing $m-1$ positive eigenvalues of $-L$.

(5) Finally, we look at the eigenvalue $\lambda = 0$. Then (4.4) yields

$$|\mu^{1/2}D(u)|_2^2 = 0,$$

hence $D(u) = 0$. Korn's inequality yields $\nabla u = 0$ and then we have $u_2 = 0$ by the no-slip condition on $\partial\Omega$, $u_1 = 0$ by $P_\Sigma[\![u]\!] = 0$. This implies further that the pressures are constant in the phases and $[\![\pi]\!] = -\sigma\mathcal{A}_\Sigma h$. Thus the dimension of the eigenspace for eigenvalue $\lambda = 0$ is the same as the dimension of the manifold of equilibria, namely $mn+1$ if Ω_1 has $m \ge 1$ components. The kernel of L is spanned by $e_R = (0, 1)$, $e_{jk} = (0, Y_j^k)$ with the spherical harmonics Y_j^k of degree one for the spheres Σ_k, $j = 1, \ldots, n$, $k = 1, \ldots, m$. As [9, Chapter 10.7], one shows that $\lambda = 0$ is a semi-simple eigenvalue of L.

Let us summarize what we have proved.

Theorem 4.2. *Let $p \in (1, \infty)$, $\sigma, \varrho_k, \mu_k > 0$, $\psi_k \in \mathbb{R}$, $k = 1, 2$ and suppose $\varrho_1 \ne \varrho_2$. Let L denote the linearization at $e_* := (0, \Sigma) \in \mathcal{E}$ as defined above. Then $-L$ generates a compact analytic semigroup in X_0 which has maximal L_p-regularity. The spectrum of L consists only of eigenvalues of finite algebraic multiplicity. Moreover, the following assertions are valid*:

(*i*) *The operator $-L$ has no eigenvalues $\lambda \ne 0$ with nonnegative real part if and only if Σ is connected.*
(*ii*) *If Σ is disconnected and has m components, then $-L$ has precisely $m-1$ positive eigenvalues.*

(*iii*) $\lambda = 0$ *is an eigenvalue of L and it is semi-simple.*
(*iv*) *The kernel $N(L)$ of L is isomorphic to the tangent space $T_{e_*}\mathcal{E}$ of the manifold of equilibria $\mathcal{E}$ at e_*.*

Consequently, $e_* = (0, \Sigma) \in \mathcal{E}$ is normally stable if and only if Σ is connected, and normally hyperbolic if and only if Σ is disconnected.

This result parallels [9, Theorem 10.7.2] in the isothermal case.

5. Nonlinear stability of equilibria

For the stability analysis of equilibria we follow the exposition of [9, Chapter 11]. We employ again the Hanzawa transform around the given equilibrium $e_* = (0, \Gamma_*)$ where the reference manifold is $\Sigma = \Gamma_*$.

$$\begin{aligned}
&\varrho\partial_t u - \mu\Delta u + \nabla\pi = \varrho F_u(u,h,\pi) \quad \text{in } \Omega\backslash\Sigma,\\
&\operatorname{div} u = G_d(u,h) \quad \text{in } \Omega\backslash\Sigma,\\
&P_\Sigma[\![u]\!] = G_m(u,h) \quad \text{on } \Sigma,\\
&-P_\Sigma[\![T(u,\pi)\nu_\Sigma]\!] = G_{u\tau}(u,h) \quad \text{on } \Sigma,\\
&-[\![T(u,\pi)\nu_\Sigma]\!]\cdot\nu_\Sigma + \sigma\mathcal{A}_\Sigma h = G_{u\nu}(u,h) + G_\gamma(h) \quad \text{on } \Sigma,\\
&-[\![\varrho^{-1}T(u,\pi)\nu_\Sigma\cdot\nu_\Sigma]\!] = G_s(u,h) \quad \text{on } \Sigma,\\
&u = 0 \quad \text{on } \partial\Omega,\\
&\partial_t h - [\![\varrho u\cdot\nu_\Sigma]\!]/[\![\varrho]\!] = F_h(u,h) \quad \text{on } \Sigma,\\
&u(0) = u_0 \quad \text{in } \Omega\backslash\Sigma, \qquad h(0) = h_0 \quad \text{on } \Sigma.
\end{aligned} \tag{5.1}$$

Here the nonlinearities are of class C^1 from $\mathbb{E}$ to $\mathbb{F}$, and satisfy $G_d'(0) = G_k'(0) = 0$ for all $k = m, u\tau, u\nu, \gamma, s$. Let $w := (z,\pi) := (u,h,\pi)$ and $z_0 := (u_0,h_0)$. We will frequently make use of the shorter notation

$$\mathbb{L}w = N(w), \quad z(0) = z_0.$$

We need to parametrize $\mathcal{SM}$ over its tangent space

$$\begin{aligned}
X_\gamma := \big\{&(u,h)\in L_p(\Omega)^n\times C^2(\Sigma) : u\in W_p^{2-2/p}(\Omega\backslash\Sigma)^n,\ h\in W_p^{3-2/p}(\Sigma),\\
&\operatorname{div} u = 0 \ \text{in}\ \Omega\backslash\Sigma, \quad u = 0 \text{ on } \partial\Omega\\
&P_\Sigma[\![u]\!] = 0,\ -P_\Sigma[\![\mu(\nabla u + [\nabla u]^{\mathsf T})]\!]\nu_\Sigma = 0 \text{ on }\ \Sigma\big\}.
\end{aligned}$$

For fixed $\omega > 0$ and given $\tilde z = (\tilde u, \tilde h)$ belonging to a small ball $B_r^{X_\gamma}(0)$, we solve the problem

$$\begin{aligned}
&\varrho\omega\bar u - \mu\Delta\bar u + \nabla\bar\pi = 0 \quad \text{in } \Omega\backslash\Sigma,\\
&\operatorname{div}\bar u = G_d(\bar u + \tilde u, \bar h + \tilde h) \quad \text{in } \Omega\backslash\Sigma,\\
&P_\Sigma[\![\bar u]\!] = G_m(\bar u + \tilde u, \bar h + \tilde h) \quad \text{on } \Sigma,\\
&-P_\Sigma[\![\mu(\nabla\bar u + [\nabla\bar u]^{\mathsf T})]\!]\nu_\Sigma = G_{u\tau}(\bar u + \tilde u, \bar h + \tilde h) \quad \text{on } \Sigma,\\
&-[\![T(\bar u,\bar\pi)\nu_\Sigma\cdot\nu_\Sigma]\!] = G_{u\nu}(\bar u + \tilde u, \bar h + \tilde h) \quad \text{on } \Sigma,\\
&-[\![T(\bar u,\bar\pi)\nu_\Sigma\cdot\nu_\Sigma/\varrho]\!] = G_s(\bar u + \tilde u, \bar h + \tilde h) \quad \text{on } \Sigma,\\
&\tilde u = 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{5.2}$$

We write this equation as

$$L_\omega\bar u = N(\bar z + \tilde z) \quad \text{on } W_p^{2-2/p}(\Omega\backslash\Sigma)\times W_p^{3-2/p}(\Sigma).$$

It holds that $N'(0) = 0$. L_ω is invertible by [9, Proposition 11.2.5], hence the implicit function theorem yields a unique solution ϕ satisfying $\phi'(0) = 0$. Then we define

$$\Phi(\tilde{u}, \tilde{h}) = (\tilde{u}, \tilde{h}) + (\phi(\tilde{u}, \tilde{h}), 0) = (u, h).$$

Φ is isomorphism and satisfies $\Phi'(0) = I$ and $\Phi(B_r^{X_\gamma}(0)) \subset \mathcal{SM}$.

Let (u, π, h) be a solution on its maximal time interval $[0, t_*)$ with initial value $(u_0, h_0) = (\tilde{u}_0, \tilde{h}_0) + (\phi(\tilde{u}_0, \tilde{h}_0), 0)$. We want to derive a decomposition of the form $u = \bar{u} + \tilde{u}, \pi = \bar{\pi} + \tilde{\pi}, h = \bar{h} + \tilde{h}$ with $(\tilde{u}(t), \tilde{h}(t)) \in X_\gamma$. For a given $(\tilde{u}, \tilde{\pi}, \tilde{h})$ such that $(\tilde{u}, \tilde{h})$ has a sufficiently small norm, we solve the problem

$$\begin{aligned}
&\varrho\omega\bar{u} + \varrho\partial_t\bar{u} - \mu\Delta\bar{u} + \nabla\bar{\pi} = \varrho F_u(u, h, \pi) \quad \text{in } \Omega\backslash\Sigma,\\
&\operatorname{div}\bar{u} = G_d(u, h) \quad \text{in } \Omega\backslash\Sigma,\\
&P_\Sigma[\![\bar{u}]\!] = G_m(u, h) \quad \text{on } \Sigma,\\
&-P_\Sigma[\![\mu(\nabla\bar{u} + [\nabla\bar{u}]^{\mathsf{T}})]\!]\nu_\Sigma = G_{u\tau}(u, \vartheta, h) \quad \text{on } \Sigma,\\
&-[\![T(\bar{u}, \bar{\pi})\nu_\Sigma \cdot \nu_\Sigma]\!] + \sigma\mathcal{A}_\Sigma\bar{h} = G_{u\nu}(u, h) + G_\gamma(h) - G_\gamma(\tilde{h}) \quad \text{on } \Sigma,\\
&-[\![T(\bar{u}, \bar{\pi})\nu_\Sigma \cdot \nu_\Sigma/\varrho]\!] = G_s(u, h) \quad \text{on } \Sigma,\\
&\bar{u} = 0 \quad \text{on } \partial\Omega,\\
&\omega\bar{h} + \partial_t\bar{h} - [\![\varrho\bar{u} \cdot \nu_\Sigma]\!]/[\![\varrho]\!] = F_h(u, h) \quad \text{on } \Sigma,\\
&\bar{u}(0) = \bar{u}_0 \quad \text{in } \Omega\backslash\Sigma, \qquad \bar{h}(0) = \bar{h}_0 \quad \text{on } \Sigma,
\end{aligned} \tag{5.3}$$

with $(\bar{u}_0, \bar{h}_0) = (\phi(\tilde{u}_0, \tilde{h}_0), 0)$. Define $(\tilde{u}, \tilde{\pi}, \tilde{h})$ as a solution of the problem

$$\begin{aligned}
&\varrho\partial_t\tilde{u} - \mu\Delta\tilde{u} + \nabla\tilde{\pi} = \varrho\omega\bar{u} \quad \text{in } \Omega\backslash\Sigma,\\
&\operatorname{div}\tilde{u} = 0 \quad \text{in } \Omega\backslash\Sigma,\\
&P_\Sigma[\![\tilde{u}]\!] = 0 \quad \text{on } \Sigma,\\
&-P_\Sigma[\![\mu(\nabla\tilde{u} + [\nabla\tilde{u}]^{\mathsf{T}})]\!]\nu_\Sigma = 0 \quad \text{on } \Sigma,\\
&-[\![T(\tilde{u}, \tilde{\pi})\nu_\Sigma \cdot \nu_\Sigma]\!] + \sigma\mathcal{A}_\Sigma\tilde{h} = G_\gamma(\tilde{h}) \quad \text{on } \Sigma,\\
&-[\![T(\tilde{u}, \tilde{\pi})\nu_\Sigma \cdot \nu_\Sigma/\varrho]\!] = 0 \quad \text{on } \Sigma,\\
&\tilde{u} = 0 \quad \text{on } \partial\Omega,\\
&\partial_t\tilde{h} - [\![\varrho\tilde{u} \cdot \nu_\Sigma]\!]/[\![\varrho]\!] = \omega\bar{h} \quad \text{on } \Sigma,\\
&\tilde{u}(0) = \tilde{u}_0 \quad \text{in } \Omega\backslash\Sigma, \qquad \tilde{h}(0) = \tilde{h}_0 \quad \text{on } \Sigma.
\end{aligned} \tag{5.4}$$

Employing Theorem 3.1 with the time weight parameter $\mu = 1$, we solve (5.4) as

$$(\tilde{u}, \tilde{\pi}, \tilde{h}) = \Psi(\tilde{u}_0, \tilde{h}_0, \bar{u}, \bar{h})$$

and insert it into (5.3). $\tilde{\pi}$ is given by a function of $(\tilde{u}, \tilde{h})$ in a similar manner as in subsection 4.1. By the implicit function theorem, we obtain a unique solution of (5.3) $\bar{z} := (\bar{u}, \bar{\pi}, \bar{h}) = \bar{z}(\tilde{u}, \tilde{h})$ in the function space $\mathbb{E}(J)$ on each interval $J = [0, \tau]$, provided that $\omega > 0$ is large enough. Now (5.3) can be written abstractly as

$$\dot{\tilde{z}} + L\tilde{z} = R(\bar{z}), \quad t > 0, \qquad \tilde{z}(0) = \tilde{z}_0, \tag{5.5}$$

where $\tilde{z} = (\tilde{u}, \tilde{h})$ and

$$R(\tilde{z}) := \omega((I - T_1)\bar{u}(\tilde{z}) - T_2 G_\gamma(\tilde{z}), \omega\bar{h}(\tilde{z})).$$

Therefore we may apply [10, Theorems 2.1 and 6.1] with an additional semilinear nonlocal term, using [2, Theorem 4] to obtain Theorem 1.4.

6. Global existence and convergence

There are basically two obstructions against global existence:

- *Regularity*: the norms of either $u(t)$ or $\Gamma(t)$ become unbounded;
- *Geometry*: the topology of the interface changes; or the interface touches the outer boundary of $\partial\Omega$.

Note that the compatibility conditions,

$$\begin{aligned}
&\operatorname{div} u(t) = 0 \text{ in } \Omega\backslash\Gamma(t), \quad u = 0 \text{ on } \partial\Omega, \\
&P_\Gamma[\![u(t)]\!] = P_\Gamma[\![\mu D(u)(t)]\!] = 0 \text{ on } \Gamma(t),
\end{aligned}$$

are preserved by the semiflow. Observe that the pressure does not enter as an explicit system variable, as it can be reconstructed by solving a weak transmission problem, at any time instant $t \in [0, t_+)$.

By the *uniform ball condition* we mean the existence of a radius $r_0 > 0$ such that for each t, at each point $x \in \Gamma(t)$ there exists centers $x_k \in \Omega_k(t)$ such that $B_{r_0}(x_k) \subset \Omega_k(t)$ and $\Gamma(t) \cap \bar{B}_{r_0}(x_k) = \{x\}, k = 1, 2$. Note that this condition bounds the curvature of $\Gamma(t)$, prevents parts of it to touch the outer boundary $\partial\Omega$, and to undergo topological changes. Hence if this condition holds, then the number of components of the phases are preserved.

With this property, combining the local semiflow for (1.1) with the Lyapunov functional and compactness, we obtain the following result.

Theorem 6.1. *Let $p > n + 2, \sigma, \varrho_k, \mu_k > 0, \psi_k \in \mathbb{R}, k = 1, 2,$ and suppose $\varrho_1 \neq \varrho_2$. Let (u, Γ) be a solution of (1.1) in the state manifold $\mathcal{SM}$ on its maximal time interval $[0, t_+)$. Assume there is a constant $M > 0$ such that the following conditions hold on $[0, t_+)$:*

(i) $|u(t)|_{W_p^{2-2/p}}, |\Gamma(t)|_{W_p^{3-2/p}} \leq M < \infty$;
(ii) $\Gamma(t)$ *satisfies the uniform ball condition.*

Then $t_+ = \infty$, i.e. the solution exists globally, and its limit set $\omega_+(u, \Gamma) \subset \mathcal{E}$ is non-empty. If further $(0, \Gamma_\infty) \in \omega_+(u, \Gamma)$ with Γ_∞ connected, then the solution converges in $\mathcal{SM}$ to this equilibrium.

Conversely, if $(u(t), \Gamma(t))$ is a global solution in $\mathcal{SM}$ which converges to an equilibrium $(0, \Gamma_*) \in \mathcal{E}$ in $\mathcal{SM}$ as $t \to \infty$, then (i) and (ii) are valid.

Proof. Assume that (i) and (ii) are valid. Then $\Gamma([0, t_+)) \subset W_p^{3-2/p}(\Omega, r)$ is bounded, hence relatively compact in $W_p^{3-2/p-\varepsilon}(\Omega, r)$. Thus we may cover this set by finitely many balls with centers Σ_l real analytic in such a way that

$$\operatorname{dist}_{W_p^{3-2/p-\varepsilon}}(\Gamma(t), \Sigma_j) \leq \delta,$$

for some $j = j(t), t \in [0, t_+)$. Let $J_l = \{t \in [0, t_+) : j(t) = l\}$; using for each l a Hanzawa-transformation Ξ_l, we see that the pull backs $\{(u(t, \cdot),) \circ \Xi_l : t \in J_l\}$ are bounded in $W_p^{2-2/p}(\Omega\backslash\Sigma_l)^{n+1}$, hence relatively compact in $W_p^{2-2/p-\varepsilon}(\Omega\backslash\Sigma_l)^{n+1}$. Employing now Theorem 3.1 we obtain solutions (u^1, Γ^1) with initial configurations

$(u(t), \Gamma(t))$ in the state manifold on a common time interval, say $(0, \tau]$ and by uniqueness, we have

$$(u^1(\tau), \Gamma^1(\tau)) = (u(t+\tau), \Gamma(t+\tau)).$$

By definition of t_+ this yields $t_+ = \infty$, and continuous dependence implies that the orbit of the solution $(u(\cdot), \Gamma(\cdot))$ is relative compact in $\mathcal{SM}$. The available energy is a strict Lyapunov functional, hence the limit set $\omega_+(u, \Gamma) \subset \mathcal{SM}$ of a solution is contained in the set $\mathcal{E}$ of equilibria. By compactness, $\omega_+(u, \Gamma) \subset \mathcal{SM}$ is non-empty, hence the solution comes close to $\mathcal{E}$. Then we may apply the convergence result Theorem 1.4. The converse follows by a compactness argument. □

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