Siberian Mathematical Journal, vol. 60, *no.* 2, *pp.* 272–278, 2019
Russian Text © 2019 *Borodin O.V. and Ivanova A.O.*

LIGHT MINOR 5-STARS IN 3-POLYTOPES WITH MINIMUM DEGREE 5

O. V. Borodin and A. O. Ivanova

UDC 519.17

Abstract: Attempting to solve the Four Color Problem in 1940, Henry Lebesgue gave an approximate description of the neighborhoods of 5-vertices in the class $\mathbf{P}_5$ of 3-polytopes with minimum degree 5. This description depends on 32 main parameters. Not many precise upper bounds on these parameters have been obtained as yet, even for restricted subclasses in $\mathbf{P}_5$. Given a 3-polytope P, by $w(P)$ denote the minimum of the maximum degree-sum (weight) of the neighborhoods of 5-vertices (minor 5-stars) in P. In 1996, Jendrol' and Madaras showed that if a polytope P in $\mathbf{P}_5$ is allowed to have a 5-vertex adjacent to four 5-vertices (called a *minor* $(5,5,5,5,\infty)$-*star*), then $w(P)$ can be arbitrarily large. For each P^* in $\mathbf{P}_5$ with neither vertices of degree 6 and 7 nor minor $(5,5,5,5,\infty)$-star, it follows from Lebesgue's Theorem that $w(P^*) \leq 51$. We prove that every such polytope P^* satisfies $w(P^*) \leq 42$, which bound is sharp. This result is also best possible in the sense that if 6-vertices are allowed but 7-vertices forbidden, or vice versa; then the weight of all minor 5-stars in $\mathbf{P}_5$ under the absence of minor $(5,5,5,5,\infty)$-stars can reach 43 or 44, respectively.

DOI: 10.1134/S0037446619020071

Keywords: planar map, planar graph, 3-polytope, structural properties, 5-star

1. Introduction

The degree of a vertex or face x in a convex finite 3-dimensional polytope (called a 3-*polytope*) is denoted by $d(x)$. A k-*vertex* is a vertex v with $d(v) = k$. A k^+-*vertex* (k^--*vertex*) is one of degree at least k (at most k). Similar notation is used for the faces. A class of all P_5's with minimum degree 5 is denoted by $\mathbf{P}_5$.

The *weight* of a subgraph S of a 3-polytope is the sum of degrees of the vertices of S in the 3-polytope. The *height* of a subgraph S of a 3-polytope is the maximum degree of the vertices of S in the 3-polytope. A k-star, a star with k rays, $S_k(v)$ is *minor* if its center v has degree at most 5. In particular, the neighborhoods of 5-vertices are minor 5-stars and vice versa. All stars in this note are minor. By $w(S_k)$ and $h(S_k)$ we denote the minimum weight and height, respectively, of minor k-stars in a given 3-polytope.

In 1904, Wernicke [1] proved that each P in $\mathbf{P}_5$ has a 5-vertex adjacent to a 6^--vertex. This result was strengthened by Franklin [2] in 1922 to the existence of a 5-vertex with two 6^--neighbors.

In 1940, in attempts to solve the Four Color Problem, Lebesgue [3, p. 36] gave an approximate description of the neighborhoods of 5-vertices in $\mathbf{P}_5$. In particular, this description implies the results in [1, 2] and shows that there is a 5-vertex with three 7^--neighbors.

For a class $\mathbf{P}_5$ the bounds $w(S_1) \leq 11$ (Wernicke [1]) and $w(S_2) \leq 17$ (Franklin [2]) are tight. It was proved by Lebesgue [3] that $w(S_3) \leq 24$, which was improved in 1996 by Jendrol' and Madaras [4] to the sharp bound $w(S_3) \leq 23$. Furthermore, Jendrol' and Madaras [4] gave a precise description of minor 3-stars in $\mathbf{P}_5$. Lebesgue [3] proved that $w(S_4) \leq 31$, which was strengthened by Borodin and Woodall [5] to the tight bound $w(S_4) \leq 30$. Note that $w(S_3) \leq 23$ easily implies $w(S_2) \leq 17$ and immediately follows from $w(S_4) \leq 30$ (in both cases, it suffices to delete a vertex of maximum degree from a minor star of the minimum weight). Recently, we [6] obtained a precise description of 4-stars in $\mathbf{P}_5$.

The authors were funded by the Russian Science Foundation (Grant 16–11–10054).

Novosibirsk. Translated from *Sibirskii Matematicheskii Zhurnal*, vol. 60, no. 2, pp. 351–359, March–April, 2019; DOI: 10.17377/smzh.2019.60.207. Original article submitted July 5, 2018; revised July 5, 2018; accepted August 17, 2018.

Lebesgue's approximate description [3] of 5-stars in $\mathbf{P}_5$ depends on 32 main parameters. Not many precise upper bounds on these parameters have been obtained as yet, even for restricted subclasses in $\mathbf{P}_5$.

Recently, Ivanova and Nikiforov [7–9] improved two parameters in Lebesgue's description [3] without worsening the others.

For a given 3-polytope P in $\mathbf{P}_5$, $w(P)$ is denoted the weight of neighborhoods of 5-vertices in P, and $h(P)$ is denoted the height of minor 5-stars.

Jendrol' and Madaras [4] showed that if P in $\mathbf{P}_5$ has a 5-vertex adjacent to four 5-vertices, called a *minor* $(5,5,5,5,\infty)$*-star*, then $w(P)$ can be arbitrarily large.

For each P in $\mathbf{P}_5$ that has neither 6-vertices nor minor $(5,5,5,5,\infty)$-stars, it follows from Lebesgue's Theorem that $w(P) \le 68$. Recently, this bound was lowered to $w(P) \le 55$ by Borodin, Ivanova, and Jensen [10], then to $w(P) \le 51$ by Borodin and Ivanova [11], and finally to the tight bound $w(P) \le 44$ by Borodin, Ivanova, and Vasil'eva [12].

For every P in $\mathbf{P}_5$ having neither minor $(5,5,5,5,\infty)$-stars nor vertices of degree from 6 to 9, Lebesgue's bound $w(P) \le 44$ was improved by Borodin and Ivanova [13] to the sharp bound $w(P) \le 42$.

For each P in $\mathbf{P}_5$ with neither vertices of degree from 6 to 8 nor minor $(5,5,5,5,\infty)$-star, it follows from Lebesgue's Theorem that $h(P) \le 14$ and $w(P) \le 46$, which bounds were recently improved in [14] to the best possible bounds $h(P) \le 12$ and $w(P) \le 42$.

The purpose of this note is to prove the following fact which improves or extends some of above mentioned results.

Theorem 1. *Each* 3*-polytope with minimum degree* 5 *and neither vertices of degree* 6 *or* 7 *nor* 5*-vertices adjacent to four* 5*-vertices has the minor* 5*-star* S_5 *with* $w(S_5) \le 42$*, and this bound is best possible.*

This improves the bound $w(S_5) \le 51$ that follows from Lebesgue's Theorem and is also best possible in the sense that if 6-vertices are allowed but 7-vertices forbidden, or vice versa, then the weight of all minor 5-stars in $\mathbf{P}_5$ under the absence of minor $(5,5,5,5,\infty)$-stars can reach 43 or 44, respectively (as shown in [15] and [12], respectively).

It is easily seen that a minor 5-star of weight at most 42 has height at most $42 - 4 \times 5 - 8 = 14$, so Theorem 1 implies the following fact established in [16].

Corollary 2. *Every* 3*-polytope* P *with minimum degree* 5 *and neither vertices of degree* 6 *or* 7 *nor* 5*-vertices adjacent to four* 5*-vertices satisfies* $h(P) \le 14$*, which bound is best possible.*

More information on the heights and weights of stars in 3-polytopes may be found in [5, 6, 10, 13–23] and surveys [24, 25].

2. Proof of Theorem 1

2.1. The tightness of the bounds 42 and 14. We start with the $(3,4,4,4)$ Archimedean solid $A(3,4,4,4)$, which is a 4-regular 3-polytope whose each vertex is incident with a 3-face and three 4-faces. Now cap each 4-face of $A(3,4,4,4)$ to obtain a triangulation T whose each face is incident with a 4-vertex and two 7-vertices. The dual D of T is a cubic 3-polytope, and we replace all its vertices by small 3-faces.

The resulting 3-polytope R is cubic and such that each vertex is incident with a 3-face, 8-face, and 14-face. Capping all 8^+-faces of R yields a desired 3-polytope in which every 5-vertex has a 14-neighbor and another 8-neighbor, as desired.

2.2. Discharging. Suppose that a 3-polytope P' is a counterexample to the main statement of Theorem 1. Thus each 5-vertex in P' has at most three 5-neighbors and the degree-sum of its neighbors is at least 43.

Let P be a counterexample with the most edges on the same vertices as P'.

Remark 1. P has no 4^+-face with two nonconsecutive 8^+-vertices along the boundary, for otherwise adding a diagonal between these vertices would result in a counterexample with a greater number of edges than in P.

Let V, E, and F be the sets of vertices, edges, and faces of P. Euler's formula $|V| - |E| + |F| = 2$ implies

$$\sum_{v \in V}(d(v) - 6) + \sum_{f \in F}(2d(f) - 6) = -12. \tag{1}$$

We assign an *initial charge* $\mu(v) = d(v) - 6$ to each $v \in V$ and $\mu(f) = 2d(f) - 6$ to each $f \in F$, so that only 5-vertices have negative initial charge. Using the properties of P as a counterexample to Theorem 1, we define a local redistribution of charges, preserving their sum, such that the final charge $\mu'(x)$ is nonnegative for all $x \in V \cup F$. This will contradict the fact that the sum of the final charges is, by (1), equal to -12.

The *final charge* $\mu'(x)$ whenever $x \in V \cup F$ is defined by applying the rules R1–R8 below (see Fig. 1).

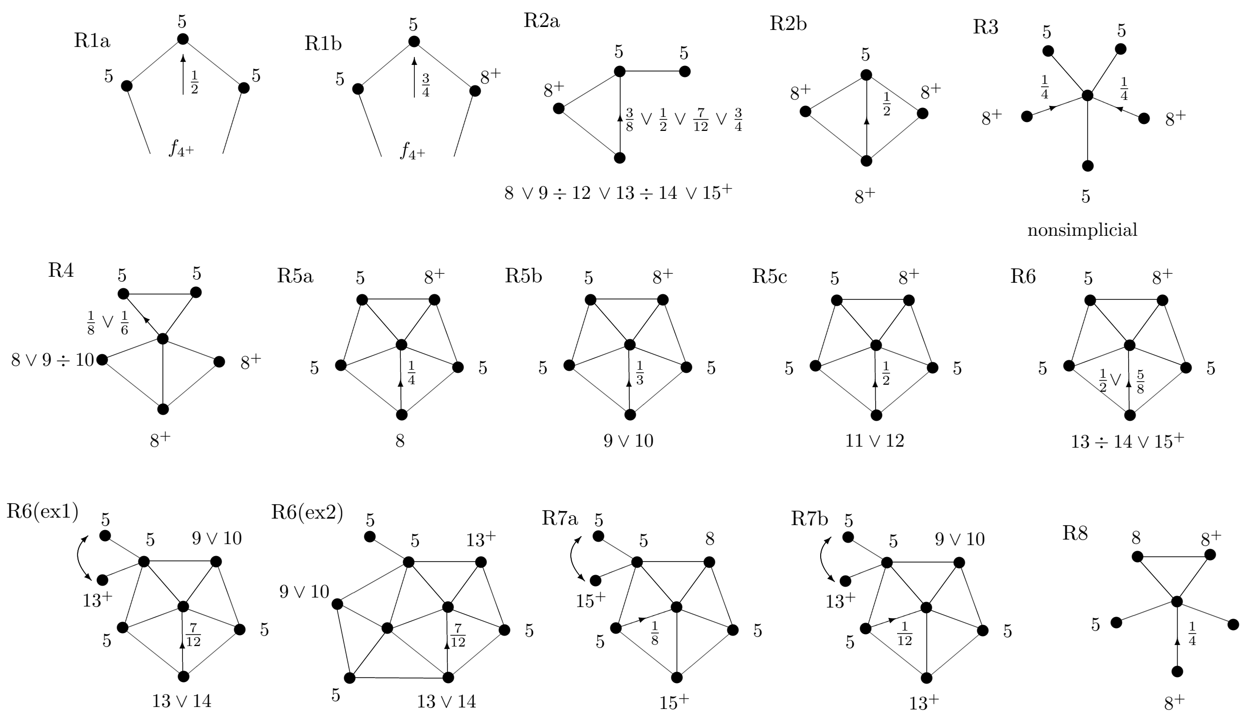

Fig. 1. Rules of discharging

For a vertex v, let $v_1, \ldots, v_{d(v)}$ be the vertices adjacent to v in a cyclic order. A vertex is *simplicial* if it is completely surrounded by 3-faces. A 5-vertex v is *strong* if $d(v_1) = d(v_2) = 5$, $d(v_3) \geq 8$, $d(v_4) \geq 8$, $d(v_5) \geq 8$, and there is a 3-face vv_1v_2. Note that v also is incident to 3-faces v_3vv_4 and v_4vv_5 due to Remark 1.

A simplicial 5-vertex v such that $d(v_1) = d(v_2) = d(v_4) = 5$, $8 \leq d(v_3) \leq 10$, and hence $d(v_5) \geq 13$ is *poor*, and v_1 is *paired* with v. We note that the poor and paired neighbors of each 13^+-vertex are in one-to-one correspondence with each other. We also see that a paired vertex v_1 is poor itself if and only if v_2 is strong.

R1. A 4^+-face $f = v_1 \ldots v_{d(f)}$ gives each incident 5-vertex v_2:

(a) $\frac{1}{2}$ if $d(v_1) = d(v_3) = 5$,

or

(b) $\frac{3}{4}$ if $d(v_1) \geq 8$ and $d(v_3) = 5$.

R2. A 5-vertex v with $d(v_1) \geq 8$ receives the following charge from its 8^+-neighbor v_2:

(a) if $d(v_3) = 5$, then $\frac{3}{8}$, $\frac{1}{2}$, $\frac{7}{12}$ or $\frac{3}{4}$ in the cases $d(v_2) = 8$, $9 \le d(v_2) \le 12$, $13 \le d(v_2) \le 14$ or $d(v_2) \ge 15$ respectively, and

(b) $\frac{1}{2}$ if $d(v_3) \ge 8$.

R3. A nonsimplicial 5-vertex v with $d(v_1) = d(v_3) = d(v_4) = 5$ receives $\frac{1}{4}$ from each of its 8^+-neighbors v_2 and v_5.

R4. A strong 5-vertex v with $d(v_1) = d(v_2) = 5$ gives $\frac{1}{8}$ or $\frac{1}{6}$ to v_1 if $d(v_5) = 8$ or $9 \le d(v_5) \le 10$, respectively, and the same is valid for v_2 depending on $d(v_3)$ by symmetry.

R5. A simplicial 5-vertex v with $d(v_1) = d(v_2) = d(v_4) = 5$ receives from v_5

(a) $\frac{1}{4}$ if $d(v_5) = 8$,

(b) $\frac{1}{3}$ if $9 \le d(v_5) \le 10$,

(c) $\frac{1}{2}$ if $11 \le d(v_5) \le 12$.

R6. If a simplicial vertex v satisfies $d(v_1) = d(v_2) = d(v_4) = 5$, $d(v_3) \ge 8$, and $d(v_5) \ge 13$, then v_5 gives $\frac{1}{2}$ or $\frac{5}{8}$ to v if $13 \le d(v_5) \le 14$ or $d(v_5) \ge 15$, respectively, with the following two exceptions:

(ex1) if $13 \le d(v_5) \le 14$, $9 \le d(v_3) \le 10$ (so v is poor) and v_2 is not strong (hence $d(v_3) \ge 13$ and v_3 has three 5-neighbors), then v_5 gives $\frac{7}{12}$ to v;

(ex2) if $13 \le d(v_5) \le 14$, v_1 is a poor vertex paired with v, and v_2 is not strong (so v_3 has three 5-neighbors), then v_5 also gives $\frac{7}{12}$ to v.

R7. Every poor 5-vertex v with a nonstrong neighbor v_2 receives from its paired vertex v_1:

(a) $\frac{1}{8}$ if v has an 8-neighbor v_3,

or

(b) $\frac{1}{12}$ if $9 \le d(v_3) \le 10$.

R8. If a vertex v satisfies $d(v_1) = d(v_3) = 5$, $d(v_2) \ge 8$, $d(v_4) \ge 8$ and $d(v_5) = 8$, then v receives $\frac{1}{4}$ from v_2.

2.3. Checking $\mu'(x) \ge 0$ whenever $x \in V \cup F$. If f is a 4^+-face, then the donation of $\frac{3}{4}$ by R1(b) may be interpreted as giving $\frac{1}{2}$ to the 5-vertex and $\frac{1}{4}$ to the neighbor 8^+-vertex along the boundary $\partial(f)$ of f. As a result, each vertex in $\partial(f)$ receives at most $2 \times \frac{1}{4}$ from f after this averaging, so we have $\mu'(f) \ge 2d(f) - 6 - d(f) \times \frac{1}{2} = \frac{3(d(f)-4)}{2} \ge 0$.

Suppose now that $v \in V$.

Case 1. $d(v) = 5$. If v is adjacent to at least four 8^+-vertices, then $\mu'(v) \ge 5 - 6 + 4 \times \frac{3}{8} > 0$ by R2, since v does not give charge away by R4 or R7.

Suppose that v has precisely three 8^+-neighbors. If they are consecutive around v, say v_1, v_2, v_3, then v receives at least $\frac{1}{2} + 2 \times \frac{3}{8} > 1$ from them by R2(a,b). Also, v can give $\frac{1}{8}$ or $\frac{1}{6}$ to each of the two 5-neighbors v_4 and v_5 by R4 provided that v is strong. More specifically, v_4 then receives $\frac{1}{8}$ if $d(v_3) = 8$ or $\frac{1}{6}$ if $9 \le d(v_3) \le 10$, and the donation to v_5 by R4 similarly depends on $d(v_1)$. Note that each of v_1 and v_3 thus brings v the total of at least $\frac{1}{4} = \frac{3}{8} - \frac{1}{8} < \frac{1}{2} - \frac{1}{6}$ by R2 combined with R4, while v_2 brings v at least $\frac{1}{2}$ by R4, so $\mu'(v) \ge -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$.

Suppose now that $d(v_1) = d(v_3) = 5$. Here, v does not give charge to v_1 and v_3 by R4 or R7, so it suffices for v to collect the total of at least 1 from its three 8^+-neighbors. If $d(v_4) \ge 9$ and $d(v_5) \ge 9$, then $\mu'(v) \ge -1 + 2 \times \frac{1}{2} = 0$ by R2(a); otherwise, we have $d(v_4) = 8$ and $d(v_5) \ge 8$ by symmetry, which yields $\mu'(v) \ge -1 + 2 \times \frac{3}{8} + \frac{1}{4} = 0$ by R2(a) combined with R8, as desired.

It remains to assume that v has precisely two 8^+-neighbors. First suppose that $d(v_1) \ge 8$ and $d(v_2) \ge 8$. If $d(v_1) = 8$, then $d(v_2) \ge 43 - 4 \times 5 - 8 = 15$, so v receives $\frac{3}{4} + \frac{3}{8}$ from v_2 and v_1 by R2(a) and possibly gives $\frac{1}{8}$ to v_3 by R7(a), which yields $\mu'(v) \ge 0$.

Suppose now that $d(v_1) \ge 9$ and $d(v_2) \ge 9$. If $d(v_1) \le 12$, then v_1 is not paired with v_5, so v simply receives $\frac{1}{2}$ from v_1 by R2(a). If $13 \le d(v_1) \le 14$, then v_1 gives $\frac{7}{12}$ to v by R2(a), and v can give to v_5 either $\frac{1}{12}$ by R7(b) if v is paired with v_5 or no charge otherwise. In both cases, v_1 brings at least $\frac{1}{2} = \frac{7}{12} - \frac{1}{12}$ to

v in total, and the same is true for v_2, so $\mu'(v) \geq -1 + 2 \times \frac{1}{2} = 0$. Finally, if $d(v_1) \geq 15$, then v_1 brings at least $\frac{3}{4} - \frac{1}{8}$ to v by R2(a), so again $\mu'(v) \geq -1 + 2 \times \frac{1}{2} = 0$.

From now on suppose that $d(v_1) \geq 8$ and $d(v_3) \geq 8$. If v is not simplicial, then v receives $2 \times \frac{1}{4}$ from v_1 and v_3 by R3 and at least $\frac{1}{2}$ from an incident 4^+-face by R1. Thus we are done unless v gives $\frac{1}{12}$ or $\frac{1}{8}$ to at least one of v_4 and v_5 by R7, which can happen only if the face $f = \dots v_4 v v_5$ is a triangle. However, then v actually receives $\frac{3}{4}$ by R1(b) at least once, and we have $\mu'(v) \geq -1 + \frac{3}{4} + 2 \times \frac{1}{4} - 2 \times \frac{1}{8} = 0$.

Finally, suppose that v is simplicial. If v gives $\frac{1}{8}$ or $\frac{1}{12}$ to v_5 by R7, so that v is paired with a poor vertex v_5, then $d(v_1) \geq 15$ or $d(v_1) \geq 13$, respectively, since $w(S_5(v)) \geq 43$ by assumption. (Hereafter, we consider two possibilities in parallel, depending on whether v_5 has an 8-neighbor or a neighbor of degree 9 or 10). Furthermore, v_4 is not strong, which implies that v_4 has a 5-neighbor different from v and v_5. In turn, this means that $d(v_3) \geq 15$ or $d(v_3) \geq 13$, respectively, since otherwise we would have $w(S_5(v_4)) \leq 42$; a contradiction.

Thus v receives from v_1 either $\frac{5}{8}$ by R6 or $\frac{7}{12}$ by R6(ex2), respectively, and hence v_1 brings the total of $\frac{1}{2} = \frac{5}{8} - \frac{1}{8} = \frac{7}{12} - \frac{1}{12}$ to v. By symmetry, the same is true for v_3: no matter whether it is paired with v_4 or not, it brings $\frac{1}{2}$ either by R6 or by R6(ex2) combined with R7.

Thus we have $\mu'(v) = -1 + 2 \times \frac{1}{2} = 0$ when v gives away $\frac{1}{8}$ or $\frac{1}{12}$ at least once to a poor neighbor according to R7, so from now we can assume that v is not a donator of charge by R7.

We know that each 11^+-neighbor gives v at least $\frac{1}{2}$ by R5(c) and R6, so it remains to assume that $d(v_1) \leq 10$, which means that v is poor.

First suppose that $d(v_1) = 8$; then $d(v_3) \geq 15$ since $w(S_5(v)) \geq 43$ by assumption. No matter whether v_5 is strong or otherwise, our v receives $\frac{1}{8}$ either from v_5 by R4 or from its paired vertex v_4 by R7(a), respectively. Also, v receives $\frac{1}{4}$ from v_1 by R5(a) and $\frac{5}{8}$ from v_3 by R6, so we have $\mu'(v) = 0$ in both options.

Now, if $9 \leq d(v_1) \leq 10$ then $d(v_3) \geq 13$ by the same reason. Furthermore, if v_5 is strong, then v receives $\frac{1}{6}$ from v_5 by R4, $\frac{1}{3}$ from v_1 by R5(b), and $\frac{1}{2}$ from v_3 by R6, so we have $\mu'(v) = 0$. Otherwise, v receives $\frac{1}{12}$ from v_4 by R7(b) and $\frac{1}{3}$ from v_1. Also, v receives from v_3 either $\frac{7}{12}$ by R6(ex1) if $d(v_3) \leq 14$ or $\frac{5}{8}$ (which is greater than $\frac{7}{12}$) by R6 if $d(v_3) \geq 15$. This again makes $\mu'(v) \geq 0$, as desired.

Case 2. $d(v) = 8$. We can average donations of v to its 5-neighbors according to R2, R3, R5(a), and R8 as follows. If $d(v_1) = d(v_2) = 5$ and $d(v_3) \geq 8$, which is the situation of R2(a), then v instead gives $\frac{1}{4}$ to v_2 and $\frac{1}{8}$ to v_3. Similarly, instead of giving $\frac{1}{2}$ to a 5-neighbor v_2 by R2(b), our v now gives $\frac{1}{4}$ to v_2 and $\frac{1}{8}$ to each of the 8^+-vertices v_1 and v_3. As a result, each neighbor receives at most $\frac{1}{4} = \frac{1}{8} + \frac{1}{8} = \frac{3}{8} - \frac{1}{8}$ from v after averaging, so $\mu'(v) \geq d(v) - 6 - \frac{d(v)}{4} = \frac{3(d(v)-8)}{4} \geq 0$, as desired.

Case 3. $9 \leq d(v) \leq 10$. We now average donations of v to its 5-neighbors according to R2, R3, R5(b), and R8 in the same fashion. Instead of giving $\frac{1}{2}$ to a 5-neighbor v_2 by R2(b), our v gives $\frac{1}{6}$ to each of the vertices v_1, v_2, and v_3. If $d(v_1) = d(v_2) = 5$ and $d(v_3) \geq 9$, which happens in R2(a), then v rather gives $\frac{1}{3}$ to v_2 and $\frac{1}{6}$ to v_3. As a result, each neighbor receives at most $\frac{1}{3} = \frac{1}{6} + \frac{1}{6} = \frac{1}{2} - \frac{1}{6}$ from v, so $\mu'(v) \geq d(v) - 6 - \frac{d(v)}{3} = \frac{2(d(v)-9)}{3} \geq 0$, and we are done.

Case 4. $11 \leq d(v) \leq 12$. We note that v gives each neighbor at most $\frac{1}{2}$ by R2, R3, R5(c), and R8, so $\mu'(v) \geq d(v) - 6 - \frac{d(v)}{2} = \frac{d(v)-12}{2}$, which settles the case $d(v) = 12$.

So suppose that $d(v) = 11$. If v has an 8^+-neighbor, then $\mu'(v) \geq 11 - 6 - 10 \times \frac{1}{2} = 0$. Thus we can assume that v is completely surrounded by 5-vertices. If v is incident with a 4^+-face $\dots v_1 v v_2$, then each of v_1 and v_2 receives $\frac{1}{4}$ from v by R3. Indeed, if the neighbors of v_1 in a cyclic order are $\dots x_1, v, y_1$, then $d(x_1) = d(y_1) = 5$ due to Remark 1, and the same argument works for v_2. This implies $\mu'(v) \geq 5 - 2 \times \frac{1}{4} - (11-2) \times \frac{1}{2} = 0$.

Therefore, it remains to assume in addition that v is simplicial. Now if there is a 4^+-face $\dots v_1' v_1 v_2 v_2'$, then each of v_1 and v_2 receives at most $\frac{1}{4}$ from v either by R3, which happens when v_1 has three 5-neighbors, or possibly by R8, otherwise. So again $\mu'(v) \geq 0$.

Thus we are done unless there are vertices $w_1, \dots, w_{11}$ in 3-faces $w_k v_k v_{k+1}$ whenever $1 \le k \le 11$ (addition mod 11 throughout proving Case 4). If so, then we cannot have $d(w_k) \le 8 \ge d(w_{k+1})$ for any k, for otherwise $w(S_5(v_{k+1})) \le 3 \times 5 + 2 \times 8 + 11 = 42$, which is impossible. By the oddness of 11, this implies that, say, $d(w_1) \ge 9$ and $d(w_2) \ge 9$. It follows from Remark 1 that there is a 3-face $w_1 v_2 w_2$, and it suffices to observe that v gives no charge to v_2 by R8 or any other our rule. Hence $\mu'(v) \ge 5 - 10 \times \frac{1}{2} = 0$.

Case 5. $13 \le d(v) \le 14$. We know that v gives at most $\frac{7}{12}$ to each adjacent 5-vertex by R1–R8. Since $\mu(v) = d(v) - 6 - \frac{7d(v)}{12} = \frac{5d(v)-72}{12}$, it follows that $\mu'(v) \ge -\frac{2}{12}$ for $d(v) = 14$, and $\mu'(v) \ge -\frac{7}{12}$ for $d(v) = 13$. Therefore, we use some additional reasons to improve these rough estimations in order to prove $\mu'(v) \ge 0$.

First of all, we can assume that v is completely surrounded by 5-vertices, for otherwise $\mu'(v) \ge d(v) - 6 - \frac{7(d(v)-1)}{12} = \frac{5(d(v)-13)}{12} \ge 0$, as desired.

Secondly, if v is not simplicial then v gives at most $\frac{1}{4}$ to each of at least two vertices incident with a common 4^+-face with v due to the argument used in Case 4, which means that in fact

$$\mu'(v) \ge d(v) - 6 - 2 \times \frac{1}{4} - \frac{7(d(v)-2)}{12} \ge \frac{5(d(v)-13)}{12} + \frac{1}{12} > 0.$$

Thus we are done unless v is simplicial and completely surrounded by 5-vertices. Furthermore, if there is a 4^+-face $\dots v_1' v_1 v_2 v_2'$, then we similarly have $\mu'(v) \ge \frac{1}{12}$.

So again there is a cyclic sequence $W_{d(v)} = w_1, \dots, w_{d(v)}$ such that there are 3-faces $w_k v_k v_{k+1}$ whenever $1 \le k \le d(v)$ (addition mod $d(v)$). There are no two consecutive 5-vertices in $W_{d(v)}$ since each v_k must have an 8^+-neighbor other than v.

If there is an 8-vertex in $W_{d(v)}$, say w_2, then $d(w_1) \ge 8$ and $d(w_3) \ge 8$, since $43 - 3 \times 5 - 13 - 8 = 7$. Thus, in fact each of v_2 and v_3 receives at most $\frac{1}{4}$ from v by R3 and R8 rather than $\frac{7}{12}$, and we again have $\mu'(v) \ge d(v) - 6 - 2 \times \frac{1}{4} - \frac{7(d(v)-2)}{12} > 0$, as above. In what follows, we can assume that $d(w_i) \ge 9$ or $d(w_i) = 5$ whenever $1 \le k \le d(v)$.

If there are two consecutive 9^+-vertices in $W_{d(v)}$, say w_1 and w_2, then v_2 receives no charge from v by R1–R8, so we can improve our rough estimation $\mu'(v) \ge -\frac{7}{12}$ to $\mu'(v) \ge -\frac{7}{12} + \frac{7}{12} \ge 0$, as desired. This complete the proof for $d(v) = 13$ due to the oddness of 13.

So suppose that $d(v) = 14$, all neighbors of v are simplicial, and $d(w_1) = d(w_3) = \cdots = d(w_{13}) = 5$, for otherwise v gives at most $\frac{1}{4}$ to one of its neighbors, and we already have $\mu'(v) \ge -\frac{2}{12} + \frac{7}{12} - \frac{1}{4} > 0$.

Now if at least one of 5-vertices in W_{14}, say w_1, is strong, i.e. w_1 has an 8^+-neighbor outside W_{14}; then each of v_1 and v_2 receives $\frac{1}{2}$ by R6 rather than $\frac{7}{12}$ by R6(ex1) or R6(ex2), which yields $\mu'(v) \ge 8 - 2 \times \frac{1}{2} - 12 \times \frac{7}{12} = 0$.

Thus we can assume that all $w_1, w_3, \dots, w_{13}$ are nonstrong, that is each of them has a 5-neighbor outside W_{14}. Among the seven 9^+-vertices $w_2, w_4, \dots, w_{14}$, there are no two consecutive (cyclically) 10^--vertices, for otherwise we would have a minor 5-star with weight at most 40, which is impossible.

By parity reasons and symmetry, we can assume that $d(w_{14}) \ge 11$ and $d(w_2) \ge 11$. So each of v_1 and v_2 obeys the general rule R6 rather than its exceptions R6(ex1) or R6(ex2). This means that again $\mu'(v) \ge 14 - 6 - 2 \times \frac{1}{2} - 12 \times \frac{7}{12} = 0$, as desired.

Case 6. $d(v) \ge 15$. We know that v gives at most $\frac{5}{8}$ to each adjacent 5-vertex by R1–R8, except for giving $\frac{3}{4}$ in R2(a).

We now average these donations so that each 8^+-neighbor will receive at most $2 \times \frac{1}{8}$ from v, while each 5-neighbor will receive at most $\frac{5}{8}$. To this end, it suffices to switch $\frac{1}{8}$ from the donation of $\frac{3}{4}$ to a 5-vertex v_2 by R2(a) to the neighbor 8^+-vertex v_1.

Since $\mu(v) = d(v) - 6 - \frac{5d(v)}{8} = \frac{3(d(v)-16)}{8}$, it follows that the above averaging results in $\mu'(v) \ge 0$ for $d(v) \ge 16$.

Finally, suppose that $d(v) = 15$. If v has an 8^+-neighbor or a nonsimplicial 5-neighbor, then $\mu'(v) \ge 15 - 6 - \frac{1}{4} - 14 \times \frac{5}{8} = 0$ by R1–R8.

Thus we can assume that v is completely surrounded by simplicial 5-vertices, which means that the sequence W_{15} in Case 5 is actually a 15-cycle. Again, W_{15} has no two consecutive 5-vertices, which implies by parity reasons and symmetry that $d(w_1) \geq 8$ and $d(w_2) \geq 8$. Since v_2 receives $\frac{1}{4}$ from v by R8 and nothing by any other our rule, we are done.

Thus we have proved $\mu'(x) \geq 0$ whenever $x \in V \cup F$, which contradicts (1) and completes the proof of Theorem 1.

References

1. Wernicke P., "Über den kartographischen Vierfarbensatz," Math. Ann., vol. 58, 413–426 (1904).
2. Franklin Ph., "The four colour problem," Amer. J. Math., vol. 44, 225–236 (1922).
3. Lebesgue H., "Quelques conséquences simples de la formule d'Euler," J. Math. Pures Appl., vol. 19, 27–43 (1940).
4. Jendrol' S. and Madaras T., "On light subgraphs in plane graphs of minimal degree five," Discuss. Math. Graph Theory, vol. 16, no. 2, 207–217 (1996).
5. Borodin O. V. and Woodall D. R., "Short cycles of low weight in normal plane maps with minimum degree 5," Discuss. Math. Graph Theory, vol. 8, no. 2, 159–164 (1998).
6. Borodin O. V. and Ivanova A. O., "Describing 4-stars at 5-vertices in normal plane maps with minimum degree 5," Discrete Math., vol. 313, no. 17, 1710–1714 (2013).
7. Ivanova A. O. and Nikiforov D. V., "The structure of neighborhoods of 5-vertices in plane triangulation with minimum degree 5" (Russian), Mat. Zametki YaSU, vol. 20, no. 2, 66–78 (2013).
8. Ivanova A. O. and Nikiforov D. V., "Combinatorial structure of triangulated 3-polytopes with minimum degree 5," in: *Proceedings of the Scientific Conference of Students, Graduate Students, and Young Researchers.* XVII and XVIII Lavrentiev's Reading, Yakutsk, Kirov, International Center for Research Projects, 2015, 2015, 22–27.
9. Nikiforov D. V., "The structure of neighborhoods of 5-vertices in normal plane maps with minimum degree 5" (Russian), Mat. Zametki YaSU, vol. 23, no. 1, 56–66 (2016).
10. Borodin O. V., Ivanova A. O., and Jensen T. R., "5-Stars of low weight in normal plane maps with minimum degree 5," Discuss. Math. Graph Theory, vol. 34, no. 3, 539–546 (2014).
11. Borodin O. V. and Ivanova A. O., "Light and low 5-stars in normal plane maps with minimum degree 5," Sib. Mat. Zh., vol. 57, no. 3, 470–475 (2016).
12. Borodin O. V., Ivanova A. O., and Vasil'eva E. I., "Light minor 5-stars in 3-polytopes with minimum degree 5 and no 6-vertices," Discuss. Math. Graph Theory (to be published).
13. Borodin O. V. and Ivanova A. O., "Light neighborhoods of 5-vertices in 3-polytopes with minimum degree 5," Sib. Èlektron. Mat. Izv., vol. 13, 584–591 (2016).
14. Borodin O. V., Ivanova A. O., and Nikiforov D. V., "Low and light 5-stars in 3-polytopes with minimum degree 5 and restrictions on the degrees of major vertices," Sib. Math. J., vol. 58, no. 4, 600–605 (2017).
15. Borodin O. V., Ivanova A. O., and Kazak O. N., "Describing neighborhoods of 5-vertices in 3-polytopes with minimum degree 5 and without vertices of degrees from 7 to 11," Discuss. Math. Graph Theory, vol. 38, no. 3, 615–625 (2018).
16. Borodin O. V., Ivanova A. O., Kazak O. N., and Vasileva E. I., "Heights of minor 5-stars in 3-polytopes with minimum degree 5 and no vertices of degree 6 and 7," Discrete Math., vol. 341, no. 3, 825–829 (2018).
17. Borodin O. V. and Ivanova A. O., "Low 5-stars in normal plane maps with minimum degree 5," Discrete Math., vol. 340, no. 2, 18–22 (2017).
18. Borodin O. V. and Ivanova A. O., "On light neighborhoods of 5-vertices in 3-polytopes with minimum degree 5," Discrete Math., vol. 340, no. 9, 2234–2242 (2017).
19. Borodin O. V., Ivanova A. O., and Nikiforov D. V., "Low minor 5-stars in 3-polytopes with minimum degree 5 and no 6-vertices," Discrete Math., vol. 340, no. 7, 1612–1616 (2017).
20. Ferencová B. and Madaras T., "Light graph in families of polyhedral graphs with prescribed minimum degree, face size, edge and dual edge weight," Discrete Math., vol. 310, no. 12, 1661–1675 (2010).
21. Harant J. and Jendrol' S., "On the existence of specific stars in planar graphs," Graphs Comb., vol. 23, no. 5, 529–543 (2007).
22. Jendrol' S., "A structural property of convex 3-polytopes," Geom. Dedicata, vol. 68, 91–99 (1997).
23. Jendrol' S. and Madaras T., "Note on an existence of small degree vertices with at most one big degree neighbour in planar graphs," Tatra Mt. Math. Publ., vol. 30, 149–153 (2005).
24. Borodin O. V. and Ivanova A. O., "New results about the structure of plane graphs: a survey," in: *AIP Conf. Proc.*, 2017, vol. 1907, no. 1. 030051. doi.org/10.1063/1.5012673
25. Jendrol' S. and Voss H.-J., "Light subgraphs of graphs embedded in the plane—a survey," Discrete Math., vol. 313, no. 4, 406–421 (2013).

O. V. Borodin; A. O. Ivanova
Sobolev Institute of Mathematics, Novosibirsk, Russia
E-mail address: `brdnoleg@math.nsc.ru`; `shmgnanna@mail.ru`