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ON SHEMETKOV'S THEOREM ABOUT THE COMPLEMENTEDNESS OF THE RESIDUAL

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Abstract: A formation $\mathfrak{F}$ of finite groups is called a GWP-formation if the $\mathfrak{F}$-residual of the group generated by two $\mathfrak{F}$-subnormal subgroups is the subgroup generated by their $\mathfrak{F}$-residuals. The main aim of the article is to find some sufficient conditions for a finite group to split over its $\mathfrak{F}$-residual.

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1. Introduction

The notion of $\mathfrak{F}$-residual, which characterizes the degree of the membership of a group in a formation $\mathfrak{F}$, naturally initiated the study of the problem of the splittability of a group over the $\mathfrak{F}$-residual. A central place in solving this problem is occupied by the following result by Shemetkov in [1, 2]:

Let $\mathfrak{F}$ be a local formation and let G be a finite group. If, for every prime p dividing $|G : G^{\mathfrak{F}}|$, a Sylow p-subgroup of $G^{\mathfrak{F}}$ is abelian then $G^{\mathfrak{F}}$ is complemented in G.

The universality of this theorem is manifested as follows:

(1) no constraints are imposed on the finite group G (apart from the abelianity of the corresponding Sylow subgroups in the $\mathfrak{F}$-residual);

(2) the theorem holds for an arbitrary local formation $\mathfrak{F}$;

(3) the theorem includes practically all available results on the complementedness of $\mathfrak{F}$-residuals (including the Schur–Zassenhaus Theorem on the complementedness of a normal Hall subgroup; Gaschütz's Theorem on the complementedness of an abelian $\mathfrak{F}$-residual (see [3]); P. Hall's Theorem on the complementedness of the commutant of a soluble group with abelian Sylow subgroups; Huppert's Theorem (see [4]) on the complementedness of an $\mathfrak{N}_p$-residual having an abelian Sylow p-subgroup).

As examples show, the condition of the abelianity of the corresponding Sylow subgroups of the $\mathfrak{F}$-residual in the above Shemetkov Theorem is substantial. Therefore, one of the approaches aimed to weaken abelianity can be given by introducing additional constraints either on the group G or the formation $\mathfrak{F}$. This approach, initiated by [5], was recently developed in [6–9]. This article also realizes the approach. Here, for an arbitrary GWP-formation $\mathfrak{F}$, we study the existence of complements to the $\mathfrak{F}$-residual of a group generated by a system of its K-$\mathfrak{F}$-subnormal subgroups. Thus the condition of the abelianity of the corresponding Sylow subgroups of the $\mathfrak{F}$-residual of the whole group G in the Shemetkov Theorem is relaxed to the condition of the abelianity of the Sylow subgroups of the $\mathfrak{F}$-residuals of the K-$\mathfrak{F}$-subnormal subgroups generating G. The main goal of the article is to prove the two theorems:

Theorem 1.1. *Let $\mathfrak{F}$ be a GWP-formation and let G be a finite group with the properties:*

(1) *$G = \langle A, B\rangle$, where A and B are K-$\mathfrak{F}$-subnormal subgroups in G;*

(2) *for some prime p, the Sylow p-subgroups of $A^{\mathfrak{F}}$ and $B^{\mathfrak{F}}$ are abelian.*

Then the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G does not include G-chief $\mathfrak{F}$-central pd-factors.

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Theorem 1.2. *Let $\mathfrak{F}$ be a GWP-formation and let G be a finite group representable as $G = \langle A, B\rangle$, where A and B are K-$\mathfrak{F}$-subnormal subgroups in G. If the subgroups $A^{\mathfrak{F}}$ and $B^{\mathfrak{F}}$ are $\pi(\mathfrak{F})$-soluble and, for each prime $p \in \pi(\mathfrak{F})$, the Sylow p-subgroups of $A^{\mathfrak{F}}$ and $B^{\mathfrak{F}}$ are abelian then each $\mathfrak{F}$-normalizer of G is a complement of the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ in G.*

2. Definitions and the Results Used

We consider only finite groups and use the definitions and notations as in [10, 11].

Recall that a *formation* is a group class closed under homomorphic images and finite subdirect products. If $\mathfrak{F}$ is a nonempty formation then denote by $G^{\mathfrak{F}}$ the intersection of all those normal subgroups N of G for which $G/N \in \mathfrak{F}$ (the subgroup $G^{\mathfrak{F}}$ is called the $\mathfrak{F}$-*residual* of G).

Lemma 2.1 [10, Lemma 1.2]. *Let $\mathfrak{F}$ be a nonempty formation and let N be a normal subgroup in G. Then*

(1) $(G/N)^{\mathfrak{F}} = G^{\mathfrak{F}}N/N$;

(2) *if $G = HN$ then $H^{\mathfrak{F}}N = G^{\mathfrak{F}}N$.*

A subgroup H of a group G is called $\mathfrak{F}$-*subnormal*, where $\mathfrak{F}$ is a nonempty group class, if either $H = G$ or there exists a maximal chain of subgroups

$$H = H_0 \subset H_1 \subset \dots \subset H_n = G$$

such that $H_i/\operatorname{Core}_{H_i}(H_{i-1}) \in \mathfrak{F}$ for all $i = 1, 2, \dots, n$.

Another concept, developing the idea of the transitive closure of normality and combining subnormal and $\mathfrak{F}$-subnormal subgroups, is based on the following definition:

If $\mathfrak{F}$ is a nonempty group class then a subgroup H of G is called *Kegel* $\mathfrak{F}$-*subnormal* (or simply K-$\mathfrak{F}$-*subnormal*) if there exists a chain of subgroups

$$H = H_0 \subseteq H_1 \subseteq \dots \subseteq H_n = G$$

such that either H_{i-1} is normal in H_i or $H_i/\operatorname{Core}_{H_i}(H_{i-1}) \in \mathfrak{F}$ for all $i = 1, 2, \dots, n$.

Note that if $\mathfrak{F} = \mathfrak{N}$ is the class of all nilpotent groups then H is K-$\mathfrak{F}$-subnormal in G if and only if H is subnormal in G.

The following three lemmas give information on the general properties of K-$\mathfrak{F}$-subnormal subgroups. Proofs of these lemmas can be found in [12, 13].

Lemma 2.2. *Let $\mathfrak{F}$ be a nonempty formation. Suppose that H and N are subgroups of a group G and N is normal in G. Then*

(1) *if H is K-$\mathfrak{F}$-subnormal in G then HN/N is K-$\mathfrak{F}$-subnormal in G/N and HN is K-$\mathfrak{F}$-subnormal in G*;

(2) *if $N \subseteq H$ then H is K-$\mathfrak{F}$-subnormal in G if and only if H/N is K-$\mathfrak{F}$-subnormal in G/N.*

Lemma 2.3. *Let $\mathfrak{F}$ be a nonempty hereditary formation. Then*

(1) *if the $\mathfrak{F}$-residual of G lies in H then H is a K-$\mathfrak{F}$-subnormal subgroup in G*;

(2) *if H and D are subgroups of G and the subgroup H is K-$\mathfrak{F}$-subnormal in G then the subgroup $H \cap D$ is K-$\mathfrak{F}$-subnormal in D*;

(3) *if a subgroup H is K-$\mathfrak{F}$-subnormal in D and a subgroup D is K-$\mathfrak{F}$-subnormal in G then H is K-$\mathfrak{F}$-subnormal in G.*

Lemma 2.4. *Let $\mathfrak{F}$ be a nonempty hereditary formation. If H is K-$\mathfrak{F}$-subnormal in G then $H^{\mathfrak{F}}$ is subnormal in G.*

A formation $\mathfrak{F}$ is said to *induce the Wielandt functor on $\mathfrak{F}$-subnormal subgroups* [12] if $\langle A, B\rangle^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$ for every two $\mathfrak{F}$-subnormal subgroups A and B of each subgroup G. In [13], such a formation is called a *formation with generalized Wielandt property for residuals*, or, briefly, a *GWP-formation.*

Every GWP-formation $\mathfrak{F}$ is a hereditary Fitting formation [12]; i.e., $\mathfrak{F}$ is closed under subgroups and the equality $G = AB$, where A and B are normal $\mathfrak{F}$-subgroups in G, always implies that $G \in \mathfrak{F}$. Moreover, as follows from [14], any GWP-formation $\mathfrak{F}$ is saturated; i.e., closed under Frattini extensions. Hence, in view of [13, Theorem 6.5.45], $\mathfrak{F}$ is a GWP-formation if and only if $\langle A, B\rangle^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}}\rangle$ for every two K-$\mathfrak{F}$-subnormal subgroups A and B in every group G.

The definition of Fitting class $\mathfrak{F}$ implies that each group G includes the $\mathfrak{F}$*-radical* $G_{\mathfrak{F}}$, i.e. the greatest normal subgroup in G belonging to $\mathfrak{F}$ (it coincides with the product of all normal $\mathfrak{F}$-subgroups in G). In what follows, we will rely upon the following result which establishes relationship between K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups of a group G with the $\mathfrak{F}$-radical of G.

Lemma 2.5 [12, Lemma 3.3.5]. *Let $\mathfrak{F}$ be a GWP-formation. Then each K-$\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroup of G lies in the $\mathfrak{F}$-radical $G_{\mathfrak{F}}$ of G.*

All soluble GWP-formations are described in [15]. Some sufficiently wide classes of nonsoluble GWP-formations are presented in [15, 16]. The problem of counting all GWP-formations remains open.

At the same time, a GWP-formation is always a *lattice formation* [15], i.e., it possesses the property that the set of all $\mathfrak{F}$-subnormal subgroups in every group constitutes a sublattice in the lattice of all subgroups of this group. Thus, all GWP-formations lie in the class of all hereditary saturated formations, which are described in [17].

Lemma 2.6 [17]. *Let $\mathfrak{F}$ be a hereditary saturated formation. Then $\mathfrak{F}$ is a lattice formation if and only if $\mathfrak{F}$ satisfies the conditions:*

(1) $\mathfrak{F} = \mathfrak{M} \times \mathfrak{H}$, $\pi(\mathfrak{M}) \cap \pi(\mathfrak{H}) = \varnothing$;

(2) *there exists a partition $\{\pi_i \mid i \in I\}$ of $\pi(\mathfrak{H})$ into pairwise disjoint sets such that $\mathfrak{H} = \times_{i\in I}\mathfrak{S}_{\pi_i}$;*

(3) $\mathfrak{M} = \mathfrak{S}_{\pi(\mathfrak{M})}\mathfrak{M}$ *is a hereditary saturated formation that is a Fitting class normal in $\mathfrak{M}^2$;*

(4) *each noncyclic $\mathfrak{M}$-critic group G with trivial Frattini subgroup is a primitive group with a single nonabelian normal subgroup $N = G^{\mathfrak{M}}$, where G/N is a cyclic primary group.*

Recall that if $\{\mathfrak{X}_i \mid i \in I\}$ is a family of group classes then $\times_{i\in I}\mathfrak{X}_i$ stands for the class of all groups representable as $H_{i_1} \times \cdots \times H_{i_t}$, where $i_k \in I$, $H_{i_k} \in \mathfrak{X}_{i_k}$ for all $k = 1, 2, \dots, t$. Below $\mathfrak{S}$ stands for the formation of all soluble groups (respectively, $\mathfrak{S}_\pi$ is the formation of all soluble π-groups, where π is a set of primes).

Let P be the set of all primes. Then the function

$$f : \mathrm{P} \to \{\text{formations of finite groups}\}$$

is called a *formation function*.

For a formation function f, a chief factor A/B of G is called f*-central* (f*-excentral*) if

$$G/C_G(A/B) \cong \mathrm{Aut}_G(A/B) \in f(p)$$

for all primes $p \in \pi(A/B)$ (respectively, $G/C_G(A/B)$ does not belong to $f(p)$ at least for one prime $p \in \pi(A/B)$). A group class $\mathfrak{F} = LF(f)$ is called a *local formation* if $\mathfrak{F}$ consists of all G such that either $G = 1$ or $G \neq 1$ and every chief factor A/B of G is f-central. In this event, the local formation $\mathfrak{F}$ is said to be *defined with the use of the formation function* f and f is called the *local definition* of $\mathfrak{F}$.

Suppose that $\mathfrak{N}_p$ is the class of all p-groups, f is a formation function, and $\mathfrak{F} = LF(f)$. Then the function f is called

(a) *inner* if $f(p) \subseteq \mathfrak{F}$ for all $p \in \mathrm{P}$;

(b) *full* if $f(p) = \mathfrak{N}_p f(p)$ for all $p \in \mathrm{P}$;

(c) *canonical* if f is full and inner.

As was shown in [11, Theorem IV.3.7], for every local formation $\mathfrak{F}$ there exists a unique canonical formation function f such that $\mathfrak{F} = LF(f)$. This function is called the *canonical local definition* of $\mathfrak{F}$.

Following Definition 5.5 in [10], refer to a chief factor H/K as $\mathfrak{F}$*-central* ($\mathfrak{F}$*-excentral*) if H/K is f-central (respectively, f-excentral) for some inner local definition f of $\mathfrak{F}$.

Below we will need the following information about the chief factors of the $\mathfrak{F}$-residual. Here by a chief *pd-factor* of a group we mean a chief factor whose order divides by a prime p.

Lemma 2.7 [10, Theorem 11.6]. *Let $\mathfrak{F}$ be a local formation. If, for some prime p, a Sylow p-subgroup in $G^{\mathfrak{F}}$ is abelian then the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G has no G-chief $\mathfrak{F}$-central pd-factors.*

Lemma 2.8 [9, Lemma 1.8]. *If $G = \langle A, B\rangle$, where A and B are subnormal subgroups in a group G, and, for some prime p, the subgroup $A^{\mathfrak{N}}$ has no A-chief central pd-factors, and the subgroup $B^{\mathfrak{N}}$ has no B-chief central pd-factors then the $\mathfrak{N}$-residual $G^{\mathfrak{N}}$ of G has no G-chief central pd-factors.*

Following [10], give the definition of $\mathfrak{F}$-normalizer of an arbitrary finite group for the case when $\mathfrak{F}$ is a local formation.

A normal subgroup R in a group G is called an $\mathfrak{F}$-*limit normal subgroup* if $R/R \cap \Phi(G)$ is an $\mathfrak{F}$-excentral chief factor of G. A maximal subgroup M in G is called $\mathfrak{F}$-*critic* in G if G has an $\mathfrak{F}$-limit normal subgroup R with $MR = G$. A subgroup H is called an $\mathfrak{F}$-*normalizer* of G if $H \in \mathfrak{F}$ and there exists a maximal chain

$$H = H_0 \subset H_1 \subset \cdots \subset H_n = G,$$

in which the subgroup H_{i-1} is $\mathfrak{F}$-critical in H_i for all $i = 1, 2, \dots, n$. By definition, every group G possesses at least one $\mathfrak{F}$-normalizer.

For proving Theorem 1.2, we will also need the following information on the properties of $\mathfrak{F}$-normalizers:

Let H be a subgroup and let A/B be a normal factor of a group G. The subgroup H is said to

(1) *cover* A/B if $A \subseteq HB$;

(2) *isolate* A/B if $H \cap A \subseteq B$.

Lemma 2.9 [10, Corollary 21.1.1]. *Let $\mathfrak{F}$ be a local formation and let G be a group with $\pi(\mathfrak{F})$-soluble $\mathfrak{F}$-residual. If F is an $\mathfrak{F}$-normalizer of G then F covers each $\mathfrak{F}$-central and isolates each $\mathfrak{F}$-excentral chief factor of G.*

3. Proof of Theorem 1.1

If p does not belong to $\pi(\mathfrak{F})$ then the theorem is obvious. Therefore, we will assume that $p \in \pi(\mathfrak{F})$. Suppose that the theorem fails. Let G be a group that satisfies the hypothesis of the theorem but fails to satisfy its claim and is such that the number $t = |G| + |G : A| + |G : B|$ is minimal for a group G with such properties. Clearly, $G^{\mathfrak{F}} \neq 1$.

If $G = A$ then, by hypothesis, the Sylow p-subgroups in $G^{\mathfrak{F}} = A^{\mathfrak{F}}$ are abelian, and so, by Lemma 2.7, the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G has no G-chief $\mathfrak{F}$-central *pd*-factors; a contradiction. Therefore, we further assume that $G \neq A$ and $G \neq B$.

Let N be a minimal normal subgroup in G. If N does not lie in $G^{\mathfrak{F}}$, then, by Lemma 2.1, $(G/N)^{\mathfrak{F}} = G^{\mathfrak{F}}N/N \simeq G^{\mathfrak{F}}$. Hence, basing on the choice of G, we imply that the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G has no G-chief $\mathfrak{F}$-central *pd*-factors; a contradiction. Hence, every minimal normal subgroup of G lies in the subgroup $G^{\mathfrak{F}}$. Moreover, by the choice of G, the $\mathfrak{F}$-residual $G^{\mathfrak{F}}/N$ of G/N has no G-chief $\mathfrak{F}$-central *pd*-factors.

If L is the only minimal normal subgroup in G different from N then it is shown similarly that the $\mathfrak{F}$-residual $G^{\mathfrak{F}}/L$ of G/L has no G-principal $\mathfrak{F}$-central *pd*-factors. But then, in view of the isomorphism $G^{\mathfrak{F}}/N \cap L \simeq G^{\mathfrak{F}}$, it follows that the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G has no G-chief $\mathfrak{F}$-central *pd*-factors. We get a contradiction to the choice of G.

Thus, N is the only minimal normal subgroup in G and N lies in $G^{\mathfrak{F}}$. Moreover, each chief *pd*-factor of G from N to $G^{\mathfrak{F}}$ is $\mathfrak{F}$-excentral. Since the theorem fails for G, the minimal normal subgroup N of G is a *pd*-subgroup which is $\mathfrak{F}$-central in G. If f is the canonical local definition of $\mathfrak{F}$ then $G/C_G(N) \in f(p) \subseteq \mathfrak{F}$. Hence, $G^{\mathfrak{F}} \subseteq C_G(N)$, and so $N \subseteq Z(G^{\mathfrak{F}})$. This in particular implies that N is an abelian p-group.

Suppose that $A^{\mathfrak{F}} = 1$. Since $\mathfrak{F}$ is a GWP-formation, by the K-$\mathfrak{F}$-subnormality of A in G and the fact that $A \in \mathfrak{F}$, by Lemma 2.5, we have $A \subseteq G_{\mathfrak{F}}$ and $G = AB = G_{\mathfrak{F}}B$. Therefore, $G^{\mathfrak{F}} = (G_{\mathfrak{F}})^{\mathfrak{F}}B^{\mathfrak{F}} = B^{\mathfrak{F}}$; and so, by Lemma 2, the subgroup $G^{\mathfrak{F}}$ has no G-chief $\mathfrak{F}$-central *pd*-factors; a contradiction. Therefore, we will assume that $A^{\mathfrak{F}} \neq 1$ and $B^{\mathfrak{F}} \neq 1$.

Put $D = A^{\mathfrak{F}} \cap N$. Suppose that $D \neq 1$. By [10, Theorem 4.7], $f(p)$ is a hereditary formation. Therefore, $G/C_G(N) \in f(p)$ implies that $AC_G(N)/C_G(N) \in f(p)$. By the isomorphism $AC_G(N)/C_G(N) \simeq A/A \cap C_G(N) = A/C_A(N)$, we infer that $A/C_A(N) \in f(p)$. Since $C_A(N) \subseteq C_A(D)$, $A/C_A(D) \in f(p)$. Consequently, all A-chief factors of the subgroup $A^{\mathfrak{F}}$ on the interval $[1, D]$ are $\mathfrak{F}$-central in A. We get a contradiction to Lemma 2.7. Thus, $A^{\mathfrak{F}} \cap N = 1$. It is shown analogously that $B^{\mathfrak{F}} \cap N = 1$.

Since $\mathfrak{F}$ is a GWP-formation, $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$. By Lemma 2.4, $A^{\mathfrak{F}}$ and $B^{\mathfrak{F}}$ are subnormal subgroups in G. Moreover, the Sylow p-subgroups in $(A^{\mathfrak{F}})^{\mathfrak{N}}$ and $(B^{\mathfrak{F}})^{\mathfrak{N}}$ are abelian, and so, by Lemma 2.7, the subgroup $(A^{\mathfrak{F}})^{\mathfrak{N}}$ has no $A^{\mathfrak{F}}$-chief central pd-factors, and the subgroup $(B^{\mathfrak{F}})^{\mathfrak{N}}$ has no $B^{\mathfrak{F}}$-chief central pd-factors. Therefore, by Lemma 2.8. the $\mathfrak{N}$-residual $(G^{\mathfrak{F}})^{\mathfrak{N}}$ of $G^{\mathfrak{F}}$ has no $G^{\mathfrak{F}}$-chief central pd-factors. Consequently, $(G^{\mathfrak{F}})^{\mathfrak{N}} \cap N = 1$. If $(G^{\mathfrak{F}})^{\mathfrak{N}} \neq 1$ then $G^{\mathfrak{F}}$ includes a minimal normal subgroup different from N; a contradiction. Therefore, $(G^{\mathfrak{F}})^{\mathfrak{N}} = 1$, i.e., $G^{\mathfrak{F}}$ is a nilpotent subgroup in G. Since N is the only minimal normal subgroup in G; in particular, $G^{\mathfrak{F}}$ is a p-group.

Since $\mathfrak{F}$ is a lattice formation, $\mathfrak{F}$ satisfies conditions 1–4 of Lemma 2.6. Therefore, $p \in \pi(\mathfrak{F})$ implies that either $p \in \pi(\mathfrak{M})$ or $p \in \pi_i$ for some $i \in I$.

If $p \in \pi(\mathfrak{M})$ then the construction of $\mathfrak{F}$ implies that $G/G^{\mathfrak{F}} = (S/G^{\mathfrak{F}}) \times (F/G^{\mathfrak{F}})$, where $S/G^{\mathfrak{F}}$ is a Hall $\pi(\mathfrak{M})$-subgroup of $G/G^{\mathfrak{F}}$ belonging to $\mathfrak{M}$ and $F/G^{\mathfrak{F}}$ is a Hall $\pi(\mathfrak{H})$-subgroup in $G/G^{\mathfrak{F}}$ belonging to $\mathfrak{H}$. Since $\mathfrak{M} = \mathfrak{S}_{\pi(\mathfrak{M})}\mathfrak{M}$ and $G^{\mathfrak{F}}$ is a p-group, where $p \in \pi(\mathfrak{M})$, we have $S \in \mathfrak{M} \subseteq \mathfrak{F}$.

If $p \in \pi_i$ for some $i \in I$ then the construction of $\mathfrak{F}$ implies that $G/G^{\mathfrak{F}} = (S/G^{\mathfrak{F}}) \times (F/G^{\mathfrak{F}})$, where $S/G^{\mathfrak{F}}$ is a soluble Hall π_i-subgroup in $G/G^{\mathfrak{F}}$ and $F/G^{\mathfrak{F}}$ is a Hall π_i'-subgroup in $G/G^{\mathfrak{F}}$ belonging to $\mathfrak{F}$. Since $\mathfrak{S}_{\pi_i}$ is closed under extensions and $G^{\mathfrak{F}}$ is a p-group, where $p \in \pi_i$; therefore, $S \in \mathfrak{S}_{\pi_i} \subseteq \mathfrak{F}$.

Thus, in both cases, $G/G^{\mathfrak{F}}$ is representable as $G/G^{\mathfrak{F}} = (S/G^{\mathfrak{F}}) \times (F/G^{\mathfrak{F}})$, where $S/G^{\mathfrak{F}}$ and $F/G^{\mathfrak{F}}$ are Hall subgroups in $G/G^{\mathfrak{F}}$ belonging to $\mathfrak{F}$. Moreover, $S \in \mathfrak{F}$.

Consider the subgroups $AG^{\mathfrak{F}}$ and $BG^{\mathfrak{F}}$. By Lemma 2.2, they are K-$\mathfrak{F}$-subnormal in G. Moreover, $G = \langle AG^{\mathfrak{F}}, BG^{\mathfrak{F}} \rangle$. Since $\mathfrak{F}$ is a GWP-formation, $(AG^{\mathfrak{F}})^{\mathfrak{F}} = \langle A, G^{\mathfrak{F}} \rangle^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, (G^{\mathfrak{F}})^{\mathfrak{F}} \rangle = A^{\mathfrak{F}}$ and $(BG^{\mathfrak{F}})^{\mathfrak{F}} = B^{\mathfrak{F}}$; i.e., the $\mathfrak{F}$-residuals of the subgroups $AG^{\mathfrak{F}}$ and $BG^{\mathfrak{F}}$ are abelian. If either $A \subset AG^{\mathfrak{F}}$ or $B \subset BG^{\mathfrak{F}}$ then

$$|G| + |G : AG^{\mathfrak{F}}| + |G : BG^{\mathfrak{F}}| < t$$

and, in view of the choice of G and the subgroups A and B of G, we infer that the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G has no G-chief $\mathfrak{F}$-central pd-factors; a contradiction. Consequently, $G^{\mathfrak{F}} \subseteq A$ and $G^{\mathfrak{F}} \subseteq B$. This in particular implies that the subgroups $A^{\mathfrak{F}}$ and $B^{\mathfrak{F}}$ are normal in $G^{\mathfrak{F}}$. Since $\mathfrak{F}$ is a GWP-formation and $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$; therefore, $G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}$.

Owing to the isomorphism

$$G^{\mathfrak{F}}/A^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}/A^{\mathfrak{F}} \simeq B^{\mathfrak{F}}/A^{\mathfrak{F}} \cap B^{\mathfrak{F}}$$

and the abelianity of $B^{\mathfrak{F}}$, we conclude that the commutant $[G^{\mathfrak{F}}, G^{\mathfrak{F}}]$ of $G^{\mathfrak{F}}$ lies in $A^{\mathfrak{F}}$. Obviously, $[G^{\mathfrak{F}}, G^{\mathfrak{F}}]$ is a normal subgroup in G. If $[G^{\mathfrak{F}}, G^{\mathfrak{F}}] \neq 1$ then from the uniqueness of the minimal normal subgroup N in G we obtain that $N \subseteq [G^{\mathfrak{F}}, G^{\mathfrak{F}}] \subseteq A^{\mathfrak{F}}$. We get a contradiction to the fact that $A^{\mathfrak{F}} \cap N = 1$. Consequently, $[G^{\mathfrak{F}}, G^{\mathfrak{F}}] = 1$; i.e., the subgroup $G^{\mathfrak{F}}$ is abelian. But then, by Lemma 2.7, the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G has no G-chief $\mathfrak{F}$-central pd-factors. We again get a contradiction to the choice of G. The theorem is proved.

4. Proof of Theorem 1.2

Obviously, the set $\mathfrak{H}$ of all $\pi(\mathfrak{F})$-soluble groups is a Fitting formation. Since $\mathfrak{F}$ is a GWP-formation, $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$. By Lemma 2.4, $A^{\mathfrak{F}}$ and $B^{\mathfrak{F}}$ are subnormal subgroups in G. Hence, $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle \subseteq G_{\mathfrak{H}}$; i.e., $G^{\mathfrak{F}}$ is $\pi(\mathfrak{F})$-soluble.

Let F be an $\mathfrak{F}$-normalizer of G. By Theorem 1.1, all G-chief factors of $G^{\mathfrak{F}}$ are $\mathfrak{F}$-excentral in G.

Let

$$1 = H_0 \subset H_1 \subset \cdots \subset H_n = G^{\mathfrak{F}} \tag{1}$$

be a G-chief series of $G^{\mathfrak{F}}$. By Lemma 2.9, the $\mathfrak{F}$-normalizer F isolates all factors of series (1); i.e., $F \cap H_i \subseteq H_{i-1}$ for all $i = 1, 2, \ldots, n$. This implies that

$$F \cap G^{\mathfrak{F}} = F \cap H_n \subseteq F \cap H_{n-1} \subseteq \cdots \subseteq F \cap H_0 = 1,$$

i.e., $F \cap G^{\mathfrak{F}} = 1$. Moreover, the definition of $\mathfrak{F}$-normalizer implies that $FG^{\mathfrak{F}} = G$. Hence, the $\mathfrak{F}$-normalizer F is a complement to $G^{\mathfrak{F}}$ in G. The theorem is proved.

5. Corollaries

By induction on the number of K-$\mathfrak{F}$-subnormal subgroups generating G, we have

Corollary 5.1. *Let $\mathfrak{F}$ be a GWP-formation and let G be a group with the properties:*
(1) $G = \langle A_1, A_2, \ldots, A_n \rangle$, where $A_1, A_2, \ldots, A_n$ are K-$\mathfrak{F}$-subnormal subgroups in G;
(2) the Sylow p-subgroups in $A_i^{\mathfrak{F}}$ are abelian for some prime p and all $i = 1, 2, \ldots, n$.
Then the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G has no G-chief $\mathfrak{F}$-central pd-factors.

Corollary 5.2. *Let $\mathfrak{F}$ be a GWP-formation and let G be a group with the properties:*
(1) $G = \langle A_1, A_2, \ldots, A_n \rangle$, where $A_1, A_2, \ldots, A_n$ are K-$\mathfrak{F}$-subnormal subgroups in G;
(2) the Sylow p-subgroups in $A_i^{\mathfrak{F}}$ are abelian for each prime $p \in \pi(\mathfrak{F})$ and all $i = 1, 2, \ldots, n$.
If the subgroup $A_i^{\mathfrak{F}}$ is $\pi(\mathfrak{F})$-soluble for each $i = 1, 2, \ldots, n$ the each $\mathfrak{F}$-normalizer of G is a complement to the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ in G.

Let us give the two particular results as consequences of Theorems 1.1 and 1.2:

Corollary 5.3. *Let $\mathfrak{F}$ be a GWP-formation and let G be a group with the properties:*
(1) $G = \langle A_1, A_2, \ldots, A_n \rangle$, where $A_1, A_2, \ldots, A_n$ are K-$\mathfrak{F}$-subnormal subgroups in G;
(2) the subgroup $A_i^{\mathfrak{F}}$ is abelian for all $i = 1, 2, \ldots, n$.
Then the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G has no G-chief $\mathfrak{F}$-central factors.

Corollary 5.4. *Let $\mathfrak{F}$ be a GWP-formation and let G be a group with the properties:*
(1) $G = \langle A_1, A_2, \ldots, A_n \rangle$, where $A_1, A_2, \ldots, A_n$ are K-$\mathfrak{F}$-subnormal subgroups in G;
(2) the subgroup $A_i^{\mathfrak{F}}$ is abelian for all $i = 1, 2, \ldots, n$.
Then every $\mathfrak{F}$-normalizer of G is a complement to the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ in G.

As was observed above, the subnormal and $\mathfrak{F}$-subnormal subgroups are K-$\mathfrak{F}$-subnormal for every formation $\mathfrak{F}$.

Corollary 5.5. *Let $\mathfrak{F}$ be a GWP-formation and let G be a group with the properties:*
(1) $G = \langle A_1, A_2, \ldots, A_n \rangle$, where $A_1, A_2, \ldots, A_n$ are subnormal subgroups in G;
(2) the Sylow p-subgroups in $A_i^{\mathfrak{F}}$ are abelian for each prime $p \in \pi(\mathfrak{F})$ and all $i = 1, 2, \ldots, n$.
If a subgroup $A_i^{\mathfrak{F}}$ is $\pi(\mathfrak{F})$-soluble for each $i = 1, 2, \ldots, n$ then each $\mathfrak{F}$-normalizer of G is a complement to the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ in G.

Corollary 5.6. *Let $\mathfrak{F}$ be a GWP-formation and let G be a group with the properties:*
(1) $G = \langle A_1, A_2, \ldots, A_n \rangle$, where $A_1, A_2, \ldots, A_n$ are $\mathfrak{F}$-subnormal subgroups in G;
(2) the Sylow p-subgroups in $A_i^{\mathfrak{F}}$ are abelian for each prime $p \in \pi(\mathfrak{F})$ and all $i = 1, 2, \ldots, n$.
If $A_i^{\mathfrak{F}}$ is $\pi(\mathfrak{F})$-soluble for each $i = 1, 2, \ldots, n$ then each $\mathfrak{F}$-normalizer of G is a complement to the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ in G.

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