

Character average of fourth power of Dirichlet L -series at unity

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Abstract. For a Dirichlet character modulo an integer $q \geq 3$, we use a highly simple elementary method to give an asymptotic formula for $\sum_{\chi \neq \chi_0(\bmod q)} |L(1, \chi)|^4$, where $\chi_0(\bmod q)$ is the principal character. This result seems to be new.

Keywords. Hurwitz zeta function; Dirichlet character; Dirichlet L -series.

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Let $s = \sigma + it$ be a complex variable, where σ, t are real. For an integer $q \geq 1$, let $\chi(\bmod q)$ denote a Dirichlet character and let $\chi_0(\bmod q)$ denote the principal character. Let $L(s, \chi)$ be the corresponding Dirichlet L -series so that $L(s, \chi) = \sum_{n \geq 1} \chi(n)n^{-s}$ for $\text{Re } s > 1$; and its analytic continuation. For $0 < \alpha \leq 1$, let $\zeta(s, \alpha)$ be the Hurwitz zeta function defined by $\zeta(s, \alpha) = \sum_{n \geq 0} (n + \alpha)^{-s}$ for $\text{Re } s > 1$; and its analytic continuation. For $0 \leq \alpha \leq 1$, let $\zeta_1(s, \alpha) = \sum_{n \geq 1} (n + \alpha)^{-s}$ for $\text{Re } s > 1$; and its analytic continuation. Note that $\zeta_1(s, \alpha) = \zeta(s, \alpha + 1) = \zeta(s, \alpha) - \alpha^{-s}$ for $\alpha \neq 0$. Let $\zeta(s)$ denote the Riemann zeta function. Thus $\zeta(s, 1) = \zeta_1(s, 0) = \zeta(s)$. Also note that $L(s, \chi) = q^{-s} \sum_{a=1}^q \chi(a) \zeta(s, \frac{a}{q})$.

In what follows, $\phi(n)$ denotes the Euler's totient function, $d(n)$ denotes the divisor function and p denotes a prime number. Also, γ stands for the Euler's constant. For u real, $[u]$ stands for its integral part.

In what follows \sum_a' (or $\sum_{a=1}^q'$) shall mean the summation $\sum_{a=1}^q$ with the restriction $(a, q) = 1$. The summation \sum_b' , \sum_l' and \sum_k' will have similar meanings. In what follows, the O and \ll constants will be absolute, unless stated otherwise. In effect, we shall be using only Euler's summation formula.

Our object is to prove the following theorem.

Theorem. For an integer $q \geq 3$, we have

$$\sum_{\chi \neq \chi_0(\bmod q)} |L(1, \chi)|^4 = \phi(q) \cdot \sum_{\substack{n=1 \\ (n,q)=1}}^q \frac{d^2(n)}{n^2} + O\left(\frac{\phi(q)}{\sqrt{q}} \log^4(q+3)\right),$$

where $\chi_0(\bmod q)$ is the principal character and $d(n)$ is the number of divisors of n .

Remark. Our theorem above for $s = 1$ seems to be the best result in this direction. Our method is highly simple and elementary. A corresponding result for $\sum_{\chi \neq \chi_0(\bmod q)} |L(1, \chi)|^2$ has been proved in [1].

Incidentally, we make following observations: We know

$$\zeta(s+1, \alpha) = \frac{1}{s} + \sum_{n \geq 0} \gamma_n(\alpha) s^n$$

for complex s with $|s| < 1$ and for complex α with $|\alpha| < 1$ and $\alpha \neq 1$. We also know that $\zeta(s, \alpha+1) = \zeta(s) + \sum_{n \geq 1} s(s+1) \dots (s+n-1) \zeta(s+n) \frac{(-\alpha)^n}{n!}$ for complex α with $|\alpha| < 1$. Note that $\zeta(s, \alpha+1) = \zeta(s, \alpha) - \alpha^{-s}$ for $\alpha \neq 0$. Letting $s \rightarrow 1$, this gives for $|\alpha| < 1$,

$$\begin{aligned} \gamma_0(\alpha) &= \lim_{s \rightarrow 1} \left(\zeta(s, \alpha) - \frac{1}{s-1} \right) \\ &= \lim_{s \rightarrow 1} \left\{ \alpha^{-s} + \left(\zeta(s) - \frac{1}{s-1} \right) \right. \\ &\quad \left. + \sum_{n \geq 1} \frac{(-1)^n s(s+1) \dots (s+n-1) \zeta(s+n)}{n!} \alpha^n \right\}. \end{aligned}$$

Thus $\gamma_0(\alpha) = \frac{1}{\alpha} + \gamma + \sum_{n \geq 1} (-1)^n \zeta(n+1) \alpha^n$ for $|\alpha| < 1$.

Next, we state the following results:

Result 1. For real $s \neq 1$ and for $0 \leq \alpha \leq 1$ and for integral $n \geq 1$, we have $\zeta(s, \alpha+1) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} s(s+1) \dots (s+k-1) \zeta(s+k) \alpha^k + \frac{(-1)^n}{n!} s(s+1) \dots (s+n-1) \zeta(s+n, \theta_1 \alpha)$ for some θ_1 with $0 < \theta_1 < 1$.

Result 2. We have for real s, α with $|s| < 1$ and $|\alpha| < 1$,

$$\begin{aligned} \zeta(s, \alpha+1) &= \zeta(0) + (s\zeta'(0) - \alpha) + \left(\frac{s^2}{2} \zeta''(0) - \gamma \alpha s \right) \\ &\quad + \left(\frac{s^3}{6} \zeta'''(0) - \gamma_1 \alpha s^2 + \frac{\zeta(2)}{2} \alpha^2 s \right) \\ &\quad + \dots + \frac{1}{(n-1)!} \left(s \frac{\partial}{\partial s} + \alpha \frac{\partial}{\partial \alpha} \right)^{n-1} \zeta(0, 1) \\ &\quad + \frac{1}{n!} \left(s \frac{\partial}{\partial s} + \alpha \frac{\partial}{\partial \alpha} \right)^n \zeta(\theta_2 s, 1 + \theta_2 \alpha) \end{aligned}$$

for some θ_2 with $0 < \theta_2 < 1$.

Result 1 follows on expanding the real-valued function $\zeta(s, \alpha+1)$ into Taylor series as a function of variable α for a fixed s and on noting that

$$\frac{\partial^k}{\partial \alpha^k} \zeta(s, \alpha) = \frac{(-1)^k}{k!} s(s+1) \dots (s+k-1) \zeta(s+k, \alpha).$$

Result 2 follows on expanding $\zeta(s, \alpha + 1)$ into the Taylor series as a function of two real variables s and α about the point $(0, 1)$ as

$$\zeta(s, \alpha + 1) = \sum_{k=0}^{n-1} \frac{\left(s \frac{\partial}{\partial s} + \alpha \frac{\partial}{\partial \alpha}\right)^k}{k!} \zeta(0, 1) + \frac{1}{n!} \left(s \frac{\partial}{\partial s} + \alpha \frac{\partial}{\partial \alpha}\right)^n \zeta(\theta_2 s, 1 + \theta_2 \alpha)$$

for some $0 < \theta_2 < 1$.

Result 2 involves values of $\zeta^{(r)}(0)$ for integral $r \geq 0$. Actually $\zeta^{(r)}(0)$ can be evaluated in terms of $\gamma_\ell = \gamma_\ell(1)$ for $0 \leq \ell \leq r$ from the functional equation of $\zeta^{(r)}(s)$, namely $\zeta^{(r)}(s) = 2\{(2\pi)^{s-1} \Gamma(1-s) \sin \frac{\pi s}{2} \cdot \zeta(1-s)\}^{(r)}$, by letting $s \rightarrow 0$.

Here the subscript (r) denotes r -th order derivative with respect to s . Next, we prove the above theorem.

Proof of Theorem. For a Dirichlet character $\chi \neq \chi_0(\bmod q)$ and for $\operatorname{Re} s > 1$, we have

$$L^2(s, \chi) = \sum_{n \geq 1} \frac{d(n)\chi(n)}{n^s} = \sum_{\ell}' \chi(\ell) \cdot \sum_{r \equiv \ell(\bmod q)} \frac{d(r)}{r^s}$$

We shall write $Z_2(s, \ell, q) = \sum_{\substack{r \geq 1 \\ r \equiv \ell(\bmod q)}} \frac{d(r)}{r^s}$ for $\operatorname{Re} s > 1$; and its analytic continuation.

Henceforth in what follows, in any congruence relation, for brevity, we write q in place of $\bmod q$.

Let ℓ be an integer with $1 \leq \ell \leq q$ and $(\ell, q) = 1$. Thus for $\sigma > 1$,

$$\begin{aligned} Z_2(s, \ell, q) &= \sum_{\substack{r \geq 1 \\ r \equiv \ell(q)}} \frac{d(r)}{r^s} = \sum_{mn \equiv \ell(q)} \sum (mn)^{-s} \\ &= \sum'_{ab \equiv \ell(q)} \sum' \left(\sum_{m \equiv a(q)} m^{-s} \right) \left(\sum_{n \equiv b(q)} n^{-s} \right) \\ &= \sum'_{ab \equiv \ell(q)} q^{-s} \zeta\left(s, \frac{a}{q}\right) q^{-s} \zeta\left(s, \frac{b}{q}\right). \end{aligned}$$

Thus for $(\ell, q) = 1$, $\chi \neq \chi_0(\bmod q)$ and for $s \neq 1$, we have

$$L^2(s, \chi) = \sum_{\ell}' \chi(\ell) \sum'_{ab \equiv \ell(q)} \sum' q^{-s} \zeta\left(s, \frac{a}{q}\right) q^{-s} \zeta\left(s, \frac{b}{q}\right).$$

□

Next, we need following lemmas:

Lemma 1. We have for real $s > 0$, $s \neq 1$ and for α with $0 < \alpha \leq 1$,

$$\zeta(s, \alpha) = \alpha^{-s} + \frac{1}{s-1} + \phi(s, \alpha),$$

where $\phi(s, \alpha) = O(1)$ with the O -constant absolute.

Proof of Lemma 1. From the expression $\zeta(s, \alpha) = \sum_{n \geq 0} (n + \alpha)^{-s}$ for $\sigma > 1$, on using Euler's summation formula, for arbitrary $x > 0$ and for $\sigma > 1$ and for $0 < \alpha \leq 1$, we have

$$\begin{aligned} \zeta(s, \alpha) &= \sum_{0 \leq n \leq x - \alpha} (n + \alpha)^{-s} + \frac{x^{1-s}}{s-1} + \left(x - \alpha - [x - \alpha] - \frac{1}{2}\right) x^{-s} \\ &\quad - s \int_x^\infty \frac{(u - \alpha - [u - \alpha] - \frac{1}{2})}{u^{s+1}} du. \end{aligned}$$

Note that $x - \alpha > -1$. Here empty sum is treated as zero.

This expression is valid for $\sigma > 0$ and for $s \neq 1$. We choose $x = 1$. This gives for $\text{Re } s > 0$ and $s \neq 1$,

$$\zeta(s, \alpha) = \alpha^{-s} + \frac{1}{s-1} + \left(\frac{1}{2} - \alpha\right) - s \int_1^\infty \frac{(u - \alpha - [u - \alpha] - \frac{1}{2})}{u^{s+1}} du.$$

Incidentally, this gives

$$\begin{aligned} \gamma_0(\alpha) &= \lim_{s \rightarrow 1} \left(\zeta(s, \alpha) - \frac{1}{s-1} \right) = \alpha^{-1} + \left(\frac{1}{2} - \alpha\right) \\ &\quad - \int_1^\infty \frac{(u - \alpha - [u - \alpha] - \frac{1}{2})}{u^2} du. \end{aligned}$$

In what follows, s is real positive unless stated otherwise. For $s > 0$,

$$s \int_1^\infty \frac{u - \alpha - [u - \alpha] - \frac{1}{2}}{u^{s+1}} du < s \int_1^\infty \frac{du}{u^{s+1}} < 1,$$

where the underlying constant is absolute. Thus for $0 < s \neq 1$ and for $0 < \alpha \leq 1$, we have

$$\zeta(s, \alpha) = \alpha^{-s} + \frac{1}{s-1} + \left(\frac{1}{2} - \alpha\right) + 0(1) = \alpha^{-s} + \frac{1}{s-1} + \phi(s, \alpha),$$

where $\phi(s, \alpha) = 0(1)$. □

Lemma 2. We have for $0 < s < 1$ and for $\chi \neq \chi_0 \pmod{q}$,

$$L^2(s, \chi) = \sum'_\ell \chi(\ell) \sum'_{ab \equiv \ell(q)} \sum' q^{-2s} \left(\frac{q^s}{a^s} + \phi\left(s, \frac{a}{q}\right) \right) \left(\frac{q^s}{b^s} + \phi\left(s, \frac{b}{q}\right) \right).$$

Proof of Lemma 2. We have

$$L^2(s, \chi) = \sum'_\ell \chi(\ell) \sum'_{ab \equiv \ell(q)} \sum' q^{-s} \zeta\left(s, \frac{a}{q}\right) \cdot q^{-s} \zeta\left(s, \frac{b}{q}\right).$$

Next,

$$\begin{aligned}\zeta\left(s, \frac{a}{q}\right) \zeta\left(s, \frac{b}{q}\right) &= \left(\frac{q^s}{a^s} + \frac{1}{s-1} + \phi\left(s, \frac{a}{q}\right)\right) \left(\frac{q^s}{b^s} + \frac{1}{s-1} + \phi\left(s, \frac{b}{q}\right)\right) \\ &= \left(\left(\frac{q^s}{a^s} + \phi\left(s, \frac{a}{q}\right)\right) \left(\frac{q^s}{b^s} + \phi\left(s, \frac{b}{q}\right)\right)\right. \\ &\quad \left.+ \frac{1}{s-1} \left(\frac{q^s}{a^s} + \phi\left(s, \frac{a}{q}\right)\right)\right. \\ &\quad \left.+ \frac{1}{s-1} \left(\frac{q^s}{b^s} + \phi\left(s, \frac{b}{q}\right)\right) + \frac{1}{(s-1)^2}\right).\end{aligned}$$

Consider

$$\begin{aligned}\sum'_{ab \equiv \ell(q)} \sum'_{\ell(q)} \frac{1}{s-1} \left(\frac{q^s}{a^s} + \phi\left(s, \frac{a}{q}\right)\right) &= \frac{1}{s-1} \sum'_{ab \equiv \ell(q)} \sum'_{\ell(q)} \frac{q^s}{a^s} \\ &\quad + \frac{1}{s-1} \sum'_{ab \equiv \ell(q)} \sum'_{\ell(q)} \phi\left(s, \frac{a}{q}\right) \\ &= \frac{1}{s-1} \sum'_a \frac{q^s}{a^s} \sum'_{\substack{b \\ ab \equiv \ell(q)}} 1 \\ &\quad + \frac{1}{s-1} \sum'_{a=1}^q \phi\left(s, \frac{a}{q}\right) \sum'_{\substack{b=1 \\ ab \equiv \ell(q)}}^q 1 \\ &= \frac{1}{s-1} \sum'_a \frac{q^s}{a^s} \\ &\quad + \frac{1}{s-1} \sum'_a \phi\left(s, \frac{a}{q}\right),\end{aligned}$$

which is independent of ℓ . Hence for $s \neq 1$ and $\chi \neq \chi_0(\bmod q)$, we have

$$\sum'_\ell \chi(\ell) \left(\sum'_{ab \equiv \ell(q)} \sum'_{\ell(q)} \frac{1}{s-1} \left(\frac{q^s}{a^s} + \phi\left(s, \frac{a}{q}\right)\right) \right) = 0.$$

Similarly,

$$\sum'_\ell \chi(\ell) \left(\sum'_{ab \equiv \ell(q)} \sum'_{\ell(q)} \frac{1}{s-1} \left(\frac{q^s}{b^s} + \phi\left(s, \frac{b}{q}\right)\right) \right) = 0$$

for $s \neq 1$ and for $\chi \neq \chi_0(\bmod q)$. Similarly, we have

$$\sum'_\ell \chi(\ell) \sum'_{ab \equiv \ell(q)} \sum'_{\ell(q)} \frac{1}{(s-1)^2} = 0$$

for $s \neq 1$ and for $\chi \neq \chi_0 \pmod{q}$. Thus for real $s \neq 1$ and for $\chi \neq \chi_0 \pmod{q}$, we have

$$\begin{aligned} L^2(s, \chi) &= \sum_{\ell}' \chi(\ell) \sum_{ab \equiv \ell(q)}' q^{-s} \zeta\left(s, \frac{a}{q}\right) \cdot q^{-s} \zeta\left(s, \frac{b}{q}\right) \\ &= \sum_{\ell}' \chi(\ell) \sum_{ab \equiv \ell(q)}' q^{-2s} \left(\frac{q^s}{a^s} + \phi\left(s, \frac{a}{q}\right) \right) \left(\frac{q^s}{b^s} + \phi\left(s, \frac{b}{q}\right) \right). \end{aligned}$$

□

Lemma 3. *If*

$$A(s, \ell, q) = \sum_{ab \equiv \ell(q)}' q^{-s} \left(\frac{q^s}{a^s} + \phi\left(s, \frac{a}{q}\right) \right) \cdot q^{-s} \left(\frac{q^s}{b^s} + \phi\left(s, \frac{b}{q}\right) \right),$$

then for $0 < s \neq 1$,

$$\sum_{\chi \neq \chi_0 \pmod{q}} |L(s, \chi)|^4 = \phi(q) \sum_{\ell}' A^2(s, \ell, q) - \left(\sum_{\ell}' A(s, \ell, q) \right)^2.$$

Proof of Lemma 3. From Lemma 2, we have $0 < s \neq 1$,

$$L^2(s, \chi) = \sum_{\ell}' \chi(\ell) A(s, \ell, q).$$

This gives

$$\begin{aligned} \sum_{\chi \neq \chi_0 \pmod{q}} |L(s, \chi)|^4 &= \sum_{\ell_1}' \sum_{\ell_2}' A(s, \ell_1, q) A(s, \ell_2, q) \cdot \sum_{\chi \neq \chi_0(q)} \chi(\ell_1) \overline{\chi}(\ell_2) \\ &= \phi(q) \sum_{\ell}' A^2(s, \ell, q) - \left(\sum_{\ell}' A(s, \ell, q) \right)^2 \quad \text{for } s \neq 1. \end{aligned}$$

□

Lemma 4. *We have for $0 < s \neq 1$,*

$$\sum_{\ell}' A(s, \ell, q) = \left(L(s, \chi_0) - \frac{\phi(q)}{q^s(s-1)} \right)^2$$

with

$$\lim_{s \rightarrow 1} \left(L(s, \chi_0) - \frac{\phi(q)}{q^s(s-1)} \right) = \frac{\phi(q)}{q} \left\{ \log q + \sum_{p/q} \frac{\log p}{p-1} + \gamma \right\}$$

so that

$$\lim_{s \rightarrow 1} \left(\sum_{\ell}' A(s, \ell, q) \right)^2 \ll \log^4(q+2) (\log \log(q+10))^4.$$

Proof of Lemma 4. We have

$$\begin{aligned} \sum_{\ell}' A(s, \ell, q) &= \sum_{\ell}' \sum_{ab \equiv \ell(q)}' \sum_{\ell}' q^{-2s} \left(\frac{q^s}{a^s} + \phi \left(s, \frac{a}{q} \right) \right) \left(\frac{q^s}{b^s} + \phi \left(s, \frac{b}{q} \right) \right) \\ &= q^{-2s} \sum_a' \left(\frac{q^s}{a^s} + \phi \left(s, \frac{a}{q} \right) \right) \left(\sum_{\ell}' \sum_{b \equiv \ell(q)}' \left(\frac{q^s}{b^s} + \phi \left(s, \frac{b}{q} \right) \right) \right). \end{aligned}$$

Note that for a fixed a , we have

$$\sum_{\ell}' \sum_{b \equiv \ell(q)}' \left(\frac{q^s}{b^s} + \phi \left(s, \frac{b}{q} \right) \right) = \sum_k' \left(\frac{q^s}{k^s} + \phi \left(s, \frac{k}{q} \right) \right).$$

Thus we have

$$\begin{aligned} \sum_{\ell}' A(s, \ell, q) &= \left(\sum_a' q^{-s} \left(\frac{q^s}{a^s} + \phi \left(s, \frac{a}{q} \right) \right) \right)^2 \\ &= \left(q^{-s} \sum_a' \left(\zeta \left(s, \frac{a}{q} \right) - \frac{1}{s-1} \right) \right)^2 \\ &= \left(L(s, \chi_0) - \frac{\phi(q)}{q^s(s-1)} \right)^2 \end{aligned}$$

for real $s \neq 1$. This gives

$$\left(\sum_{\ell}' A(s, \ell, q) \right)^2 = \left(L(s, \chi_0) - \frac{\phi(q)}{q^s(s-1)} \right)^4.$$

Note that

$$\sum_{\chi \neq \chi_0 \pmod{q}} |L(1, \chi)|^4 = \phi(q) \sum_{\ell}' A^2(1, \ell, q) - \lim_{s \rightarrow 1} \left(\sum_{\ell}' A(s, \ell, q) \right)^2.$$

Next

$$\lim_{s \rightarrow 1} \left(\sum_{\ell}' A(s, \ell, q) \right)^2 = \left(\lim_{s \rightarrow 1} \left(L(s, \chi_0) - \frac{\phi(q)}{q^s(s-1)} \right) \right)^4.$$

Thus we need $\lim_{s \rightarrow 1} \left(L(s, \chi_0) - q^{-s} \cdot \frac{\phi(q)}{s-1} \right)$. Note that

$$\begin{aligned}
\lim_{s \rightarrow 1} \left(L(s, \chi_0) - \frac{\phi(q)}{q^s} \cdot \frac{1}{s-1} \right) &= \lim_{s \rightarrow 1} \left(\zeta(s) \prod_{p/q} (1 - p^{-s}) - \frac{\phi(q)}{q^s(s-1)} \right) \\
&= \lim_{s \rightarrow 1} \left(\zeta(s) - \frac{1}{s-1} \right) \cdot \prod_{p/q} (1 - p^{-s}) \\
&\quad + \lim_{s \rightarrow 1} \frac{1}{s-1} \left(\prod_{p/q} (1 - p^{-s}) - \frac{\phi(q)}{q^s} \right) \\
&= \gamma \prod_{p/q} (1 - p^{-1}) + \lim_{s \rightarrow 1} \frac{1}{s-1} \left(\prod_{p/q} (1 - p^{-s}) - \frac{\phi(q)}{q^s} \right) \\
&= \gamma \frac{\phi(q)}{q} + \lim_{s \rightarrow 1} \frac{1}{s-1} \left(\prod_{p/q} (1 - p^{-s}) - \frac{\phi(q)}{q^s} \right)
\end{aligned}$$

Next, we obtain $\lim_{s \rightarrow 1} \frac{\prod_{p/q} (1 - p^{-s}) - \frac{\phi(q)}{q^s}}{s-1}$. Note that as $s \rightarrow 1$ through real values, this limit is of the form $\frac{\prod_{p/q} (1 - p^{-1}) - \frac{\phi(q)}{q}}{1-1} = \frac{0}{0}$. Thus, using L'Hospital's rule, we have

$$\begin{aligned}
\lim_{s \rightarrow 1} \frac{d}{ds} \left(\prod_{p/q} (1 - p^{-s}) - \frac{\phi(q)}{q^s} \right) &= \lim_{s \rightarrow 1} \left\{ \left(\prod_{p/q} (1 - p^{-s}) \right) \sum_{p/q} \frac{\log p}{p^s - 1} + \frac{\phi(q)}{q^s} \log q \right\} \\
&= \left(\prod_{p/q} \left(1 - \frac{1}{p} \right) \right) \left(\sum_{p/q} \frac{\log p}{p-1} \right) + \frac{\phi(q)}{q} \log q \\
&= \frac{\phi(q)}{q} \left(\log q + \sum_{p/q} \frac{\log p}{p-1} \right)
\end{aligned}$$

This gives

$$\lim_{s \rightarrow 1} \left(L(s, \chi_0) - \frac{\phi(q)}{q^s(s-1)} \right) = \frac{\phi(q)}{q} \left(\log q + \sum_{p/q} \frac{\log p}{p-1} + \gamma \right).$$

Thus

$$\begin{aligned}
\lim_{s \rightarrow 1} \left(\sum_{\ell}' A(s, \ell, q) \right)^2 &= \frac{\phi^4(q)}{q^4} \left(\log q + \sum_{p/q} \frac{\log p}{p-1} + \gamma \right)^4 \\
&<< (\log^4(q+2)) (\log \log(q+10))^4.
\end{aligned}$$

Next, we continue with the proof of the theorem. We have

$$\sum_{\chi \neq \chi_0 \pmod{q}} |L(1, \chi)|^4 = \phi(q) \sum_{\ell}' A^2(1, \ell, q) + O(\log^4 q \cdot (\log \log(q+10))^4)$$

Next we estimate

$$\sum_{\ell}' A^2(1, \ell, q) = \sum_{\ell}' \left(\sum_{ab}' \sum_{\ell(q)}' q^{-2} \left(\frac{q}{a} + \phi \left(1, \frac{a}{q} \right) \right) \left(\frac{q}{b} + \phi \left(1, \frac{b}{q} \right) \right) \right)^2.$$

Consider

$$\begin{aligned} & \sum_{ab}' \sum_{\ell(q)}' q^{-2} \left(\frac{q}{a} + \phi \left(1, \frac{a}{q} \right) \right) \cdot \left(\frac{q}{b} + \phi \left(1, \frac{b}{q} \right) \right) \\ &= \sum_{ab}' \sum_{\ell(q)}' \left(\frac{1}{a} + \frac{1}{q} \phi \left(1, \frac{a}{q} \right) \right) \cdot \left(\frac{1}{b} + \frac{1}{q} \phi \left(1, \frac{b}{q} \right) \right) \\ &= \sum_{ab \equiv \ell \pmod{q}}' \sum_{(q)}' \left(\frac{1}{a} + O \left(\frac{1}{q} \right) \right) \left(\frac{1}{b} + O \left(\frac{1}{q} \right) \right) \\ &= \sum_{ab}' \sum_{\ell(q)}' \frac{1}{ab} + O \left(\frac{1}{q} \sum_{ab}' \sum_{\ell(q)}' \frac{1}{a} \right) \\ & \quad + O \left(\frac{1}{q} \sum_{ab}' \sum_{\ell(q)}' \frac{1}{b} \right) + O \left(\frac{1}{q^2} \sum_a' \sum_{ab \equiv \ell(q)}' 1 \right) \\ &= \sum_{ab}' \sum_{\ell(q)}' \frac{1}{ab} + O \left(\frac{1}{q} \sum_a' \frac{1}{a} \right) + O \left(\frac{1}{q} \sum_b' \frac{1}{b} \right) + O \left(\frac{\phi(q)}{q^2} \right) \\ &= \sum_{\substack{n \leq q^2 \\ n \equiv \ell(q)}} \frac{a(n)}{n} + O \left(\frac{\log(q+3)}{q} \right). \end{aligned}$$

Here $a(n) = a_q(n) = \sum_{uv=n} 1$ with $1 \leq u, v \leq q$ so that $a(n) \leq d(n)$, where $d(n)$ is the divisor function. Thus

$$\begin{aligned} \phi(q) \sum_{\ell}' A^2(1, \ell, q) &= \phi(q) \sum_{\ell}' \left(\sum_{\substack{n \leq q^2 \\ n \equiv \ell \pmod{q}}} \frac{a(n)}{n} + O \left(\frac{\log(q+3)}{q} \right) \right)^2 \\ &= \phi(q) \sum_{\ell}' \left(\sum_{\substack{n \leq q^2 \\ n \equiv \ell \pmod{q}}} \frac{a(n)}{n} \right)^2 + O \left(\frac{\phi(q)}{q} \log(q+3) \cdot \sum_{\substack{n \leq q^2 \\ (n,q)=1}} \frac{a(n)}{n} \right) \\ & \quad + O(\log^2(q+3)) \end{aligned}$$

$$= \phi(q) \sum_{\ell}' \left(\sum_{\substack{n \leq q^2 \\ n \equiv \ell \pmod{q}}} \frac{a(n)}{n} \right)^2 + O(\log^3(q+3)),$$

as

$$\sum_{n \leq q^2} \frac{a(n)}{n} \ll \sum_{n \leq q^2} \frac{d(n)}{n} \ll \log^2(q+3) \quad .$$

Next, we have

$$\begin{aligned} \sum_{\ell}' \left(\sum_{\substack{n \leq q^2 \\ n \equiv \ell \pmod{q}}} \frac{a(n)}{n} \right)^2 &= \sum_{\ell}' \left(\frac{d(\ell)}{\ell} + \sum_{\substack{q < n \leq q^2 \\ n \equiv \ell \pmod{q}}} \frac{a(n)}{n} \right)^2 \\ &= \sum_{\ell}' \frac{d^2(\ell)}{\ell^2} + 2 \sum_{\ell}' \frac{d(\ell)}{\ell} \cdot \sum_{\substack{q < n \leq q^2 \\ n \equiv \ell \pmod{q}}} \frac{a(n)}{n} \\ &\quad + \sum_{\ell}' \left(\sum_{\substack{q < n \leq q^2 \\ n \equiv \ell \pmod{q}}} \frac{a(n)}{n} \right)^2 \end{aligned}$$

Then we estimate $\sum_{\ell}' \left(\sum_{\substack{q < n \leq q^2 \\ n \equiv \ell \pmod{q}}} \frac{a(n)}{n} \right)^2$. On using Schwarz's inequality, we have

$$\begin{aligned} \left(\sum_{\substack{q < n \leq q^2 \\ n \equiv \ell \pmod{q}}} \frac{a(n)}{n} \right)^2 &\leq \sum_{\substack{q < n \leq q^2 \\ n \equiv \ell \pmod{q}}} \frac{1}{n} \cdot \sum_{\substack{q < n \leq q^2 \\ n \equiv \ell \pmod{q}}} \frac{a^2(n)}{n} \\ &\leq \sum_{1 \leq m \leq q} \frac{1}{mq + \ell} \cdot \sum_{\substack{q < n \leq q^2 \\ n \equiv \ell \pmod{q}}} \frac{a^2(n)}{n} \\ &\leq \sum_{1 \leq m \leq q} \frac{1}{mq} \sum_{\substack{q < n \leq q^2 \\ n \equiv \ell \pmod{q}}} \frac{a^2(n)}{n} \\ &\leq \left(\frac{\log(q+3)}{q} \right) \left(\sum_{\substack{q < n \leq q^2 \\ n \equiv \ell \pmod{q}}} \frac{a^2(n)}{n} \right) \end{aligned}$$

Thus

$$\begin{aligned} \sum_{\ell}' \left(\sum_{\substack{q < n \leq q^2 \\ n \equiv \ell(q)}} \frac{a(n)}{n} \right)^2 &\ll \left(\frac{\log(q+2)}{q} \right) \sum_{\ell}' \sum_{\substack{q < n \leq q^2 \\ n \equiv \ell(q)}} \frac{d^2(n)}{n} \\ &\ll \left(\frac{\log(q+2)}{q} \right) \cdot \sum_{q < n \leq q^2} \frac{d^2(n)}{n} \\ &\ll \frac{\log^5(q+2)}{q} \end{aligned}$$

Note $\sum_{\ell}' \frac{d^2(\ell)}{\ell^2} < \log^3(q+2)$. Hence by Schwarz's inequality,

$$\begin{aligned} \sum_{\ell}' \frac{d(\ell)}{\ell} \cdot \sum_{\substack{q < n \leq q^2 \\ n \equiv \ell(q)}} \frac{a(n)}{n} &\ll \left((\log^3(q+2)) \left(\frac{\log^5(q+2)}{q} \right) \right)^{\frac{1}{2}} \\ &\ll \frac{\log^4(q+2)}{\sqrt{q}}. \end{aligned}$$

Thus

$$\phi(q) \sum_{\ell}' \left(\sum_{\substack{n \leq q^2 \\ n \equiv \ell(q)}} \frac{a(n)}{n} \right)^2 = \phi(q) \sum_{\ell}' \frac{d^2(\ell)}{\ell^2} + O\left(\frac{\phi(q)}{\sqrt{q}} \cdot \log^4(q+2) \right).$$

Hence

$$\sum_{\chi \neq \chi_0(q)} |L(1, \chi)|^4 = \phi(q) \sum_{\ell}' \frac{d^2(\ell)}{\ell^2} + O\left(\frac{\phi(q)}{\sqrt{q}} \cdot \log^4(q+2) \right).$$

This completes the proof of the theorem. \square

Reference

- [1] Rane V V, Character average of second and fourth powers of Dirichlet L-series at unity, [arXiv:0809.207401](https://arxiv.org/abs/0809.207401)

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