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REDUCTION OF VECTOR BOUNDARY VALUE PROBLEMS ON RIEMANN SURFACES TO ONE-DIMENSIONAL PROBLEMS

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Abstract: This article lays foundations for the theory of vector conjugation boundary value problems on a compact Riemann surface of arbitrary positive genus. The main constructions of the classical theory of vector boundary value problems on the plane are carried over to Riemann surfaces: reduction of the problem to a system of integral equations on a contour, the concepts of companion and adjoint problems, as well as their connection with the original problem, the construction of a meromorphic matrix solution. We show that each vector conjugation boundary value problem reduces to a problem with a triangular coefficient matrix, which in fact reduces the problem to a succession of one-dimensional problems. This reduction to the well-understood one-dimensional problems opens up a path towards a complete construction of the general solution of vector boundary value problems on Riemann surfaces.

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1. Introduction and Statement of the Problem

The author established in [1] a connection between vector boundary value problems on a Riemann surface and holomorphic vector bundles. Namely, it was shown that a general method for solving homogeneous vector boundary value problems is actually equivalent to a complete classification of holomorphic vector bundles up to equivalence, while a general method for solving inhomogeneous problems, to description of the first cohomology group with coefficients in the vector bundle. In particular, one manifestation of this link is the equivalence of the well-known statements for the Riemann surface of genus zero, the Riemann sphere or the plane: the reduction of a vector problem to a problem with diagonal matrix [2] and the equivalence of every vector bundle to a direct sum of one-dimensional bundles by Grothendieck's Theorem [3].

By a complete classification of bundles (boundary value problems) we understand the introduction of a system of invariants, meaning the characteristics shared by equivalent bundles and/or equivalent boundary coefficients which uniquely determine the equivalence class of bundles and respectively the coefficients in boundary conditions up to equivalence. For instance, for one-dimensional bundles (one-dimensional boundary value problems) there are two invariants: The first is the index of boundary coefficients, equal to the degree of the divisor of sections of the bundle, equal to the Chern class of the bundle; The second is the element of the Jacobi group, uniquely determining the class of equivalent coefficients of index zero, equal to the element of the Picard group uniquely defining the class of equivalent divisors of order zero (bundles with vanishing Chern class); see [1]. For vector bundles over the Riemann sphere these invariants are the Chern classes of one-dimensional bundles, equal to the special indices of the coefficient matrix [1].

The classification problem for vector bundles over an arbitrary compact Riemann surface is thoroughly studied in [4]. Once this problem attracted much attention mainly due to the work of Tyurin, for instance [5–7]. Essentially, by now it is clear that not all vector bundles on a Riemann surface of arbitrary genus are equivalent to the direct sums of one-dimensional ones; but no complete system of invariants classifying holomorphic vector bundles is available yet.

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As for vector boundary value problems, previously they were barely considered on Riemann surfaces beyond the one-dimensional ones, which are understood quite well; see [8–11] for instance.

This article attempts to lay foundations for the theory of vector boundary value problems on Riemann surfaces. To this end, we firstly carry over to Riemann surfaces the main constructions of the classical theory of vector boundary value problems on the plane, in particular, the reduction of a vector problem to a system of integral equations on a contour, done in Section 2, and the construction of a linearly independent system of meromorphic solutions in Section 3. This adaptation of results from the plane, a surface of genus zero, to the surfaces of arbitrary genus is not quite trivial and requires some additional effort.

Section 4 establishes that each vector boundary value problem reduces to a problem with a triangular matrix. This enables us to reduce the solution of the boundary value problem to the successive solution of one-dimensional boundary value problems, for which a good theory is available. This is equivalent to the statement proved in [4, p. 63] that, up to equivalence, each bundle can be defined by triangular transition matrices. Important here, however, is not the reduction to a problem with a triangular matrix by itself, but the technique of the article for analyzing boundary value problems with the use of meromorphic matrices and their divisors, which enables us to construct solutions to boundary value problems by singular integrals as in the classical theory, and in particular, opening up a path to a complete classification of boundary coefficients.

Let us emphasize the conclusion that was implied by [1]. It is shown there that boundary value problems on an arbitrary smooth curve, including even not closed and not simple, reduce to a construction of sections of holomorphic vector bundles. More exactly, this reduction requires that a Riemann surface with a cut along this curve split into finitely many connected components (possibly one) with piecewise smooth boundary. In particular, this means that the general method for solving problems is independent of the choice of the curve; therefore, to construct a general solution, and to classify bundles, we can choose a rather arbitrary curve.

Take a Riemann surface D of genus $\rho > 0$ and a simple smooth contour L splitting D into two connected parts $D^{\pm}$; the orientation of L agrees with D^{+}. Consider the homogeneous boundary value problem of linear conjugation

$$g^{+}(t) = G(t)g^{-}(t), \quad t \in L, \tag{1}$$

where $g^{\pm}(t)$ are the boundary values on L of the required analytic or meromorphic on $D^{\pm}$ vector $g^{\pm}(z)$ of dimension n, and $G(t)$ is a given Hölder matrix of size $n \times n$ on L which is nondegenerate meaning that $\det G(t) \neq 0$ for $t \in L$. Refer to $G(t)$ as the *coefficient* of the boundary value problem. Simultaneously we consider similar problems for the differentials

$$w^{+}(t) = G^{T}(t)w^{-}(t), \quad t \in L, \tag{2}$$

where $w^{\pm}(t)$ are the boundary values on L of the n-dimensional abelian differential $w^{\pm}(z)$ for $z \in D^{\pm}$. Problem (2) for vector differentials reduces to a vector bundle as well, i.e. eventually to a problem for vector functions of the form (1). We consider the problem for differentials separately only for technical convenience.

Remark 1. If need be, we may assume that one of the domains D^{+} and D^{-} is simply-connected, i.e. conformally equivalent to a part of the plane.

Below, in accordance with generally accepted terminology [2], we call a point of the surface D a *pole vector* if it is a pole of at least one of its coordinates, while the order of a pole is assumed to be the maximal order of poles among all coordinates. Accordingly, the zero of a vector is the zero of all its coordinates simultaneously, while the order of a zero is the minimal order among all coordinates. This way we define the divisor of each meromorphic vector. Since the multiplication by an arbitrary meromorphic function preserves the solution to problems (1) and (2), the divisors of meromorphic solutions are defined up to principal divisors. Finally, since locally problems (1) and (2) are equivalent to problems on the plane, their arbitrary zeros and poles, including those on the contour L, are of integer orders.

2. Reduction of Problems to Systems of Integral Equations. Companion and Adjoint Problems

Here we carry over to a Riemann surface the basic theory of boundary value problems on the plane [2].

Let us start with the Cauchy kernel $K_0(p,q)$ for the one-dimensional boundary value problem [10, 11]: $K_0(p,q)$ is an abelian differential with respect to p which is a multiple of the divisor $p_1 \cdots p_\rho p_0^{-1} q^{-1}$ with residues 1 at q and -1 at p_0, and a meromorphic function with respect to q which is a multiple of $p^{-1} p_0 p_1^{-1} \cdots p_\rho^{-1}$; here $p_1, \dots, p_\rho$ is an arbitrary tuple of points of D such that no abelian differential of genus 1 on D has zeros at all these points. Henceforth we call p_j for $j = \overline{1,\rho}$ the *poles* of the Cauchy kernel, and p_0 its *zero*. We can choose the zero and poles rather arbitrarily; in particular, we can always choose poles in a neighborhood of an arbitrary given point of D and choose the zero arbitrarily, only avoiding the poles [10].

Properties of Cauchy kernels. 1. Suppose that $g(t)$ is a Hölder function on L and put

$$\Phi^{\pm}(z) = \frac{1}{2\pi i} \int\limits_{t_0 \in L} K_0(t_0, z) g(t_0), \quad z \in D^{\pm}.$$

The following hold:

(a) $\Phi^{\pm}(z)$ is analytic on $D^{\pm}$, except for p_j for $j = \overline{1,\rho}$ where $\Phi^{\pm}(z)$ has simple poles.

(b) If

$$\int\limits_L g(t) w(t) = 0$$

for every genus 1 abelian differential $w(z)$ on D then $\Phi^{\pm}(z)$ is analytic on $D^{\pm}$.

(c) The boundary values on L are as follows:

$$\Phi^{\pm}(t) = \frac{1}{2\pi i} \int\limits_{t_0 \in L} K_0(t_0, t) g(t_0) \pm \frac{g(t)}{2}, \quad t \in L.$$

2. Suppose that $w(t)$ is a differential on L and

$$W^{\pm}(z) = \frac{1}{2\pi i} \int\limits_{t_0 \in L} K_0(z, t_0) w(t_0), \quad z \in D^{\pm}.$$

The following hold:

(a) The differential $W^{\pm}(z)$ is analytic on $D^{\pm}$ outside p_0, where it has a simple pole; furthermore,

$$\operatorname{Res} W(z)|_{z=p_0} = -\frac{1}{2\pi i} \int\limits_L w(t).$$

(b) The boundary values are

$$W^{\pm}(t) = \frac{1}{2\pi i} \int\limits_{t_0 \in L} K_0(t, t_0) w(t_0) \mp \frac{w(t)}{2}, \quad t \in L.$$

Indeed, it is shown in [10, Chapter 1, § 4] that in the plane of the local parameter t_0, the Cauchy kernel as $z \to t_0$ is of the form

$$K(t_0, z) = \frac{dt_0}{t_0 - z} + F(t_0, z)\, dt_0,$$

where $F(t_0, z)$ is analytic in both variables, and so all presented expressions for the boundary values of $\Phi^{\pm}(t)$ and $W^{\pm}(t)$ have the usual meaning of singular integrals in the sense of principal values, while all statements concerning these boundary values follow straightforwardly from the properties of the limit values of the Cauchy-type integral. All remaining properties are either established directly in [10, Chapter 1, § 4] or follow easily from the results of [10].

Moreover, we borrow from [10, Chapter 2, § 7] the functions $\psi^{\pm}(z) = \psi(p \mid z)$ for $z \in D^{\pm}$ which are analytic on $D^{\pm}$ for $p \in D^{\pm}$ and have unique simple zeros at p.

We use some standard notation: $r[\Delta]$ is the number of linearly independent meromorphic functions that are multiples of a divisor Δ, and $i[\Delta]$ is the number of linearly independent abelian differentials that are multiples of Δ. The Riemann–Roch Theorem asserts [4, 10, 12] that $r[\Delta^{-1}] = i[\Delta] + \deg\Delta - \rho + 1$.

Let us turn to the conditions for the analytic continuation of functions and differential from a contour L to the domain $D^{\pm}$. Below we use the two Cauchy kernels: the kernel $K_1(p,q)$ with poles $p_{j1} \in D^-$ for $j = \overline{1,\rho}$ and zero $p_{01} \in D^+$, and the kernel $K_2(p,q)$ with poles $p_{j2} \in D^+$ for $j = \overline{1,\rho}$ and zero $p_{02} \in D^-$.

Theorem 1 (analytic continuation of functions and differentials). *Suppose that a function $g(t)$ of $t \in L$ is given.*

1. *The function $g(t)$ continues analytically into D^+, i.e. $g(t) = g^+(t)$, if and only if*

$$\frac{1}{2\pi i}\int\limits_{t_0 \in L} K_1(t_0,t)g(t_0) - \frac{g(t)}{2} = C = \text{const}\,.$$

Furthermore,

$$g^+(z) = \frac{1}{2\pi i}\int\limits_{L} K_1(t,z)g(t) - C, \quad C = -g^+(p_{01}).$$

2. *The function $g(t)$ continues analytically into D^-, i.e. $g(t) = g^-(t)$, if and only if*

$$\frac{1}{2\pi i}\int\limits_{t_0 \in L} K_2(t_0,t)g(t_0) + \frac{g(t)}{2} = C = \text{const}\,.$$

Furthermore,

$$g^-(z) = -\frac{1}{2\pi i}\int\limits_{L} K_2(t,z)g(t) + C, \quad C = g^-(p_{02}).$$

By analogy, suppose that a differential $w(t)$ is given on L.

1. *The differential $w(t)$ continues analytically into D^+ if and only if there exist constants $C_1, \dots, C_\rho$ such that*

$$\frac{1}{2\pi i}\int\limits_{t_0 \in L} K_2(t,t_0)w(t_0) + \frac{w(t)}{2} = \sum_{k=1}^{\rho} C_k w_k(t).$$

2. *The differential $w(t)$ continues analytically into D^- if and only if there exist constants $C_1, \dots, C_\rho$ such that*

$$\frac{1}{2\pi i}\int\limits_{t_0 \in L} K_1(t,t_0)w(t_0) - \frac{w(t)}{2} = \sum_{k=1}^{\rho} C_k w_k(t).$$

Here $w_k(z)$ for $k = \overline{1,\rho}$ is a basis for genus 1 abelian differentials on D.

Proof. Take $g(t) = g^+(t)$ and suppose that $g^+(z)$ is analytic on D^+. Then for every genus 1 abelian differential $w(z)$ on D, we obviously have

$$\int\limits_{L=\partial D^+} g^+(t)w(t) = 0,$$

which implies that the function

$$\Phi_1(z) = \frac{1}{2\pi i} \int\limits_L K_1(t,z) g^+(t)$$

is an analytic solution in $D^\pm$ of the jump problem $\Phi^+(t) - \Phi^-(t) = g(t)$, for $t \in L$. Another solution of the same problem is

$$\Phi_2(z) = \begin{cases} g^+(z), & z \in D^+, \\ 0, & z \in D^-. \end{cases}$$

Then $\Phi_1(z) - \Phi_2(z)$ is clearly analytic on D, and therefore constant. Thus,

$$\frac{1}{2\pi i} \int\limits_L K_1(t,z) g^+(t) - g^+(z) = C, \quad z \in D^+,$$

$$\frac{1}{2\pi i} \int\limits_L K_1(t,z) g^+(t) = C, \quad z \in D^-,$$

whence

$$\frac{1}{2\pi i} \int\limits_L K_1(t_0,t) g(t_0) - \frac{g(t)}{2} = C.$$

Conversely, suppose that the last equality holds for some constant C. Consider the function

$$g^+(z) = \frac{1}{2\pi i} \int\limits_L K_1(t,z) g(t) - C.$$

Since $K_1(t, p_{01}) \equiv 0$, it follows that $g^+(p_{01}) = -C$. The function $g^+(z)$ is meromorphic on $D^\pm$ and has poles at $p_{j1} \in D^-$, for $j = \overline{1,\rho}$, i.e., it is analytic on D^+ and its boundary value is obviously

$$g^+(t) = \frac{1}{2\pi i} \int\limits_L K_1(t_0,t) g(t_0) + \frac{g(t)}{2} - C = g(t).$$

The claims of the theorem on continuation into D^-, as well as the claims on the continuation of differentials have similar proofs.

REMARK 2. The statements and proofs remain absolutely the same if we replace the function $g(t)$ by dimension n vector functions $g(t) = (g_1(t), \dots, g_n(t))$, or similarly replace the differential $w(t)$ by a vector composed of differentials; we just have to replace the constants C_1, C_2, C_{k1}, and C_{k2} for $k = \overline{1,\rho}$ with constant vectors.

Henceforth we tacitly assume that $g(t)$ is a vector function, $w(t)$ is a vector differential, C_1, C_2, C_{k1}, and C_{k2} for $k = \overline{1,\rho}$ are constant vectors.

Corollary of Theorem 1. 1. *Problem* (1) *for functions is equivalent to the following system of equations for the vector function* $g(t) = g^-(t)$,

$$\frac{1}{2\pi i} \int\limits_{t_0 \in L} K_1(t_0,t) G(t_0) g(t_0) - \frac{G(t) g(t)}{2} = C_1, \quad \frac{1}{2\pi i} \int\limits_{t_0 \in L} K_2(t_0,t) g(t_0) + \frac{g(t)}{2} = C_2, \tag{3}$$

which implies

$$g(t) + \frac{1}{2\pi i} \int\limits_{t_0 \in L} [K_2(t_0,t) - K_1(t_0,t) G^{-1}(t) G(t_0)] g(t_0)$$
$$= C_2 - G^{-1}(t), \quad C_1, C_1, C_2 = \text{const}\,. \tag{4}$$

2. *Problem (2) for differentials is equivalent to the following system of equations for the vector differential* $w(t) = w^+(t)$,

$$\frac{1}{2\pi i}\int\limits_{t_0\in L} K_2(t,t_0)w(t_0) + \frac{w(t)}{2} = \sum_{k=1}^{\rho} C_{k2}w_k(t),$$

$$\frac{1}{2\pi i}\int\limits_{t_0\in L} K_1(t,t_0)(G^T(t_0))^{-1}w(t_0) - \frac{(G^T(t))^{-1}w(t)}{2} = \sum_{k=1}^{\rho} C_{k1}w_k(t), \tag{5}$$

which implies

$$w(t) + \frac{1}{2\pi i}\int\limits_{t_0\in L} [K_2(t,t_0) - K_1(t,t_0)G^T(t)(G^T(t_0))^{-1}]w(t_0)$$

$$= \sum_{k=1}^{\rho}(C_{k2} - G^T(t)C_{k1})w_k(t), \quad C_{k1}, C_{k2} = \text{const},\ k = \overline{1,\rho}. \tag{6}$$

Remark 3. Note that (4) and (5) are systems of Fredholm integral equations of the second kind since their kernels have singularities of order less than 1; i.e., the integral operators are compact. Furthermore, these systems are conjugate, i.e., in particular, (4) has a solution if and only if its right-hand side is orthogonal to every solution of the homogeneous equation (6):

$$\int\limits_L (C_2 - G^{-1}(t)C_1)w(t) = 0$$

for every solution to (6) with $C_{k1} = C_{k2} = 0$ for $k = \overline{1,\rho}$.

Verify that under certain conditions the solutions to (4) or (6) yield solutions to the boundary value problems (1) or (2) respectively. Introduce the functions

$$\psi_0^+(z) = \psi(p_{01} \mid z), \quad \psi_1^+(z) = \prod_{j=1}^{\rho}\psi(p_{j2} \mid z), \quad z \in D^+,$$

$$\psi_0^-(z) = \psi(p_{02} \mid z), \quad \psi_1^-(z) = \prod_{j=1}^{\rho}\psi(p_{j1} \mid z), \quad z \in D^-,$$

and respectively $\psi_j^\pm(t)$ for $t \in L$ and $j = \overline{0,1}$.

Given $g(t)$ for $t \in L$, put

$$h^+(z) = \psi_1^+(z)\biggl(\frac{1}{2\pi i}\int\limits_{t\in L} K_2(t,z)g(t) - C_2\biggr), \quad z \in D^+;$$

$$h^-(z) = \psi_1^-(z)\biggl(\frac{1}{2\pi i}\int\limits_{t\in L} K_1(t,z)G(t)g(t) - C_1\biggr), \quad z \in D^-,$$

where $C_1, C_2 = \text{const}$. The function $h^+(z)$ is analytic on D^+ because the integral has only simple poles at the points p_{j2}, for $j = \overline{1,\rho}$, which compensate for the zeros of ψ_1^+, while $h^-(z)$ is analytic on D^-. In turn, we have

$$\frac{h^+(t)}{\psi_1^+(t)} = \frac{1}{2\pi i}\int\limits_{t_0\in L} K_2(t_0,t)g(t_0) + \frac{g(t)}{2} - C_2,$$

$$\frac{h^-(t)}{\psi_1^-(t)} = \frac{1}{2\pi i}\int\limits_{t_0\in L} K_1(t_0,t)G(t_0)g(t_0) - \frac{G(t)g(t)}{2} - C_1.$$

But then (4) means that

$$\frac{h^+(t)}{\psi_1^+(t)} - G^{-1}(t)\frac{h^-(t)}{\psi_1^-(t)} = 0$$

or

$$h^+(t) = G^{-1}(t)\frac{\psi_1^+(t)}{\psi_1^-(t)}h^-(t), \quad t \in L. \tag{7}$$

Following the terminology of [2], call this the *companion problem* to (1). If the companion problem (7) has only the trivial solution then $h^{\pm}(t) = 0$, which implies that (3) are satisfied; i.e., the solution to (4) yields a solution to (1).

Similarly, for problem (2) for differentials and equation (6) respectively we obtain the companion problem

$$w^+(t) = \frac{\psi_0^+(t)}{\psi_0^-(t)}(G^T(t))^{-1}w^-(t), \quad t \in L. \tag{8}$$

Refer to this problem, as in [2], as the *adjoint problem* to (1).

Observe that the form of companion and adjoint problems for a surface of genus $\rho > 0$ differs substantially from the case of the plane [2].

Theorem 2. 1. *If* (7) *has only the trivial solution in the class of analytic functions then every solution* $g(t)$ *to* (4) *yields a solution to* (1)*:* $g^-(t) = g(t)$ *and* $g^+(t) = G(t)g(t)$ *in the class of analytic functions.*

2. *If* (8) *has only the trivial solution in the class of analytic differentials on* $D^{\pm}$ *then every solution* $w(t)$ *to* (6) *yields a solution to* (2) *for differentials:* $w^+(t) = w(t)$ *and* $w^-(t) = (G^T(t))^{-1}w(t)$.

3. *If both companion and adjoint problems have only the trivial solutions then the original problem* (1) *has an analytic solution.*

Proof. Only claim 3 needs a proof. Consider (4) with $C_1 = 0$, i.e., with the right-hand side C_2. It is solvable, see Remark 3, if and only if

$$\int_L C_2 w(t) = 0$$

for every solution $w(t)$ of the homogeneous equation (6), meaning $C_{1k} = C_{2k} = 0$ for $k = \overline{1,\rho}$. However, since (8) has only the trivial solution, every solution to (6), including the homogeneous case, yields a solution to (2) with $w(t) = w^+(t)$, and the solvability condition, which is obviously satisfied, becomes

$$\int_{L=\partial D^+} C_2 w^+(t) = 0.$$

Thus, a solution to (4) with $C_1 = 0$ and arbitrary vector C_2 exists. However, since (7) has only the trivial solution, the solution to (4) yields an analytic solution to (1).

3. Construction of a Meromorphic Matrix Solution

Consider the matrix $\Phi^{\pm}(z) = \big(g_1^{\pm}(z) \cdots g_n^{\pm}(z)\big)$ composed of meromorphic solutions $g_j^{\pm}(z)$, for $j = \overline{1,n}$, to (1), meaning that $\Phi^+(t) = G(t)\Phi^-(t)$ for $t \in L$. Assume that $\det \Phi^{\pm}(z) \not\equiv 0$ for $z \in D^{\pm}$. Call Φ the *meromorphic matrix solution* to (1).

The construction of meromorphic matrix solution, more exactly, the proof of its existence, is the starting point for classifying the coefficients of a boundary value problem and constructing the general solution in the classical theory of vector boundary value problems on the plane [2]. In this section we verify that the matrix solution exists on a Riemann surface as well.

Theorem 3. *The analytic solutions to* (1) *and* (2) *constitute a finite-dimensional linear space, and the order of the divisor of each analytic solution to* (1) *and* (2) *is at most the constant* $m = m(G, \rho)$.

PROOF. Since (4) is a system of Fredholm equations, the number of its linearly independent solutions for arbitrary constant vectors C_1 and C_2 is finite, and so is the number of linearly independent analytic solutions to (1). Denote this number by $s = s(G)$. Suppose that an analytic solution $g^{\pm}(z)$ of (1) has divisor Δ, consisting only of zeros, and that $\deg \Delta > 2\rho - 2$. Then [10, 12] imply that $i[\Delta] = 0$, and the Riemann–Roch Theorem yields $r = r[\Delta^{-1}] = \deg \Delta - \rho + 1$. But if $f_j(z)$ for $j = \overline{1, r}$ are linearly independent meromorphic functions which are multiples of Δ^{-1} then the vectors $f_j(z) g^{\pm}(z)$ are linearly independent analytic solutions to (1), i.e., $r \leq s$, whence $\deg \Delta \leq s + \rho - 1$, and finally $\deg \Delta \leq m = \max(s(G) + \rho - 1, 2\rho - 2)$.

The proof of the theorem for problem (2) and the system (6) is similar.

Lemma 1. 1. *The vector* $g^{\pm}(z)$ *is a meromorphic solution to* (1) *with poles* q_j, *for* $j = \overline{1, s}$, *if and only if* $g_0^{\pm}(z) = g^{\pm}(z) \psi^{\pm}(z)$ *is an analytic solution to* (1) *with coefficients* $G_0(t) = G(t) \psi^{+}(t) / \psi^{-}(t)$, *where*

$$\psi^{\pm}(z) = \prod_{q_j \in D^{\pm}} \psi(q_j \mid z), \quad z \in D^{\pm}.$$

2. *In turn, if* $g_0^{\pm}(z)$ *is an analytic solution to* (1) *with the coefficient* $G_0(t) = G(t) \psi^{-}(t) / \psi^{+}(t)$ *then* $g^{\pm}(z) = g_0^{\pm}(z) \psi^{\pm}(z)$ *is an analytic solution to* (1) *with the zero* q_j, *for* $j = \overline{1, s}$.

The claims of Lemma 1 are obvious.

Theorem 4 (existence of meromorphic matrix solution). *Introduce the upper bounds for the number of zeros,* $m_1 = m(G^{-1} \psi_1^{+} / \psi_1^{-}, \rho)$ *for the companion problem and* $m_2 = m((G^T)^{-1} \psi_0^{+} / \psi_0^{-}, \rho)$ *for the adjoint problem, and take the divisor* $\Delta = \prod q_j$ *with* $q_j \in D^{-}$. *Moreover,* $\deg \Delta > \max(m_1, m_2)$.

Then there exist solutions to (1) *which are multiples of* Δ^{-1}. *These solutions constitute a finite-dimensional space of dimension at least n; furthermore, for each point* q_j *there are n solutions* $g_1, \dots, g_n$, *whose leading constants at this point are linearly independent. By the leading constants we understand the constant vectors* C *in the expansion*

$$g(z) = \frac{C}{(z - q_j)^{k_j}} + \cdots$$

in a neighborhood of q_j, *where* k_j *is the order of the zero* q_j *in* Δ.

These n solutions constitute a meromorphic matrix solution to (1).

PROOF. Take

$$\psi^{-}(z) = \prod_{q_j} \psi(q_j \mid z), \ z \in D^{-}; \quad \psi^{+}(z) \equiv 1.$$

Problem (1) in the class of vectors which are multiples of Δ^{-1} reduces by Lemma 1 to (1) with the coefficient $G_0(t) = G(t) / \psi^{-}(t)$ in the class of analytic functions, whose solutions, and so the solutions to (1) which are multiples of Δ^{-1}, constitute a finite-dimensional space by Theorem 3.

Let us show that this space is nonempty and contains solutions $g_1, \dots, g_n$ with linearly independent leading constants. The companion and adjoint problems corresponding to the coefficient $G_0(t)$ respectively have the coefficients

$$G_1 = G^{-1} \psi^{-} \psi_1^{+} / \psi_1^{-},$$
$$G_2 = (G^T)^{-1} \psi^{-}(t) \psi_0^{+} / \psi_0^{-}.$$

For instance, if (1) with the coefficient G_1 has an analytic solution $g_1^{\pm}(z)$; then, again by Lemma 1, $g^{\pm}(z) = \psi^{\pm}(z) g_1^{\pm}(z)$ is an analytic solution to (1) with the coefficient $G^{-1} \psi_1^{+} / \psi_1^{-}$. Furthermore, this solution has zeros at the points of Δ; i.e., it has the order of $m_0 \geq \deg \Delta > m_1 = m(G^{-1} \psi_1^{+} / \psi_1^{-}, \rho)$, which is impossible by Theorem 3. Similarly, there exists no analytic solution to (2) with the coefficient G_2. Therefore, for (1) with the coefficient $G_0(t)$ neither companion nor adjoint problems have analytic solutions; then by Theorem 2 this problem has an analytic solution $g_0^\pm(z)$.

Observe that for the solution g_0 we can choose an arbitrary vector $C_2 = g_0^-(p_{02})$; see Theorem 1 and the proof of Theorem 2. In turn, the validity of Theorem 2 is independent of the choice of Cauchy kernels $K_{1,2}(p,q)$, i.e., of the choice of their poles p_{j1} and p_{j2} for $j = \overline{1,\rho}$, and zeros p_{01} and p_{02}. In particular, for every fixed point q_j of the divisor Δ we can choose $p_{02} = q_j$. Then, taking n linearly independent values of the vector $C = g_0^-(q_j)$, we obtain n linearly independent solutions to (1) with the coefficient $G_0(t)$. This, again by Lemma 1, yields n linearly independent solutions to (1) with the original coefficients $G(t)$; furthermore, the correspondence $g^\pm(z) = g_0^\pm(z)/\psi^\pm(z)$ means that the chosen values of C are the leading constants at q_j.

The last claim of Theorem 4 is straightforward from the previous one.

Thus, the meromorphic matrix solution to (1) exists always.

4. Reduction of the Coefficient in the Boundary Condition to Triangular Form

Recall [1, 2] that the coefficients $G_1(t)$ and $G_2(t)$ are called equivalent, written $G_1 \sim G_2$, whenever $G_1(t) = X^+(t)G_2(t)(X^-(t))^{-1}$, for $t \in L$, where the matrices $X^\pm(z)$ are analytic on $D^\pm$ and $\det X^\pm(z) \neq 0$ for $z \in D^\pm$. Conjugation problems with equivalent coefficients reduce to each other with the zeros and poles kept in place, and the coefficients in the boundary condition are classified precisely up to this equivalence.

It is known that on a genus $\rho = 0$ surface we can assume, up to multiplication by a meromorphic function, that all zeros and poles of a meromorphic solution lie at one fixed point; on the plane they are usually placed at infinity; see [2]. This enables us [2] to establish on the plane a representation for the matrix solution $\Phi^\pm(z) = X^\pm(z)\Gamma^\pm(z)$, where $X^\pm(z)$ are analytic on $D^\pm$ with $\det X^\pm(z) \neq 0$ for $z \in D^\pm$, while $\Gamma^\pm = \operatorname{diag}(\gamma_1^\pm, \dots, \gamma_n^\pm)$ is a diagonal matrix. This implies that

$$G(t) \sim \Gamma^+(t)(\Gamma^-(t))^{-1} = \Gamma(t) = \operatorname{diag}(\gamma_1, \dots, \gamma_n);$$

i.e., each coefficient is equivalent to a diagonal matrix, and every boundary value problem reduces to a solution of independent one-dimensional problems. Passing from the homogeneous problem to a vector bundle [1], we find that, up to equivalence, the coordinates of the sections of each holomorphic vector bundle are simply sections of one-dimensional bundles, i.e., recover the well-known Theorem by Grothendieck [3].

On a Riemann surface of positive genus $\rho > 0$ placing all zeros/poles of solution at one point is impossible in general, as it is impossible to reduce every coefficient to a diagonal matrix. In this section we verify that every coefficient is equivalent to a triangular matrix.

Consider a meromorphic matrix solution $\Phi^\pm = (g_1^\pm \cdots g_n^\pm)$ and introduce the divisors of its columns

$$(g_j^\pm) = \Delta_j^\pm = \prod_l (s_{jl}^\pm)^{k_{jl}^\pm}, \quad s_{jl}^\pm \in D^\pm, \ j = \overline{1,n},$$

and the functions which are multiples of these divisors:

$$\psi_j^\pm(z) = \psi(\Delta_j^\pm \mid z) = \prod_l \bigl(\psi(s_{jl}^\pm \mid z)\bigr)^{k_{jl}^\pm}, \quad z \in D^\pm, \ j = \overline{1,n}.$$

It is obvious that $g_j^\pm(z) = \psi_j^\pm(z)\varphi_j^\pm(z)$ for $j = \overline{1,n}$, where the vectors $\varphi_j^\pm(z)$ lack zeros and poles. Hence,

$$\Phi^\pm = (\varphi_1^\pm \cdots \varphi_n^\pm) \cdot \operatorname{diag}(\psi_1^\pm, \dots, \psi_n^\pm) = \Phi_0^\pm \Gamma_0^\pm,$$

the matrices $\Phi_0^\pm$ are analytic on $D^\pm$, and their columns lack zeros. But $\det \Phi_0^\pm$ can have zeros. Consider for definiteness $\Phi_0^+(z) = \Phi_0(z) = (\varphi_1(z) \cdots \varphi_n(z))$ for $z \in D^+$ and suppose that $(\det \Phi_0) = \Delta = s_1 \cdots s_l$, where $s_j \in D^+$ for $j = \overline{1,l}$ are the zeros of $\det \Phi_0$. Since $\det \Phi_0(s_1) = 0$, there exist some coefficients α_j for $j = \overline{1,n}$, not all vanishing, such that $\sum \alpha_j \varphi_j(s_1) = 0$ or

$$\varphi_m(s_1) = \sum_{j<m} \alpha_j \varphi_j(s_1),$$

where m is the maximal index of a nonzero coefficient. Then

$$\Phi_0 = (\varphi_1 \cdots \varphi_m \cdots \varphi_n) = \left(\varphi_1 \cdots \varphi_m^0 \cdots \varphi_n\right) \begin{pmatrix} 1 & 0 & \cdots & 0 & \alpha_1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \alpha_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_{m-1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

where

$$\varphi_m^0(z) = \varphi_m(z) - \sum_{j<m} \alpha_j \varphi_j(z).$$

But then $\varphi_m^0(s_1) = 0$, and so $\varphi_m^0(z) = \psi_1(z)\varphi_m^1(z)$, where $\psi_1(z) = \psi(s_1 \mid z)$, and furthermore $\varphi_m^1(s_1) \neq 0$; more exactly, it has a smaller order of zero in s_1. Therefore,

$$\Phi_0 = \left(\varphi_1 \cdots \varphi_m^1 \cdots \varphi_n\right) \begin{pmatrix} 1 & 0 & \cdots & 0 & \alpha_1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \alpha_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_{m-1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \psi_1(z) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \Phi_1(z)\Gamma_1(z),$$

where $\Gamma_1(z)$ is an upper triangular matrix, $\Phi_1(z)$ is analytic and has the divisor of the determinant $(\det \Phi_1) = \Delta/s_1 = s_2 \cdots s_l$. Repeating this process, we arrive at the representation

$$\Phi^+(z) = X^+(z)\Gamma^+(z), \quad z \in D^+,$$

where $X^+(z)$ is analytic on D^+ and $\det X^+(z) \neq 0$, while $\Gamma^+ = \Gamma_l \cdots \Gamma_1 \Gamma_0$ is an upper triangular matrix. A rather similar representation results for $\Phi^-(z)$, whence

$$X^+(t)\Gamma^+(t) = \Phi^+(t) = G(t)\Phi^-(t) = G(t)X^-(t)\Gamma^-(t),$$

i.e.,

$$G(t) = X^+(t)\Gamma(t)(X^-(t))^{-1}, \quad \Gamma(t) = \Gamma^+(t)(\Gamma^-(t))^{-1}.$$

Theorem 5. *Each coefficient $G(t)$ is equivalent to an upper triangular matrix $\Gamma(t)$; furthermore, the diagonal elements of $\Gamma(t)$ are*

$$\gamma_j(t) = \psi^+(\Delta^+ \mid t)/\psi^-(\Delta^- \mid t),$$

where $\Delta^\pm \subset D^\pm$.

Remark 4. It is easy to show that, up to the multiplication of meromorphic solutions by meromorphic functions on D, we may always assume that the points of the divisors $\Delta^\pm$ avoid L, and even lie only in D^+ (if $\psi^- \equiv 1$) or, conversely, only in D^- (if $\psi^+ \equiv 1$).

Theorem 5 reduces the solution of (1) to a problem with a triangular matrix, which we can solve successively, finding one coordinate at each stage. These triangular matrices themselves can be expressed as products of triangular or diagonal matrices of quite simple form. Basing on this and using the well-known method for solving one-dimensional problems, we expect to construct a general solution to the vector problem and a system of invariants classifying the coefficients $G(t)$ up to equivalence.

References

1. Semenko E. V., "Connection between holomorphic vector bundles and cohomology on a Riemann surface and conjugation boundary value problems," Sib. Math. J., vol. 58, no. 2, 310–318 (2017).
2. Vekua N. P., *Systems of Singular Integral Equations and Some Boundary Value Problems* [Russian], Gostekhizdat, Moscow (1950).
3. Grothendieck A., "Sur la classification des fibres holomorphes sur la sphere de Riemann," Amer. J. Math., vol. 79, no. 1, 121–138 (1957).
4. Gunning R. C., *Lectures on Vector Bundles over Riemann Surfaces*, Princeton Univ. Press, Princeton (1967).
5. Tyurin A. N., "On the classification of two-dimensional fibre bundles over an algebraic curve of arbitrary genus," Izv. Akad. Nauk SSSR Ser. Mat., vol. 28, no. 1, 21–52 (1964).
6. Tyurin A. N., "The classification of vector bundles over an algebraic curve of arbitrary genus," Amer. Math. Soc. Transl. Ser. 2, vol. 63, no. 3, 245–279 (1967,).
7. Tyurin A. N., "Classification of n-dimensional vector bundles over an algebraic curve of arbitrary genus," Amer. Math. Soc. Transl. Ser. 2, vol. 73, no. 6, 196–211 (1968).
8. Zverovich È. I., "Boundary value problems in the theory of analytic functions in Hölder classes on Riemann surfaces," Russian Math. Surveys, vol. 26, no. 1, 117–192 (1971).
9. Monakhov V. N., Verkhovod D. B., and Semenko E. V., *Boundary-Value Problems for Quasianalytic Functions on Compact Riemann Surfaces* [Russian], Lavrentev Inst. of Hydrodynamics, Novosibirsk (1990).
10. Monakhov V. N. and Semenko E. V., *Boundary-Value Problems and Pseudodifferential Operators on Riemann Surfaces* [Russian], Fizmatlit, Moscow (2003).
11. Semenko E. V., "Prym differentials as solutions to boundary value problems on Riemann surfaces," Sib. Math. J., vol. 57, no. 1, 124–134 (2016).
12. Forster O., *Lectures on Riemann Surfaces*, Springer-Verlag, New York (1981).

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