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Non-collinear libration points in ER3BP with albedo effect and oblateness

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Abstract. In this paper we establish a relation between direct radiations (generally called radiation factor) and reflected radiations (albedo) to show their effects on the existence and stability of non-collinear libration points in the elliptic restricted three-body problem taking into account the oblateness of smaller primary. It is discussed briefly when $\alpha = 0$ and $\sigma = 0$, the non-collinear libration points form an isosceles triangle with the primaries and as e increases the libration points $L_{4,5}$ move vertically downward (α, σ and e represents the radiation factor, oblateness factor and eccentricity of the primaries respectively). If $\alpha = 0$ but $\sigma \neq 0$, the libration points slightly displaced to the right-side from its previous location and form scalene triangle with the primaries and go vertically downward as e increases. If $\alpha \neq 0$ and $\sigma \neq 0$, the libration points $L_{4,5}$ form scalene triangle with the primaries and as e increases $L_{4,5}$ move downward and displaced to the left-side. Also, the libration points $L_{4,5}$ are stable for the critical mass parameter $\mu \leq \mu_c$.

Keywords. Elliptic restricted three-body problem—radiation pressure—albedo effect—libration points—stability.

1. Introduction

Albedo effect is a non-gravitational force having significant effects on the motion of infinitesimal mass. According to Harris and Lyle (1969), albedo is the fraction of solar energy reflected diffusely from the planet back into space. It is the measure of the reflectivity of the planet's surface. Rocco (2009) defined the albedo as the fraction of incident solar radiation returned to the space from the surface of the planet, i.e.

$$Albedo = \frac{radiation\ reflected\ back\ to\ the\ space}{incident\ radiation}.$$

Albedo is dimensionless quantity and measured on a scale from 0 to 1. A body or surface has zero albedo means the body is 'black-body' which absorbs all the incident radiations while the unity albedo of a body represents a 'white-body' which is a perfect reflector that reflects all incident radiations completely and uniformly in all directions. A high albedo surface has the lower temperature because it reflects the majority of the radiation that hits it and absorbs the rest. On the other hand a low albedo surface has the higher temperature as it reflects a small amount of the incoming radiation and absorbs the rest. For instance, fresh snow has a high albedo of 0.95 as it reflects up to 95% of the incoming radiations while water reflects about 10% of the incoming radiation, resulting in a low albedo of 0.1. On an average the albedo of Earth is 0.3 as 30% of Sun's energy is reflected by the entire Earth. Generally, dark surfaces have a low albedo and light surfaces have a high albedo. The albedo is studied by Anselmo *et al.* (1983); Nuss (1998); McInnes (2000); Bhanderi (2005); Pontus (2005); Mac Donald (2011), Gong and Li (2015), Idrisi (2017), Idrisi and Ullah (2017), and others.

In the previous studies, authors did not consider the effect of reflected radiations upon the spacecraft in restricted problem of three or more bodies. As this effect is much lesser than the direct radiations effect known as photogravitaional effect, so generally it was neglected by the authors in the last decades. But if this effect is neglected it means the primaries are considered as black-bodies which is a contradiction to the fact that there is no planet in our solar system whose albedo is zero or no planet in our solar system is a black-body. The planets with their respective average albedo are as follows:

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
Albedo	0.12	0.75	0.30	0.16	0.34	0.34	0.30	0.29

In the light of all above facts, we have decided to develop a new model for elliptic restricted three-body problem in which one primary be a source of radiation and the other one a non-black oblate body. In this paper we are interested in investigating the albedo effect on the non-collinear libration points $L_{4,5}$. This paper is divided into seven sections. In Section 2, the equations of motion are derived. In Section 3, a relation between α and β has been established. The mean-motion of the primaries is obtained in Section 4. In Section 5, this is proved that there exist only two non-collinear libration points $L_{4,5}$ and are affected by albedo. In Section 6, the stability of non-collinear libration points $L_{4,5}$ has been discussed. In our solar system the Sun is a source of radiation, so we consider a real application to Sun–Earth system in Section 7 in which we have studied the albedo effect on the infinitesimal mass taking into account the oblateness of the Earth which is a suitable example as the real application concern.

2. Equations of motion

Let m_1 and m_2 ($m_1 > m_2$) be the masses of the primaries such that m_1 is spherical in shape and a source of radiation while m_2 is an oblate spheroid with axes a' and c', are moving in the elliptic orbits around their center of mass O. An infinitesimal mass $m_3 << 1$, is moving in the plane of motion of m_1 and m_2. The distances of m_3 from m_1, m_2 and O are $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{r}$ respectively. $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are the gravitational forces acting on m_3 due to m_1 and m_2 respectively, $\boldsymbol{F}_p$ is the force due to solar radiation pressure by m_1 on m_3 and $\boldsymbol{F}_A$ is the Albedo force due to solar radiation reflected by m_2 on m_3. Let the line joining m_1 and m_2 be taken as X-axis and O their center of mass as origin. Let the line passing through O and perpendicular to OX and lying in the plane of motion m_1 and m_2 be the Y-axis. Let us consider a synodic system of co-ordinates $Oxyz$ initially coincide with the inertial system $OXYZ$, rotating with angular velocity $\dot{f}$ about Z-axis (the z-axis is coincide with Z-axis). We wish to find the equations of motion of m_3 using the terminology of Szebehely (1967) in the synodic co-ordinate system and dimensionless variables *i.e*. the distance between the primaries is unity, the unit of time t is such that the gravitational constant $G = 1$ and the sum of the masses of the primaries is unity, i.e. $m_1 + m_2 = 1$.

The forces acting on m_3 due to m_1 and m_2 are $F_1\,(1 - F_p/F_1) = F_1\,(1-\alpha)$ and $F_2\,(1 - F_A/F_2) = F_2(1-\beta)$ respectively, where $\alpha = F_p/F_1 << 1$ and $\beta = F_A\,/\,F_2 << 1$. Also, α and β can be expressed as:

$$\alpha = \frac{L_1}{2\pi G m_1 c \sigma^*};\ \beta = \frac{L_2}{2\pi G m_2 c \sigma^*};$$

where L_1 is the luminosity of the larger primary m_1, L_2 is the luminosity of smaller primary m_2, G is the gravitational constant, c is the speed of light and σ^* is mass per unit area.

The equations of motion of infinitesimal mass $m_3 << 1$ in terms of pulsating coordinates (ξ, η) are given by

$$\xi'' - 2\eta' = \Omega^*_\xi,$$
$$\eta'' + 2\xi' = \Omega^*_\eta, \tag{1}$$

where

$$\Omega^* = \frac{1}{\sqrt{1-e^2}}\left[\frac{\xi^2+\eta^2}{2} + \frac{\Omega}{n^2}\right],$$
$$\Omega = \frac{(1-\mu)(1-\alpha)}{r_1} + \frac{\mu(1-\beta)}{r_2}\left(1 + \frac{\sigma}{2r_2^2}\right).$$

n = mean-motion of the primaries,
e = common eccentricity of elliptic orbit described by the primaries ($0 < e < 1$),
$\sigma = \frac{a'^2 - c'^2}{5}$ is the oblateness factor,

$$r_1^2 = (\xi - \mu)^2 + \eta^2, \tag{2}$$
$$r_2^2 = (\xi + 1 - \mu)^2 + \eta^2, \tag{3}$$

$0 < \mu = \frac{m_2}{m_1+m_2} < \frac{1}{2} \Rightarrow m_1 = 1 - \mu;\ m_2 = \mu$,
α is radiation factor and β is the albedo factor.

Note: For $\beta = 0$, i.e. m_2 is non-luminous, then the problem reduces to elliptic photogravitational restricted three body problem when smaller primary is an oblate spheroid. If α and β both are zero, then the problem becomes elliptic restricted three-body problem, when the smaller primary is an oblate spheroid.

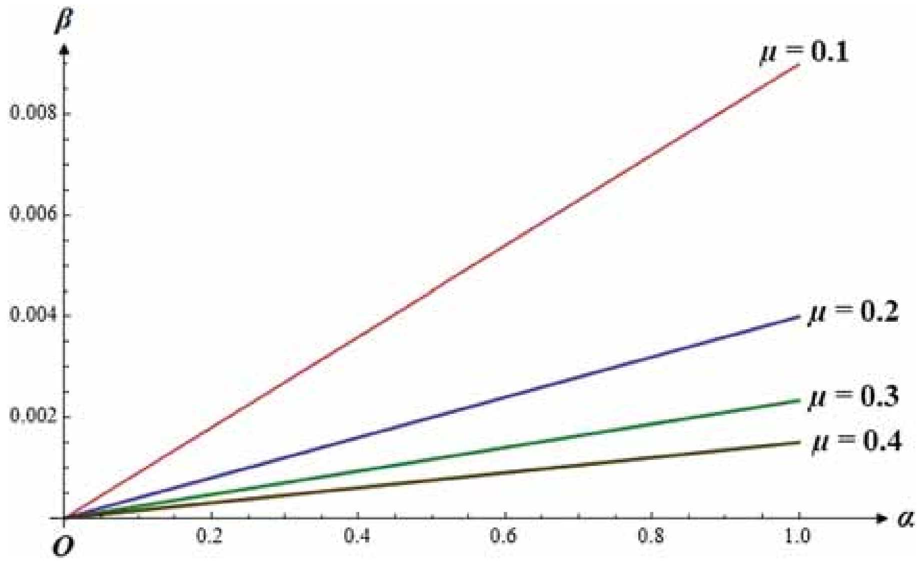

Figure 1. α versus β; $k = 0.001$.

Table 1. β in the interval $0 < \alpha < 1$; $k = 0.001$.

α		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
β	$\mu = 0.1$	0.0009	0.0018	0.0027	0.0036	0.0045	0.0054	0.0063	0.0072	0.0081	0.0090
	$\mu = 0.2$	0.0004	0.0008	0.0012	0.0016	0.0020	0.0024	0.0028	0.0032	0.0036	0.0040
	$\mu = 0.3$	0.0002	0.0004	0.0007	0.0009	0.0011	0.0014	0.0016	0.0018	0.0021	0.0023
	$\mu = 0.4$	0.0001	0.0003	0.0004	0.0006	0.0007	0.0009	0.0010	0.0012	0.0013	0.0015

3. Relation between β and α

In previous studies, many authors have taken both primaries as source of radiations and denoted these radiations factors as 'q_1' and 'q_2' but they did not establish any relation between these two factors. In this study we have shown a relation between these two factors and its effect on non-collinear libration points. Therefore,

$$\frac{\beta}{\alpha} = \frac{m_1}{m_2}\frac{L_2}{L_1} \Rightarrow \beta = \alpha\left(\frac{1-\mu}{\mu}\right)k,$$

$$k = \frac{L_2}{L_1} = \text{constant}, \; 0 < \alpha < 1 \text{ and } 0 < k < 1. \qquad (4)$$

From the relation given by equation (4), a graph between β and α is plotted (Figure 1) and the effect of α and μ is remarkable. It is observed that as α increase, β also increases for different values of μ but if α and μ increases simultaneously, β decreases. Also, β is evaluated for different values of μ and α in Table 1.

4. Mean-motion of the primaries

In the elliptic case, the distance between the primaries is $r = \frac{a(1-e^2)}{1+e\cos f}$,and the mean distance between the primaries is given by $\frac{1}{2\pi}\int_0^{2\pi} r df = \frac{a(1-e^2)}{\sqrt{1+e^2}}$, where a is semi-major axis of the elliptic orbit of one primary around the other. Since the orbits of the primaries with respect to centre of mass with semi-major axes $a_1 = am_2$ and $a_2 = am_1$ have the same eccentricity (Szebehely 1967), their equations of motion are given by

$$\frac{n^2 a_1(1-e^2)}{\sqrt{1+e^2}} = Gm_1m_2\left(1+\frac{3}{2}\sigma\right) \text{ and } \frac{n^2 a_2(1-e^2)}{\sqrt{1+e^2}} = Gm_2m_1\left(1+\frac{3}{2}\sigma\right).$$

The addition yields $n^2 = \frac{\sqrt{1+e^2}}{a(1-e^2)}\left(1+\frac{3}{2}\sigma\right)$. $m_1 + m_2 = 1$ and we choose the unit of time such that the gravitational constant $G = 1$. Consider $a = 1$ and only terms of e^2, and neglecting their product, we have

$$n^2 = 1 + \frac{3}{2}\left(\sigma + e^2\right). \qquad (5)$$

The mean-motion curve with respect to eccentricity 'e' for different values of oblateness factor 'σ' is plotted in Figure. 2 and it is observed as 'e' and 'σ' increases the mean-motion also increases.

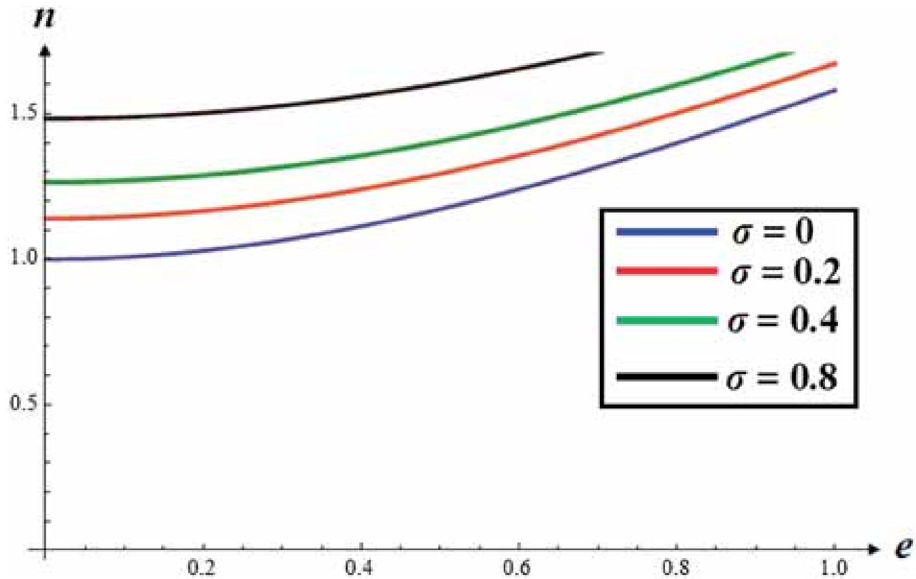

Figure 2. e versus n.

5. Non-collinear libration points

The non-collinear libration points are the solution of the equations for $\Omega^*_\xi = 0$ and $\Omega^*_\eta = 0$, $\eta \neq 0$, i.e.,

$$\xi - \frac{1}{n^2}\left\{\frac{(1-\mu)(\xi-\mu)(1-\alpha)}{r_1^3} + \frac{\mu(\xi+1-\mu)(1-\beta)}{r_2^3}\left(1+\frac{3\sigma}{2r_2^2}\right)\right\} = 0, \quad (6)$$

$$1 - \frac{1}{n^2}\left\{\frac{(1-\mu)(1-\alpha)}{r_1^3} + \frac{\mu(1-\beta)}{r_2^3}\left(1+\frac{3\sigma}{2r_2^2}\right)\right\} = 0. \quad (7)$$

On substituting $\sigma = 0$, $\alpha = 0$, $\beta = 0$ and $e = 0$, the solution of equations (6) and (7) is $r_1 = 1$, $r_2 = 1$ and from equation (3), $n = 1$. Now we assume that the solution of equations (6) and (7) for $\sigma_1 \neq 0$, $\sigma_2 \neq 0$, $\alpha \neq 0$ and $\beta \neq 0$ as $r_1 = 1 + \varepsilon_1$, $r_2 = 1 + \varepsilon_2$, ε_1, $\varepsilon_2 << 1$. Substituting these values of r_1 and r_2 in the equations (4) and (5), we get

$$\xi = \mu - \frac{1}{2} + \varepsilon_2 - \varepsilon_1$$
$$\eta = \pm\frac{\sqrt{3}}{2}\left[1 + \frac{2}{3}(\varepsilon_2 + \varepsilon_1)\right] \quad (8)$$

Now, substituting the values of $r_1 = 1+\varepsilon_1$, $r_2 = 1+\varepsilon_2$ and ξ, η from equations (8) to the equations (6) and (7) and neglecting second and higher order terms, we obtain

$$\varepsilon_1 = -\frac{1}{3}\alpha - \frac{1}{2}e^2 - \frac{1}{2}\sigma,$$
$$\varepsilon_2 = -\frac{1}{3}\beta - \frac{1}{2}e^2.$$

Thus, the coordinates of the non-collinear libration points $L_{4,5}$ are

$$\xi = \mu - \frac{1}{2} + \frac{(\alpha-\beta)}{3} + \frac{\sigma}{2},$$
$$\eta = \pm\frac{\sqrt{3}}{2}\left[1 - \frac{2}{3}\left\{\frac{(\alpha+\beta)}{3} + e^2 + \frac{\sigma}{2}\right\}\right]$$

Using the relation (4), i.e. $\beta = \alpha\,(1-\mu)\,k\,/\,\mu$, we have

$$\xi = \mu - \frac{1}{2} + \frac{\alpha}{3}\left[1 - \frac{(1-\mu)k}{\mu}\right] + \frac{\sigma}{2}, \quad (9)$$
$$\eta = \pm\frac{\sqrt{3}}{2}\left[1 - \frac{2}{3}\left\{\frac{\alpha}{3}\left(1 + \frac{(1-\mu)k}{\mu}\right) + \frac{\sigma}{2} + e^2\right\}\right] \quad (10)$$

Thus, there exist two non-collinear libration points $L_{4,5}$ forming a scalene triangle with the primaries as $r_1 \neq r_2$. The analytical solution of the equations (9 and 10) is given in Table 2.

- For $e = 0$, the results are in conformity with Idrisi et al. (2017).
- For $e = 0$ and $\sigma = 0$, the results are in conformity with Idrisi (2017).
- For $k = 0$, the results are agreed with Singh et al. (2012).

Table 2. Non-collinear libration points $L_{4,5}$ $(\xi, \pm\eta)$ for $\mu = 0.1$ and $k = 0.001$.

e	$\sigma = 0, \alpha = 0$		$\sigma = 0.1, \alpha = 0$		$\sigma = 0.1, \alpha = 0.2$	
	ξ	$\pm\eta$	ξ	$\pm\eta$	ξ	$\pm\eta$
0	−0.4	0.866025	−0.35	0.837158	−0.283933	0.798321
0.1	−0.4	0.860252	−0.35	0.831384	−0.284594	0.792936
0.2	−0.4	0.842931	−0.35	0.814064	−0.286576	0.776781
0.3	−0.4	0.814064	−0.35	0.785196	−0.289879	0.749855
0.4	−0.4	0.773649	−0.35	0.744782	−0.294504	0.712159
0.5	−0.4	0.721688	−0.35	0.692821	−0.300451	0.663693
0.6	−0.4	0.658179	−0.35	0.629312	−0.307717	0.604456
0.7	−0.4	0.583124	−0.35	0.554256	−0.316306	0.534451
0.8	−0.4	0.496521	−0.35	0.467654	−0.326216	0.453673
0.9	−0.4	0.398372	−0.35	0.369504	−0.337447	0.362125
1.0	−0.4	0.288675	−0.35	0.259808	−0.35	0.259808

- For $e = 0$ and $\alpha = 0$, the results are in conformity with those of Bhatnagar and Hallan (1979).
- For $e = 0$, $\sigma = 0$ and$k$ =0, the results are in agreement with Bhatnagar and Chawla (1979).
- For $e = 0$, $\sigma = 0$ and $\alpha = 0$, the results are totally agreed with Szebehely (1967).

When we consider the only effect of eccentricity e on the non-collinear libration points $L_{4,5}$, it is observed that the non-collinear libration points form an isosceles triangle with the primaries and as e increases the libration points $L_{4,5}$ move vertically downward (Figure 3(a)). Figure 3(b) shows the effect of oblateness and eccentricity on $L_{4,5}$, when we include the oblateness effect, the libration points slightly displaced to the right-side from its previous position (when $\sigma = 0$) and form scalene triangle with the primaries and as e increases the libration points $L_{4,5}$ move vertically downward. Figure 3(c) shows the albedo and oblateness effect on libration points $L_{4,5}$ with respect to e and it is observed that the libration points $L_{4,5}$ are forming scalene triangle with the primaries and as e increases the abscissa (ξ) and ordinate (η) of libration points $L_{4,5}$ decreases resulting$L_{4,5}$ move downward and displaced to the left-side.

6. Stability of libration points $L_{4,5}$

The variational equations are obtained by substituting $\xi = \xi_o + \varepsilon$ and $\eta = \eta_o + \delta$ in the equations of motion (equation 2), where (ξ_o, η_o) are the coordinates of libration points and $\varepsilon, \delta << 1$, i.e.

$$\begin{aligned} \varepsilon'' - 2\delta' &= \varepsilon\Omega^{*0}_{\xi\xi} + \delta\Omega^{*0}_{\xi\eta}, \\ \delta'' + 2\varepsilon' &= \varepsilon\Omega^{*0}_{\xi\eta} + \delta\Omega^{*0}_{\eta\eta}. \end{aligned} \tag{11}$$

Here we have taken only linear terms in ε and δ. The subscript in Ω^* indicates the second partial derivative of Ω^* and superscript o indicates that the derivative is to be evaluated at the libration point (ξ_o, η_o) .The characteristic equation corresponding to equation (11) is

$$\lambda^4 + \left(4 - \Omega^{*0}_{\xi\xi} - \Omega^{*0}_{\eta\eta}\right)\lambda^2 + \Omega^{*0}_{\xi\xi}\Omega^{*0}_{\eta\eta} - \left(\Omega^{*0}_{\xi\eta}\right)^2 = 0. \tag{12}$$

where

$$\begin{aligned} \Omega^{*0}_{\xi\xi} &= \frac{3}{4}\left[1 - \frac{2}{3}(1-3\mu)\alpha + \frac{2}{3}(2-3\mu)\beta\right] \\ &\quad -\frac{3}{4}\left[1 - 4\mu + \frac{1}{6}(3-19\mu) + \frac{1}{3}\mu\beta\right]\sigma \\ &\quad +\frac{9}{8}\left[1 - \frac{1}{9}(5-17\mu)\alpha + \frac{1}{9}(12-17\mu)\beta\right]e^2, \\ \Omega^{*0}_{\xi\eta} &= \frac{3\sqrt{3}}{2}\left[\mu - \frac{1}{2} + \frac{1}{9}(1+\mu)\alpha - \frac{1}{9}(2-\mu)\beta\right] \\ &\quad +\frac{\sqrt{3}}{4}\left[1 + 4\mu - \frac{1}{2}(1-\mu)\right. \\ &\quad \left.\alpha - \frac{1}{3}(4-21\mu)\beta\right]\sigma - \\ &\quad \frac{5\sqrt{3}}{8}\left[(1-2\mu) + \frac{1}{15}(9-13\mu)\alpha\right. \\ &\quad \left.+\frac{1}{15}(4-13\mu)\beta\right]e^2, \end{aligned}$$

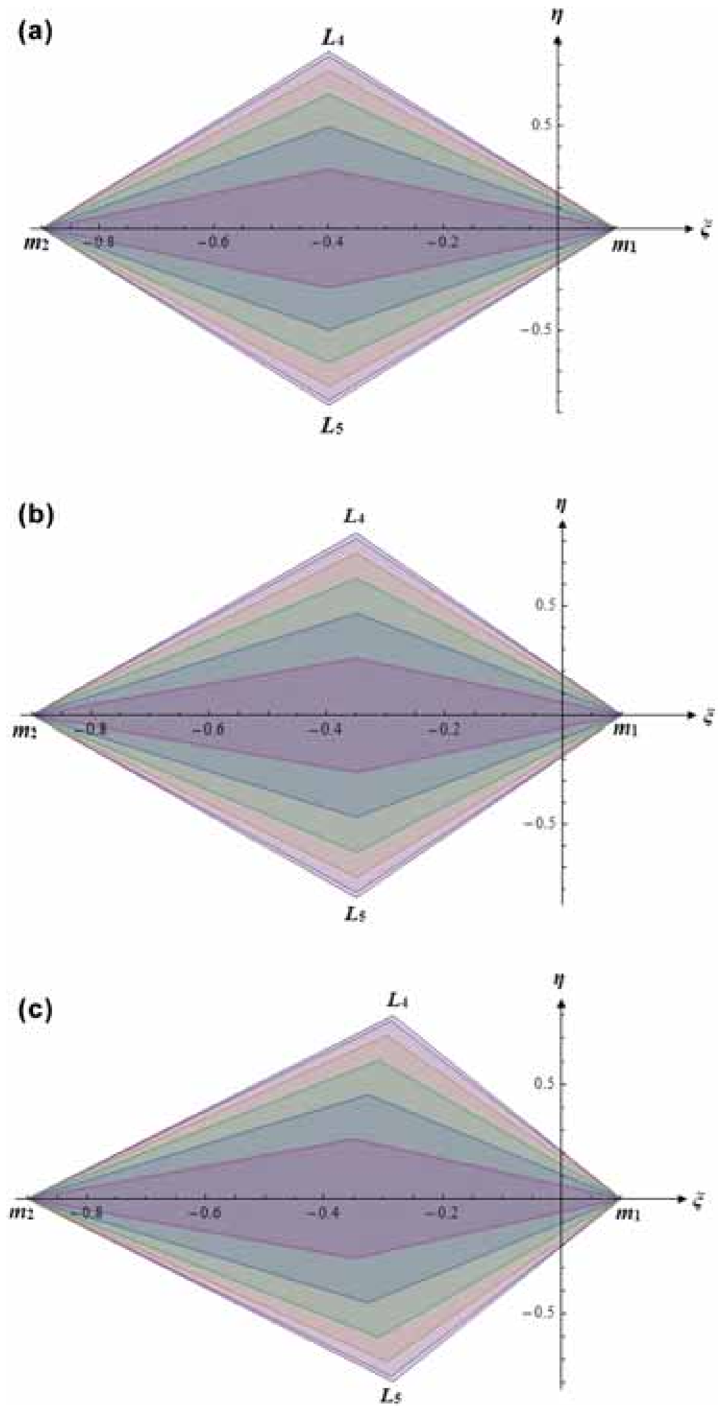

Figure 3. (*a*) $L_4, 5$ in $0 \le e \le 1$; $\mu = 0.1, k = 0.001, \sigma = 0, \alpha = 0$. (*b*) $L_4, 5$ in $0 \le e \le 1$; $\mu = 0.1, k = 0.001, \sigma = 0.1, \alpha = 0$. (*c*) $L_4, 5$ in $0 \le e \le 1$; $\mu = 0.1, k = 0.001, \sigma = 0.1, \alpha = 0.2$.

$$\begin{aligned}\Omega^{*0}_{\eta\eta} &= \frac{9}{4}\left[1 + \frac{2}{9}(1-3\mu)\alpha - \frac{2}{9}(2-3\mu)\beta\right] \\ &+ \frac{3}{4}\left[1 + \frac{1}{18}(29-45\mu)\alpha - \frac{1}{9}(8-39\mu)\beta\right]\sigma \\ &+ \frac{3}{8}\left[1 + \frac{1}{9}(19-39\mu)\alpha - \frac{1}{9}(20-39\mu)\beta\right]e^2.\end{aligned}$$

Let $\lambda^2 = \Pi$, therefore equation (12) becomes

$$\Pi^2 + q_1\Pi + q_2 = 0 \tag{13}$$

which is a quadratic equation in Π and its roots are given by

$$\Pi_{1,2} = \frac{1}{2}\left(-q_1 \pm \sqrt{D}\right) \tag{14}$$

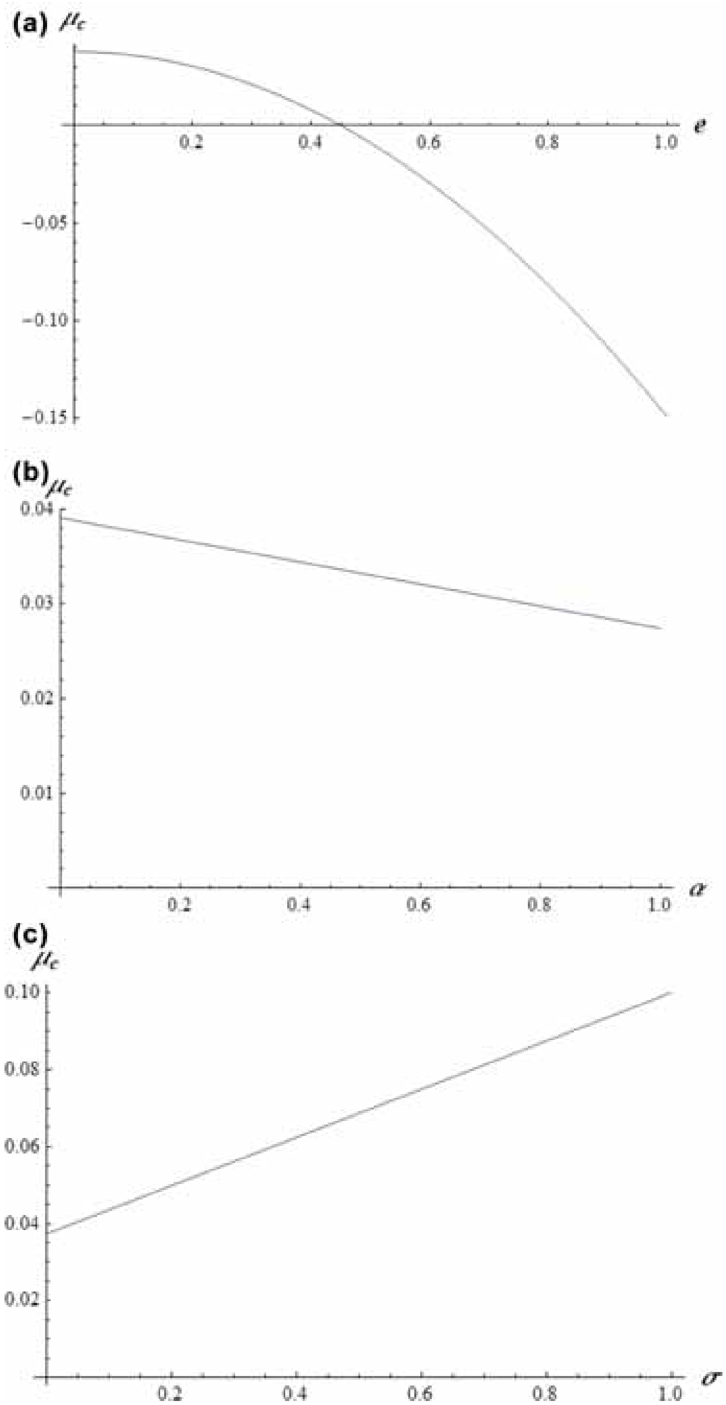

Figure 4. (*a*) e versus μ_c; $k = 0.001$; $\alpha = 0.1$; $\sigma = 0.01$. (*b*) α versus μ_c; $k = 0.001$; $e = 0.01$; $\sigma = 0.01$. (*c*) σ versus μ_c; $k = 0.001$; $\alpha = 0.1$; $e = 0.01$.

where $q_1 = 4 - \Omega^{*0}_{\xi\xi} - \Omega^{*0}_{\eta\eta}$; $q_2 = \Omega^{*0}_{\xi\xi}\Omega^{*0}_{\eta\eta} - \left(\Omega^{*0}_{\xi\eta}\right)^2$; $D = q_1^2 - 4q_2$.

The motion near the Libration point (ξ_o, η_o) is said to be bounded if $D \geq 0$, i.e.

$$\Big[1 - 27\mu + 27\mu^2 + \Big(9k - 3\frac{k}{\mu} - 6\mu - 12k\mu + 6\mu^2 + 6k\mu^2\Big)\alpha\Big]$$

$$+\left[-42\mu+36\mu^2+\left(\frac{13}{3}-\frac{130}{3}k+\frac{7}{3\mu}k-\frac{67}{2}\mu\right.\right.$$
$$\left.\left.+82k\mu+\frac{53}{2}\mu^2-41k\mu^2\right)\alpha\right]\sigma$$
$$+\left[-3-45\mu+45\mu^2+\left(\frac{17}{3}-\frac{11}{3}k-\frac{23}{6\mu}k-\frac{63}{2}\mu\right.\right.$$
$$\left.\left.+22k\mu+\frac{49}{2}\mu^2-\frac{29}{2}k\mu^2\right)\alpha\right]e^2\geq 0 \qquad (15)$$

For $\alpha = 0$, $\sigma = 0$ and $e = 0$, $\mu_c = 0.038520896504551\ldots$. is the critical value of mass parameter in classical case (Szebehely 1967). When $\alpha \neq 0$, $\sigma \neq 0$, $e \neq 0$, we suppose that $\mu_c = \mu^* + \gamma_1\alpha + \gamma_2\sigma + \gamma_{3-}\ e^2$ as the root of the equation (15), where, $\mu^* = 0.0385208965\ldots$ and $\gamma_1, \gamma_2, \gamma_3$ are to be determined in a manner such that $D = 0$. Therefore, we have

$$\gamma_1 = -\frac{k-3k\mu_o+2\mu_o^2+2k\mu_o^2}{9\mu_o(1-2\mu_o)},$$
$$\gamma_2 = \frac{2(7\mu_o-6\mu_o^2)}{9(1-2\mu_o)},\ \gamma_3 = -\frac{1+15\mu_o-15\mu_o^2}{9(1-2\mu_o)}.$$

Thus, $\mu_c = 0.0385208965\ldots - (0.00891747$
$+2.78224\ k)\alpha + 0.0627796\sigma$
$-0.187267e^2$ (16)

The critical mass parameter μ_c decreases as the eccentricity of the orbits of the primaries and luminosity factor α increase and is valid only in the range $0 \leq e \leq 0.45033$ and $0 \leq \alpha \leq 1$ respectively (Figure 4(a) and (b)), but μ_c increases uniformly as the oblateness factor σ increases (Figure 4(c)).

Table 3. Non-collinear libration points $L_{4,5}$ $(\xi, \pm\eta)$ in Sun–Earth system.

α	ξ	$\pm\eta$
0.0	−0.499997	0.865864
0.1	−0.466665	0.846619
0.2	−0.433333	0.827373
0.3	−0.400001	0.808127
0.4	−0.366671	0.788881
0.5	−0.333338	0.769635
0.6	−0.300006	0.750389
0.7	−0.266674	0.731143
0.8	−0.233342	0.711897
0.9	−0.200001	0.692652
1.0	−0.166678	0.673406

7. Application to real systems

Let us consider an example of the Sun–Earth system in the restricted three-body problem in which the smaller primary m_2 (oblate spheroid) is taken as the Earth and the bigger one m_1 as Sun. From the astrophysical data we have: Mass of the Sun $(m_1) = 1.9891 \times 10^{30}$ kg; Mass of the Earth $(m_2) = 5.9742 \times 10^{24}$ kg; axes of the Earth: $a = 6378.140$ km and $c = 6356.755$ km; mean distance of Earth from the Sun $= 1 = 1.5 \times$

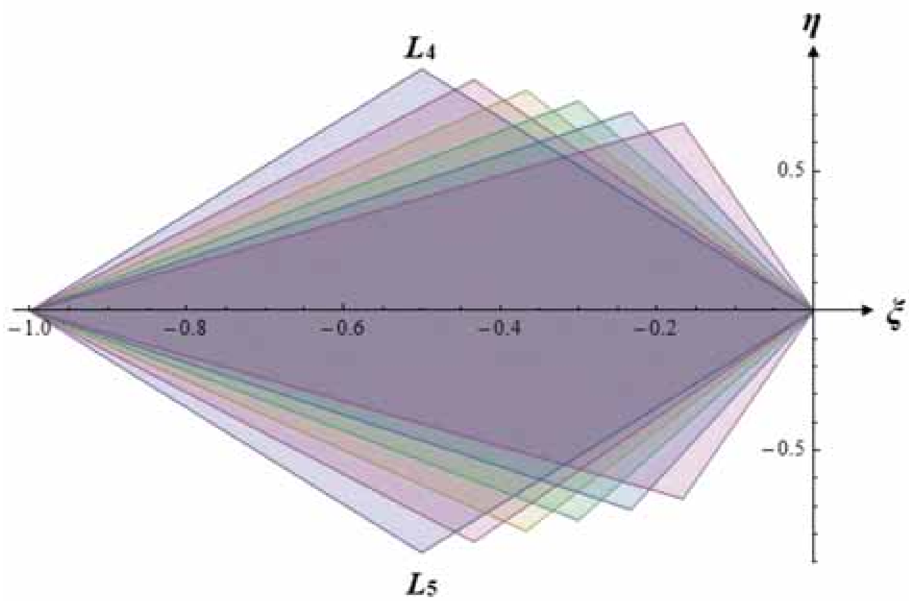

Figure 5. $L_{4,5}$ in Sun–Earth system; $0 \leq \alpha \leq 1$.

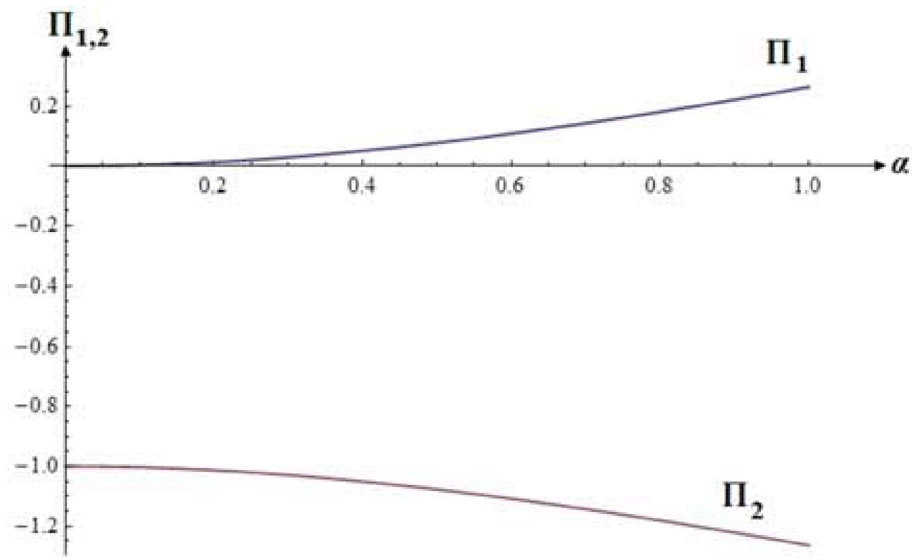

Figure 6. $\Pi_{1,2}$ in Sun–Earth system.

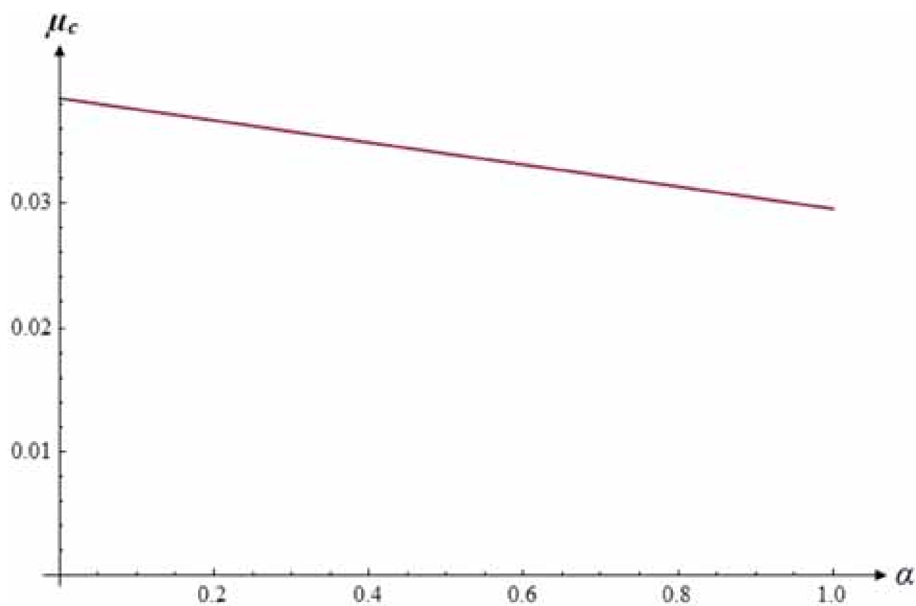

Figure 7. μ_c in Sun–Earth system.

10^{11} m; luminosity of Sun $= 3.9 \times 10^{26}$W; eccentricity of Earth $= 0.0167$; flux received by Earth from the Sun $= 1379$ W/m^2; albedo of Earth $= 0.3$, i.e. 30% of energy reflected back to space by the Earth, therefore the luminosity of Earth $= 5.2 \times 10^{16}$ W.

In dimensionless system, $\mu = 0.00000300346$, $a = 0.0000426352$, $c = 0.0000424923$, $e = 0.0167$, $k = 1.3 \times 10^{-10}$. Therefore, $\sigma = 2.43294 \times 10^{-12}$, $\beta = 0.0000443931\ \alpha$, and $n = 1 + 0.75\ e^2$.

From the equations (9) and (10), the non-collinear libration points obtained for various values of α in Sun-Earth system are given in Table 3.

From Table 3 and Figure 5, this is observed that the non-collinear libration points move towards the bigger primary m_1 as α increases in the interval [0, 1]. Also, as α increases the abscissa (ξ) of $L_{4,5}$ increases while the ordinate (η) decreases and hence, the shape of the scalene triangle formed by $L_{4,5}$ from the primaries m_1 and m_2 reduces.

Since $\Pi_1 < 0$ in the interval $0 \leq \alpha \leq 0.00723894$ and $\Pi_2 < 0$ in $0 \leq \alpha \leq 1$ (Fig. 6), the roots of the characteristic equation (13) are pure imaginary in the interval $0 \leq \alpha \leq 0.00723894$. Thus, the non-collinear libration points $L_{4,5}$ in Sun–Earth system are stable for

$0 \leq \alpha \leq 0.00723894$ and the value of critical mass parameter μ_c decreases as α increases (Figure 7).

8. Conclusion

In this paper we have established a relation between β and α and it is shown that as α increase, β also increases for different values of μ but if α and μ increases simultaneously, β decreases (Figure 1). The mean-motion of the primaries depends upon the eccentricity 'e' of the primaries and oblateness factor 'σ' and as e and σ increases the mean-motion also increases (Figure. 2). There exist two non-collinear libration points L_{45}, and the equations (9) and (10) represent the location of non-collinear libration points $L_{4,5}$ $(\xi, \pm\eta)$, i.e.

$$\xi = \mu - \frac{1}{2} + \frac{\alpha}{3}\left[1 - \frac{(1-\mu)k}{\mu}\right] + \frac{\sigma}{2},$$

$$\eta = \pm\frac{\sqrt{3}}{2}\left[1 - \frac{2}{3}\left\{\frac{\alpha}{3}\left(1 + \frac{(1-\mu)k}{\mu}\right) + \frac{\sigma}{2} + e^2\right\}\right]$$

and it is verified that for $k = 0$, the results are agreed with Singh et al. (2012); for $e = 0$, the results are in conformity with Idrisi et al. (2017); for $e = 0$ and $\sigma = 0$, the results are in conformity with Idrisi (2017); for $e = 0$, $\sigma = 0$ and $\alpha = 0$, the results are in total agreement with Szebehely (1967); for $e = 0$, $\sigma = 0$ and $k = 0$, the results are in agreement with Bhatnagar and Chawla (1979); for $e = 0$ and $\alpha = 0$, the results are in conformity with those of Bhatnagar and Hallan (1979). When we consider the only effect of eccentricity e on the non-collinear libration points $L_{4,5}$, it is observed that the non-collinear libration points form an isosceles triangle with the primaries and as e increases the libration points $L_{4,5}$ move vertically downward (Figure 3(a)). Figure 3(b) shows the effect of oblateness and eccentricity on $L_{4,5}$, when we include the oblateness effect, the libration points slightly displaced to the right-side from its previous position (when $\sigma = 0$) and form scalene triangle with the primaries and as e increases the libration points $L_{4,5}$ move vertically downward. Figure 3(c) shows the albedo and oblateness effect on libration points $L_{4,5}$ with respect to e and it is observed that the libration points $L_{4,5}$ are forming scalene triangle with the primaries and as e increases the abscissa (ξ) and ordinate (η) of libration points $L_{4,5}$ decreases resulting $L_{4,5}$ move downward and displaced to the left-side. Finally, the libration points $L_{4,5}$ are stable for the critical mass parameter $\mu_c = 0.0385208965\ldots - (0.00891747 + 2.78224\ \mathrm{k})\ \alpha + 0.0627796\,\sigma - 0.187267\,e^2$ and the critical mass parameter μ_c decreases as the eccentricity of the orbits of the primaries and luminosity factor α increase and is valid only in the range $0 \leq e \leq 0.45033$ and $0 \leq \alpha \leq 1$ respectively (Figure 4(a) and (b)) but μ_c increases uniformly as the oblateness factor σ increases (Figure 4(c)). Also, an example of Sun-Earth system is taken in the previous section as a real application and this is observed that the non-collinear libration points move towards the bigger primary m_1 as α increases in the interval [0, 1]. Also, as α increases the abscissa (ξ) of $L_{4,5}$ increases while the ordinate (η) decreases and hence, the shape of the scalene triangle formed by $L_{4,5}$ from the primaries m_1 and m_2 reduces. Since $\Pi_1 < 0$ in the interval $0 \leq \alpha \leq 0.00723894$ and $\Pi_2 < 0$ in $0 \leq \alpha \leq 1$ (Figure 6), the roots of the characteristic equation (13) are pure imaginary in the interval $0 \leq \alpha \leq 0.00723894$. Thus, the non-collinear libration points $L_{4,5}$ in Sun–Earth system are stable for $0 \leq \alpha \leq 0.00723894$ and the value of critical mass parameter μ_c decreases as α increases (Figure 7).

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