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Maximal regularity for non-autonomous stochastic evolution equations in UMD Banach spaces

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Abstract. A non-autonomous stochastic linear evolution equation in UMD Banach spaces of type 2 is considered. We construct unique strict solutions to the equation and show their maximal regularity. The abstract results are then applied to a stochastic partial differential equation.

Keywords. Evolution operators; stochastic linear evolution equations; strict solutions; maximal regularity; UMD Banach spaces of type 2.

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1. Introduction

We consider the Cauchy problem for a non-autonomous stochastic linear evolution equation

$$\begin{cases} dX + A(t)Xdt = F(t)dt + G(t)dW(t), & 0 < t \le T, \\ X(0) = \xi \end{cases} \tag{1.1}$$

in a separable UMD Banach space E of type 2 with norm $\|\cdot\|$. Here, W denotes a cylindrical Wiener process on a separable Hilbert space H, and is defined on a filtered, complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$. Operator G is an H-strongly measurable and adapted process in $L^2((0,T)\times\Omega; \gamma(H;E))$ (see subsection 2.2), whereas F is a progressively E-valued measurable process on $[0,T]$. Initial value ξ is an E-valued $\mathcal{F}_0$-measurable random variable. And $A(t), 0 \le t \le T$ are densely defined, closed linear operators in E and satisfy the condition:

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(A1) The spectrum $\sigma(A(t))$ of $A(t)$ is contained in a sectorial open domain:

$$\sigma(A(t)) \subset \Sigma_\varpi = \{\lambda \in \mathbb{C} : |\arg \lambda| < \varpi\}, \qquad 0 \le t \le T$$

with some angle $0 < \varpi < \frac{\pi}{2}$. The resolvent of $A(t)$ satisfies the estimate:

$$\|(\lambda - A(t))^{-1}\| \le \frac{M_\varpi}{|\lambda|}, \qquad \lambda \notin \Sigma_\varpi, 0 \le t \le T$$

with some $M_\varpi > 0$.

Assume further that

(A2) The domain $\mathcal{D}(A(t))$ of $A(t)$ is allowed to vary with t, but there exists $0 < \nu \le 1$ such that

$$\mathcal{D}(A(s)) \subset \mathcal{D}(A(t)^\nu), \qquad 0 \le s, t \le T.$$

(A3) $A(\cdot)^{-1}$ is Hölder continuous in t in the sense that

$$\|A(t)^\nu [A(t)^{-1} - A(s)^{-1}]\| \le N(t-s)^\mu, \qquad 0 \le s \le t \le T$$

with some $1 - \nu < \mu \le 1$ and $N > 0$.

The basic problem on existence and uniqueness of solution to (autonomous) stochastic evolution equations has been extensively studied since 1970s. The main interest was to construct unique mild solutions, see [2,3,8,9,11] and references therein. Strict solutions which have stronger regularity than mild solutions have also been shown in some special cases, e.g., the case where A is a bounded linear operator [3].

In the meantime, non-autonomous stochastic evolution equations have been studied by a few researchers. Veraar [12] investigated an equation of the form

$$dX + A(t)Xdt = F(t, X)dt + G(t, X)dW(t)$$

in a UMD Banach space of type 2. It is assumed that $A(t)$, $0 \le t \le T$, satisfy the conditions given in [1] (see also [4–6]). These conditions highlight that the domain $\mathcal{D}(A(t))$ of $A(t)$ completely varies in time. In other words, there is no relation between $\mathcal{D}(A(t))$ and $\mathcal{D}(A(s))$ for every $0 \le t \ne s \le T$ (i.e. $\nu = 0$ in (A2)). The author also required a very common condition, say Lipschitz continuity of F and G. The existence of mild solutions has been then verified.

Mild solutions to the latter equation have also been considered in an earlier work of Seidler [7], but in Hilbert space setting. The author supposed that the domain $\mathcal{D}(A(t))$ of $A(t)$ is constant in time, i.e., $\nu = 1$ in (A2).

In [10], we considered (1.1) in Banach spaces of M-type 2, where W is a standard Brownian motion. We constructed unique strict solutions from a 'solution formula'.

In this paper, we consider (1.1) in UMD Banach spaces of type 2, where W is a Wiener process depending on both time and space. In other words, W is a cylindrical Wiener process on a Hilbert space. We show the existence of an unique strict solution and the maximal regularity of the solutions. Our results are even new in Hilbert space setting.

The paper is organized as follows. Section 2 is preliminary. Section 3 investigates existence of unique strict solution and the maximal regularity for (1.1). Section 4 gives an example to illustrate our abstract results.

2. Preliminary results

2.1 *UMD Banach spaces of type* 2

DEFINITION 2.1

(i) A Banach space E is called a UMD space if for some (equivalently, for all) $1 < p < \infty$ there is a constant $c_p(E)$ such that for any L^p-integrable E-valued martingale difference $\{M_n\}_n$ (i.e. $\{\sum_{i=1}^n M_i\}_{n=1}^\infty$ is a martingale) and any $\epsilon\colon \{1, 2, 3, \ldots\} \to \{-1, 1\}$,

$$\mathbb{E}\left\|\sum_{i=1}^n \epsilon(i)M_n\right\|^p \le c_p(E)\mathbb{E}\left\|\sum_{j=1}^n M_j\right\|^p, \qquad n = 1, 2, 3, \ldots.$$

(ii) A Banach space E is said to be of type 2 if there exists $c_2(E) > 0$ such that for any Rademacher sequence $\{\epsilon_i\}_i$ on a complete probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and any finite sequence $\{x_k\}_{k=1}^n$ of E,

$$\mathbb{E}'\left\|\sum_{i=1}^n \epsilon_i x_i\right\|^2 \le c_2(E)\sum_{i=1}^n \|x_i\|^2.$$

(Recall that a Rademacher sequence is a sequence of independent symmetric random variables each one taking on the values $\{1, -1\}$.)

Example 2.2. Hilbert spaces are UMD spaces of type 2. The L^p spaces are UMD spaces with $1 < p < \infty$. They are of type 2 with $2 \le p < \infty$.

From now on, we always assume that E is a separable UMD Banach space of type 2 and H is a separable Hilbert space.

2.2 γ-*Radonifying operators*

Let us review the notion of γ-radonifying operators. For details on the subject, we refer to [11].

DEFINITION 2.3

Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis of H. Let $\{\gamma_n\}_{n=1}^\infty$ be a sequence of independent standard Gaussian random variables on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. A γ-radonifying operator from H to E is an operator, denoted by ϕ for example, in $L(H; E)$ such that the Gaussian series $\sum_{n=1}^\infty \gamma_n \phi e_n$ converges in $L^2(\Omega', E)$.

Denote by $\gamma(H; E)$ the set of all γ-radonifying operators from H to E. Define a norm in $\gamma(H; E)$ by

$$\|\phi\|_{\gamma(H;E)} = \left[\mathbb{E}'\left\|\sum_{n=1}^\infty \gamma_n \phi e_n\right\|^2\right]^{\frac{1}{2}}, \qquad \phi \in \gamma(H; E).$$

It is known that the norm is independent of the orthonormal basis $\{e_n\}_{n=1}^\infty$ and the Gaussian sequence $\{\gamma_n\}_{n=1}^\infty$. Furthermore, $(\gamma(H; E), \|\cdot\|_{\gamma(H;E)})$ is complete.

Remark 2.4. When E is a Hilbert space, the space $(\gamma(H;E), \|\cdot\|_{\gamma(H;E)})$ is isometrical to the space $L_2(H;E)$ of Hilbert–Schmidt operators.

Let (S,Σ) be a measurable space. A function $\varphi\colon S \to E$ is said to be strongly measurable if it is the pointwise limit of a sequence of simple functions. A function $\phi\colon S \to L(H;E)$ is said to be H-strongly measurable if $\phi(\cdot)h\colon S \to E$ is strongly measurable for all $h \in H$.

Denote by $\mathcal{N}^2([0,T])$ the class of all H-strongly measurable and adapted processes $\phi\colon [0,T]\times\Omega \to \gamma(H;E)$ in $L^2((0,T)\times\Omega;\gamma(H;E))$.

The following result are very often used in this paper.

Lemma 2.5. *Let $\phi_1 \in L(E)$ and $\phi_2 \in \gamma(H;E)$. Then, $\phi_1\phi_2 \in \gamma(H;E)$ and*

$$\|\phi_1\phi_2\|_{\gamma(H;E)} \le \|\phi_1\|_{L(E)}\|\phi_2\|_{\gamma(H;E)}.$$

2.3 *Cylindrical Wiener process and stochastic integrals*

DEFINITION 2.6

A family $W = \{W(t)\}_{t\ge 0}$ of bounded linear operators from H to $L^2(\Omega)$ is called a cylindrical Wiener process on H if

(i) $Wh = \{W(t)h\}_{t\ge 0}$ is a scalar Wiener process on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge 0},\mathbb{P})$ for all $h \in H$
(ii) $\mathbb{E}[W(t_1)h_1W(t_2)h_2] = \min\{t_1,t_2\}\langle h_1,h_2\rangle_H$, $0 \le t_1,t_2 < \infty$, $h_1,h_2 \in H$

For each $\phi \in \mathcal{N}^2([0,T])$, the stochastic integral $\int_0^T \phi(t)dW(t)$ is defined as a limit of integrals of adapted step processes. By a localization argument stochastic integrals can be extended to the class $\mathcal{N}([0,T])$ of all H-strongly measurable and adapted processes $\phi\colon [0,T]\times\Omega \to \gamma(H;E)$ which are in $L^2((0,T);\gamma(H;E))$ a.s. (see [11]).

Theorem 2.7. *There exists $c(E) > 0$ depending only on E such that*

$$\mathbb{E}\left\|\int_0^T \phi(t)dW(t)\right\|^2 \le c(E)\|\phi\|^2_{L^2((0,T)\times\Omega;\gamma(H;E))}, \qquad \phi \in \mathcal{N}^2([0,T]),$$

here $\|\phi\|^2_{L^2((0,T)\times\Omega;\gamma(H;E))} = \int_0^T \mathbb{E}\|\phi(s)\|^2_{\gamma(H;E)}ds$. In addition, for any ϕ in $\mathcal{N}^2([0,T])$ (resp. $\mathcal{N}([0,T])$), $\{\int_0^t \phi(s)dW(s), 0 \le t \le T\}$ is an E-valued continuous martingale (resp. local martingale) and a Gaussian process.

For the proof, see, e.g., [11].

PROPOSITION 2.8

Let B be a closed linear operator on E and $\phi\colon [0,T]\times\Omega \to \gamma(H;E)$. If both ϕ and $B\phi$ belong to $\mathcal{N}^2([0,T])$, then

$$B\int_0^T \phi(t)dW(t) = \int_0^T B\phi(t)dW(t) \qquad a.s.$$

The proof of Proposition 2.8 is very similar to the one in [3]. So, we omit it.

Let us finally restate the Kolmogorov continuity theorem. It gives a sufficient condition for a stochastic process to be Hölder continuous. For $\sigma > 0$ and an interval $I \subset \mathbb{R}$, denote by $\mathcal{C}^\sigma(I; E)$ the space of all E-valued functions which are Hölder continuous with exponent σ.

Theorem 2.9. *Let ζ be an E-valued stochastic process on $[0, T]$. Assume that for some $c > 0$, $\epsilon_1 > 1$ and $\epsilon_2 > 0$,*

$$\mathbb{E}\|\zeta(t) - \zeta(s)\|^{\epsilon_1} \le c(t-s)^{1+\epsilon_2}, \qquad 0 \le s \le t \le T. \tag{2.1}$$

Then, for any $0 < \gamma < \frac{\epsilon_2}{\epsilon_1}$,

$$\zeta \in \mathcal{C}^\gamma([0, T]; E) \qquad a.s.$$

When ζ is a Gaussian process, the condition (2.1) can be weakened.

Theorem 2.10. *Let ζ be an E-valued Gaussian process on $[0, T]$ such that $\mathbb{E}\zeta(t) = 0$ for $t \ge 0$. Assume that for some $c > 0$ and $0 < \epsilon \le 1$,*

$$\mathbb{E}\|\zeta(t) - \zeta(s)\|^2 \le c(t-s)^\epsilon, \qquad 0 \le s \le t \le T.$$

Then, for any $0 < \gamma < \frac{\epsilon}{2}$,

$$\zeta \in \mathcal{C}^\gamma([0, T]; E) \qquad a.s.$$

For the proofs of Theorems 2.9 and 2.10, see, e.g., [3].

2.4 *Deterministic linear evolution equations*

Consider the Cauchy problem for a deterministic linear evolution equation

$$\begin{cases} \frac{dX}{dt} + A(t)X = F(t), & 0 < t \le T, \\ X(0) = \xi \end{cases} \tag{2.2}$$

in E. Here, $A(t), t \ge 0$ are densely defined, closed linear equations in E satisfying the conditions (A1)–(A3). Meanwhile, F is an E-valued function on $[0, T]$ belonging to $\mathcal{F}^{\beta,\sigma}((0, T]; E)$ for some $0 < \beta \le 1$ and $0 < \sigma < \min\{\beta, \mu + \nu - 1\}$.

Here, $\mathcal{F}^{\beta,\sigma}((0, T]; E)$ denotes a weighted Hölder continuous function space introduced in [14] which consists of all E-valued continuous functions f on $(0, T]$ (resp. $[0, T]$) when $0 < \beta < 1$ (resp. $\beta = 1$) with the properties:

(i) When $\beta < 1$, $t^{1-\beta} f(t)$ has a limit as $t \to 0$;

(ii) f is Hölder continuous with exponent σ and weight $s^{1-\beta+\sigma}$, i.e.

$$\sup_{0\le s<t\le T}\frac{s^{1-\beta+\sigma}\|f(t)-f(s)\|}{(t-s)^{\sigma}} = \sup_{0\le t\le T}\sup_{0\le s<t}\frac{s^{1-\beta+\sigma}\|f(t)-f(s)\|}{(t-s)^{\sigma}} < \infty;$$

(iii)

$$\lim_{t\to 0} w_f(t) = 0,$$

where $w_f(t) = \sup_{0\le s<t}\frac{s^{1-\beta+\sigma}\|f(t)-f(s)\|}{(t-s)^{\sigma}}$.

It is clear that $\mathcal{F}^{\beta,\sigma}((0,T];E)$ is a Banach space with norm

$$\|f\|_{\mathcal{F}^{\beta,\sigma}(E)} = \sup_{0\le t\le T} t^{1-\beta}\|f(t)\| + \sup_{0\le s<t\le T}\frac{s^{1-\beta+\sigma}\|f(t)-f(s)\|}{(t-s)^{\sigma}}.$$

(For simplicity, we denote the norm by $\|\cdot\|_{\mathcal{F}^{\beta,\sigma}}$)
In addition, for $f\in\mathcal{F}^{\beta,\sigma}((0,T];E)$,

$$\begin{aligned}\|f(t)\| &\le \|f\|_{\mathcal{F}^{\beta,\sigma}} t^{\beta-1}, && 0<t\le T,\\ \|f(t)-f(s)\| &\le w_f(t)(t-s)^{\sigma}s^{\beta-\sigma-1}\\ &\le \|f\|_{\mathcal{F}^{\beta,\sigma}}(t-s)^{\sigma}s^{\beta-\sigma-1}, && 0<s\le t\le T. \end{aligned}\tag{2.3}$$

According to [13,14], the following results are known.

Theorem 2.11. *Let* (A1), (A2) *and* (A3) *be satisfied. Then, there exists a unique evolution operator* $U(t,s)$, $0\le s\le t\le T$, *having the following properties*:

(*i*) $U(t,s)$ *is a bounded linear operator on* E *with*

$$\begin{cases} U(t,r)=U(t,s)U(s,r), & 0\le r\le s\le t\le T,\\ U(s,s)=I, & 0\le s\le T. \end{cases}$$

(*ii*) *For* $0\le\theta<\mu+\nu$, $\mathcal{R}(U(t,s))\subset\mathcal{D}(A(t)^{\theta})$ *and there exists* $\iota_\theta>0$ *such that*

$$\|A(t)^{\theta}U(t,s)\|\le\iota_\theta(t-s)^{-\theta}, \qquad 0\le s<t\le T. \tag{2.4}$$

When $0\le\theta\le 1$,

$$\|A(t)^{-\theta}\|\le\iota_\theta, \qquad 0\le t\le T. \tag{2.5}$$

(*iii*) *For* $0\le\theta_1\le 1$ *and* $\theta_1\le\theta_2<\mu+\nu$, *there exists* $\kappa_{\theta_2}>0$ *such that*

$$\|A(t)^{\theta_2}U(t,s)A(s)^{\theta_1-\theta_2}\|\le\kappa_{\theta_2}(t-s)^{-\theta_1}, \qquad 0\le s<t\le T. \tag{2.6}$$

(*iv*) *There exists* $c_{\mu,\nu} > 0$ *such that*

$$\begin{aligned} &\|A(t)U(t,s)A(s)^{-1} - e^{-(t-s)A(s)}\| \\ &\quad \le c_{\mu,\nu}(t-s)^{\mu+\nu-1}, \qquad 0 \le s < t \le T, \end{aligned} \tag{2.7}$$

where $\{e^{-\tau A(s)}\}_{\tau \ge 0}$ *is the semigroup generated by* $A(s)$.

Theorem 2.12. *Let* (A1), (A2), (A3) *be satisfied. Let*

$$F \in \mathcal{F}^{\beta,\sigma}((0,T];E), \quad \xi \in \mathcal{D}(A(0)^{\beta})$$

for some $0 < \beta \le 1$ *and* $0 < \sigma < \min\{\beta, \mu+\nu-1\}$. *Then, there exists a unique solution* X *to* (2.2) *having the regularity*:

$$\begin{aligned} &A^{\beta}X \in \mathcal{C}([0,T];E), \\ &\frac{dX}{dt}, AX \in \mathcal{F}^{\beta,\sigma}((0,T];E) \end{aligned}$$

together with the estimate

$$\left|\left|\frac{dX}{dt}\right|\right|_{\mathcal{F}^{\beta,\sigma}} + \|A^{\beta}X\|_{\mathcal{C}} + \|AX\|_{\mathcal{F}^{\beta,\sigma}} \le C[\|A(0)^{\beta}\xi\| + \|F\|_{\mathcal{F}^{\beta,\sigma}}],$$

where $C > 0$ *is some constant. Furthermore,* X *is necessarily given by the formula*

$$X(t) = U(t,0)\xi + \int_0^t U(t,s)F(s)ds, \qquad 0 \le t \le T.$$

3. Maximal regularity

In this section, we construct unique strict solutions and show the maximal regularity for (1.1). Let us introduce a definition of strict solutions to (1.1).

DEFINITION 3.1

An E-valued continuous process X on $[0,T]$ is called a strict solution of (1.1) if

$$\begin{aligned} &X(t) \in \mathcal{D}(A(t)) \qquad \text{a.s.} \quad 0 < t \le T, \\ &\int_0^T \|A(s)X(s)\|ds < \infty \qquad \text{a.s.} \end{aligned}$$

and

$$X(t) = \xi - \int_0^t A(s)X(s)ds + \int_0^t F(s)ds + \int_0^t G(s)dW(s) \qquad \text{a.s.} \quad 0 < t \le T.$$

A strict solution X on $[0, T]$ is said to be unique if any other strict solution $\bar{X}$ on $[0, T]$ is indistinguishable from it, which means that

$$\mathbb{P}\{X(t) = \bar{X}(t) \text{ for every } 0 \le t \le T\} = 1.$$

3.1 *Case where* $0 < \nu < 1$

Consider the case where $0 < \nu < 1$ (ν is the exponent in (A2)). Assume that

(F) $F \in \mathcal{F}^{\beta,\sigma}((0, T]; E)$ a.s.,
where $0 < \beta \le 1$ and $0 < \sigma < \min\{\beta, \mu + \nu - 1\}$,

and

(G) there exist $\delta > \frac{1}{2}$ and a square integrable random variable ζ such that

$$G(t) \in \mathcal{D}(A(t)^\delta) \quad \text{a.s.,} \quad 0 \le t \le T$$

and

$$\|A(t)^\delta G(t) - A(s)^\delta G(s)\|_{\gamma(H;E)} \le \zeta(t - s)^\sigma \quad \text{a.s., } 0 \le s \le t \le T.$$

In addition, $\mathbb{E}\|A(0)^\delta G(0)\|^2_{\gamma(H;E)} < \infty$.

Theorem 3.2. *Let* (A1), (A2), (A3), (F) *and* (G) *be satisfied. Assume that* $\xi \in \mathcal{D}(A(0)^\beta)$ *a.s. and* $\mathbb{E}\|A(0)^\beta \xi\|^2 < \infty$. *Then, there exists a unique strict solution of* (1.1) *possessing the regularities*:

$$A^\beta X \in \mathcal{C}([0, T]; E), \quad X \in \mathcal{C}^{\gamma_1}([0, T]; E) \quad \text{a.s.},$$

and

$$AX \in \mathcal{C}^{\gamma_2}([\epsilon, T]; E) \quad \text{a.s.}$$

for any $0 < \gamma_1 \le \min\{\beta, \frac{1}{2}\}$, $\gamma_1 \ne \frac{1}{2}$, $0 < \gamma_2 < \min\{\delta - \frac{1}{2}, \sigma\}$ *and* $0 < \epsilon \le T$. *In addition,*

(i) *when* $\beta \ge \delta$,

$$\begin{aligned}&\mathbb{E}\|A(t)^\beta X(t)\|^2 \\ &\quad \le C[\mathbb{E}\|A(0)^\beta \xi\|^2 + \mathbb{E}\|F\|^2_{\mathcal{F}^{\beta,\sigma}} + \mathbb{E}\|A(0)^\delta G(0)\|^2_{\gamma(H;E)} t^{1-2(\beta-\delta)} \\ &\qquad + \mathbb{E}\|\zeta\|^2 t^{1-2(\beta-\delta)+2\sigma}], \qquad 0 \le t \le T, \end{aligned} \tag{3.1}$$

(ii) *when* $\beta < \delta$,

$$\begin{aligned}&\mathbb{E}\|A(t)^\beta X(t)\|^2 \\ &\quad \le C[\mathbb{E}\|A(0)^\beta \xi\|^2 + \mathbb{E}\|F\|^2_{\mathcal{F}^{\beta,\sigma}} + \mathbb{E}\|A(0)^\delta G(0)\|^2_{\gamma(H;E)} t \\ &\qquad + \mathbb{E}\|\zeta\|^2 t^{1+2\sigma}], \qquad 0 \le t \le T. \end{aligned} \tag{3.2}$$

Here, C is some constant depending on the exponents. Furthermore, if $G(0) = 0$, *then for any* $0 < \gamma_1 \le \min\{\frac{1+\sigma}{2}, \delta, \beta\}$, $\gamma_1 \notin \{\frac{1+\sigma}{2}, \delta\}$,

$$X \in \mathcal{C}^{\gamma_1}([0,T];E) \qquad a.s.$$

Proof. Since the proof of uniqueness of strict solutions is very similar to [10, Theorem 3.2], we omit that proof. We prove the existence and the regularity of solutions by using the solution formula and the Kolmogorov continuity theorem.

For $0 \le \theta \le 1$, denote by $(A^\theta U)\diamond G(t)$ the stochastic convolution between $A(t)^\theta U(t,\cdot)$ and G, i.e.

$$(A^\theta U)\diamond G(t) = \int_0^t A(t)^\theta U(t,s)G(s)dW(s), \qquad 0 \le t \le T.$$

Let us divide the proof into several steps. Throughout the proof, denote by C^* some universal constant.

Step 1. Let us verify that $(A^\theta U)\diamond G$ is well-defined on $[0,T]$ and satisfies

$$(A^\theta U)\diamond G(t) = A(t)^\theta [U\diamond G](t).$$

From (G), it follows that

$$\|A(t)^\delta G(t)\|_{\gamma(H;E)} \le \|A(0)^\delta G(0)\|_{\gamma(H;E)} + \zeta t^\sigma, \qquad 0 \le t \le T. \tag{3.3}$$

Therefore,

$$\begin{aligned}
&\int_0^t \mathbb{E}\|A(t)^\theta U(t,s)G(s)\|^2_{\gamma(H;E)}ds \\
&= \int_0^t \mathbb{E}\|A(t)^\theta U(t,s)A(s)^{-\delta}A(s)^\delta G(s)\|^2_{\gamma(H;E)}ds \\
&\le \int_0^t \|A(t)^\theta U(t,s)A(s)^{-\delta}\|^2 \mathbb{E}\|A(s)^\delta G(s)\|^2_{\gamma(H;E)}ds \\
&\le 2\int_0^t \|A(t)^\theta U(t,s)A(s)^{-\delta}\|^2 [\mathbb{E}\|A(0)^\delta G(0)\|^2_{\gamma(H;E)} + \mathbb{E}\zeta^2 s^{2\sigma}]ds.
\end{aligned}$$

If $\theta \ge \delta$, then (2.6) gives

$$\begin{aligned}
&\int_0^t \mathbb{E}\|A(t)^\theta U(t,s)G(s)\|^2_{\gamma(H;E)}ds \\
&\le 2\int_0^t \|A(t)^\theta U(t,s)A(s)^{\theta-\delta-\theta}\|^2 [\mathbb{E}\|A(0)^\delta G(0)\|^2_{\gamma(H;E)} + \mathbb{E}\zeta^2 s^{2\sigma}]ds \\
&\le 2\kappa_\theta^2 \int_0^t (t-s)^{-2(\theta-\delta)} [\mathbb{E}\|A(0)^\delta G(0)\|^2_{\gamma(H;E)} + \mathbb{E}\zeta^2 s^{2\sigma}]ds \\
&= \frac{2\kappa_\theta^2 \mathbb{E}\|A(0)^\delta G(0)\|^2_{\gamma(H;E)} t^{1-2(\theta-\delta)}}{1-2(\theta-\delta)} \\
&\quad + 2\mathbb{E}\zeta^2 \kappa_\theta^2 B(1+2\sigma, 1-2(\theta-\delta))t^{1-2(\theta-\delta)+2\sigma} \\
&\le C^*, \qquad 0 \le t \le T,
\end{aligned} \tag{3.4}$$

where $B(\cdot,\cdot)$ is the beta function. Meanwhile, if $\theta < \delta$, then (2.5) and (2.6) give

$$\begin{aligned}
&\int_0^t \mathbb{E}\|A(t)^\theta U(t,s)G(s)\|^2_{\gamma(H;E)}ds \\
&\le 2\int_0^t \|A(t)^\theta U(t,s)A(s)^{-\theta}\|^2 \\
&\quad \times \|A(s)^{\theta-\delta}\|^2\left[\mathbb{E}\|A(0)^\delta G(0)\|^2_{\gamma(H;E)} + \mathbb{E}\zeta^2 s^{2\sigma}\right]ds \\
&\le 2\kappa_\theta^2 \iota_{\delta-\theta}^2 \int_0^t \left[\mathbb{E}\|A(0)^\delta G(0)\|^2_{\gamma(H;E)} + \mathbb{E}\zeta^2 s^{2\sigma}\right]ds \\
&= 2\kappa_\theta^2 \iota_{\delta-\theta}^2 \left[\mathbb{E}\|A(0)^\delta G(0)\|^2_{\gamma(H;E)} t + \frac{\mathbb{E}\zeta^2 t^{1+2\sigma}}{1+2\sigma}\right] \\
&\le C^*, \qquad 0 \le t \le T.
\end{aligned} \tag{3.5}$$

Hence, $(A^\theta U)\diamond G$ is well-defined on $[0,T]$. Since $A(t)^\theta$ is closed, Proposition 2.8 yields

$$(A^\theta U)\diamond G(t) = A(t)^\theta [U\diamond G](t), \qquad 0 \le t \le T.$$

Step 2. Fix $0 < \gamma_2 < \min\{\delta - \frac{1}{2}, \sigma\}$. Let us verify that

$$(AU)\diamond G \in \mathcal{C}^{\gamma_2}([0,T];E) \qquad \text{a.s.}$$

Let $0 < \gamma_2 < \gamma_3 < \min\{\delta - \frac{1}{2}, \sigma\}$. We prove that

$$\mathbb{E}\|(AU)\diamond G(t) - (AU)\diamond G(s)\|^2 \le C^*(t-s)^{2\gamma_3}, \qquad 0 \le s \le t \le T. \tag{3.6}$$

Indeed, noting the expression

$$(AU)\diamond G(t) = \int_0^t A(t)U(t,r)A(r)^{-\delta}A(r)^\delta G(r)dW(r),$$

we have

$$\begin{aligned}
&(AU)\diamond G(t) - (AU)\diamond G(s) \\
&= \int_s^t A(t)U(t,r)A(r)^{-\delta}A(r)^\delta G(r)dW(r) \\
&\quad + \int_0^s A(t)U(t,r)A(r)^{-\delta}A(r)^\delta G(r)dW(r) \\
&\quad - \int_0^s A(s)U(s,r)A(r)^{-\delta}A(r)^\delta G(r)dW(r) \\
&= \int_s^t A(t)U(t,r)A(r)^{-\delta}[A(r)^\delta G(r) - A(t)^\delta G(t) + A(t)^\delta G(t)]dW(r) \\
&\quad + \int_0^s [A(t)U(t,s) - A(s)]U(s,r)A(r)^{-\delta}A(r)^\delta G(r)dW(r)
\end{aligned}$$

$$
\begin{aligned}
&= \int_s^t A(t)U(t,r)A(r)^{-\delta}[A(r)^{\delta}G(r) - A(t)^{\delta}G(t)]dW(r) \\
&\quad + \int_s^t A(t)U(t,r)A(r)^{-\delta}A(t)^{\delta}G(t)dW(r) \\
&\quad + \int_0^s [A(t)U(t,s) - A(s)]U(s,r)A(r)^{-\delta}A(r)^{\delta}G(r)dW(r) \\
&= J_1 + J_2 + J_3, \qquad 0 \le s \le t \le T.
\end{aligned}
$$

Thereby,

$$
\mathbb{E}\|(AU)\diamond G(t) - (AU)\diamond G(s)\|^2 \le 3\sum_{i=1}^{3}\mathbb{E}\|J_i\|^2, \qquad 0 \le s \le t \le T.
$$

The terms $\mathbb{E}\|J_i\|^2$ $(i = 1, 2, 3)$ are estimated as follows. For $\mathbb{E}\|J_1\|^2, \mathbb{E}\|J_2\|^2$ and $\mathbb{E}\|J_3\|^2$, (2.4) and (G) give

$$
\begin{aligned}
\mathbb{E}\|J_1\|^2 &\le c(E)\int_s^t \mathbb{E}\|A(t)U(t,r)A(r)^{-\delta}[A(r)^{\delta}G(r) - A(t)^{\delta}G(t)]\|^2_{\gamma(H;E)}dr \\
&\le c(E)\int_s^t \|A(t)U(t,r)A(r)^{-\delta}\|^2\mathbb{E}\|A(r)^{\delta}G(r) - A(t)^{\delta}G(t)\|^2_{\gamma(H;E)}dr \\
&\le c(E)\kappa_1^2\mathbb{E}\zeta^2\int_s^t (t-r)^{2(\delta+\sigma-1)}dr \\
&\le \frac{c(E)\kappa_1^2\mathbb{E}\zeta^2(t-s)^{2(\delta+\sigma)-1}}{2(\delta+\sigma)-1}, \qquad 0 \le s \le t \le T. \\
\mathbb{E}\|J_2\|^2 &\le c(E)\int_s^t \mathbb{E}\|A(t)U(t,r)A(r)^{-\delta}A(t)^{\delta}G(t)\|^2_{\gamma(H;E)}dr \\
&\le c(E)\int_s^t \|A(t)U(t,r)A(r)^{-\delta}\|^2\mathbb{E}\|A(t)^{\delta}G(t)\|^2_{\gamma(H;E)}dr \\
&\le 2c(E)\kappa_1^2\int_s^t (t-r)^{2(\delta-1)}[\mathbb{E}\|A(0)^{\delta}G(0)\|^2_{\gamma(H;E)} + \mathbb{E}\zeta^2 t^{2\sigma}]dr \\
&= \frac{2c(E)\kappa_1^2[\mathbb{E}\|A(0)^{\delta}G(0)\|^2_{\gamma(H;E)} + \mathbb{E}\zeta^2 t^{2\sigma}](t-s)^{2\delta-1}}{2\delta-1}, \\
&\qquad\qquad 0 \le s \le t \le T. \\
\mathbb{E}\|J_3\|^2 &\le c(E)\int_0^s \mathbb{E}\|[A(t)U(t,s) - A(s)]U(s,r)A(r)^{-\delta}A(r)^{\delta}G(r)\|^2_{\gamma(H;E)}dr \\
&\le c(E)\int_0^s \|[A(t)U(t,s)A(s)^{-1} - I]A(s)U(s,r)A(r)^{-\delta}\|^2 \\
&\quad \times \mathbb{E}\|A(r)^{\delta}G(r)\|^2_{\gamma(H;E)}dr \\
&\le c(E)\int_0^s \|[A(t)U(t,s)A(s)^{-1} - e^{-(t-s)A(s)}]A(s)U(s,r)A(r)^{-\delta}\|^2 \\
&\quad \times \mathbb{E}[\|A(0)^{\delta}G(0)\|_{\gamma(H;E)} + \zeta r^{\sigma}]^2dr
\end{aligned}
$$

$$+ c(E)\int_0^s \|[e^{-(t-s)A(s)} - I]A(s)^{-\theta}\|^2 \|A(s)^{1+\theta}U(s,r)A(r)^{-\delta}\|^2$$
$$\times \mathbb{E}[\|A(0)^{\delta}G(0)\|_{\gamma(H;E)} + \zeta r^{\sigma}]^2 dr, \qquad 0 \le s \le t \le T.$$

The terms on the right-hand side of the latter inequality can be evaluated by using (2.6) and (2.7). Thereby, for any $0 < \theta < \min\{\mu + \nu - 1, \delta - \frac{1}{2}\}$,

$$\begin{aligned}
\mathbb{E}\|J_3\|^2 &\le c(E)c_{\mu,\nu}^2\kappa_1^2 \int_0^s (t-s)^{2(\mu+\nu-1)}(s-r)^{2(\delta-1)} \\
&\quad \times \mathbb{E}[\|A(0)^{\delta}G(0)\|_{\gamma(H;E)} + \zeta r^{\sigma}]^2 dr \\
&\quad + c(E)\iota_\theta^2\kappa_{1+\theta} \int_0^s (t-s)^{2\theta}(s-r)^{2(\delta-\theta-1)} \\
&\quad \times \mathbb{E}[\|A(0)^{\delta}G(0)\|_{\gamma(H;E)} + \zeta r^{\sigma}]^2 dr \\
&\le 2c(E)c_{\mu,\nu}^2\kappa_1^2 \int_0^s (t-s)^{2(\mu+\nu-1)}(s-r)^{2(\delta-1)} \\
&\quad \times [\mathbb{E}\|A(0)^{\delta}G(0)\|_{\gamma(H;E)}^2 + \mathbb{E}\zeta^2 r^{2\sigma}] dr \\
&\quad + 2c(E)\iota_\theta^2\kappa_{1+\theta} \int_0^s (t-s)^{2\theta}(s-r)^{2(\delta-\theta-1)} \\
&\quad \times [\mathbb{E}\|A(0)^{\delta}G(0)\|_{\gamma(H;E)}^2 + \mathbb{E}\zeta^2 r^{2\sigma}] dr \\
&= 2c(E)c_{\mu,\nu}^2\kappa_1^2 \left[\frac{\mathbb{E}\|A(0)^{\delta}G(0)\|_{\gamma(H;E)}^2 s^{2\delta-1}}{2\delta - 1} \right. \\
&\quad \left. + \mathbb{E}\zeta^2 B(1+2\sigma, 2\delta-1)s^{2(\delta+\sigma)-1} \right] \\
&\quad \times (t-s)^{2(\mu+\nu-1)} \\
&\quad + 2c(E)\iota_\theta^2\kappa_{1+\theta} \left[\frac{\mathbb{E}\|A(0)^{\delta}G(0)\|_{\gamma(H;E)}^2 s^{2(\delta-\theta)-1}}{2(\delta-\theta) - 1} \right. \\
&\quad \left. + \mathbb{E}\zeta^2 B(1+2\sigma, 2(\delta-\theta)-1)s^{2(\delta-\theta+\sigma)-1} \right] (t-s)^{2\theta}, \\
&\qquad\qquad 0 \le s \le t \le T.
\end{aligned}$$

Since $\mu + \nu - 1 \ge \sigma$, we obtain

$$\mathbb{E}\|J_3\|^2 \le C^*(t-s)^{2\gamma_3}, \qquad 0 \le s \le t \le T.$$

In this way, we conclude that

$$\mathbb{E}\|(AU) \diamond G(t) - (AU) \diamond G(s)\|^2 \le C^*(t-s)^{2\gamma_3}, \qquad 0 \le s \le t \le T.$$

On the other hand, thanks to Step 1, it is possible to see that $(AU) \diamond G$ is a Gaussian process on $[0, T]$. By (3.6), Theorem 2.10 applied to $(AU) \diamond G$ yields

$$(AU) \diamond G \in \mathcal{C}^{\gamma_2}([0,T]; E) \qquad \text{a.s.}$$

Step 3. Let

$$I_1(t) = U(t,0)\xi + \int_0^t U(t,s)F(s)ds, \qquad 0 \le t \le T,$$

and

$$X(t) = I_1(t) + U \diamond G(t), \qquad 0 \le t \le T.$$

Let us verify that X is a strict solution of (1.1) possessing the regularity:

$$AX \in \mathcal{C}((0,T]; E) \qquad \text{a.s.}$$

Indeed, due to Theorem 2.12,

$$AI_1 \in \mathcal{F}^{\beta,\sigma}((0,T]; E) \qquad \text{a.s.}$$

In particular,

$$AI_1 \in \mathcal{C}((0,T]; E) \qquad \text{a.s.}$$

In addition, by (2.3),

$$\begin{aligned}\int_0^t \|A(s)I_1(s)\|ds &\le \int_0^t \|AI_1\|_{\mathcal{F}^{\beta,\sigma}} s^{\beta-1}ds \\ &= \frac{\|AI_1\|_{\mathcal{F}^{\beta,\sigma}} t^{\beta}}{\beta} < \infty, \qquad 0 < t \le T.\end{aligned}$$

Since $\frac{dI_1}{dt} = -A(t)I_1(t) + F(t)$, we then have

$$I_1(t) = \xi - \int_0^t A(s)I_1(s)ds + \int_0^t F(s)ds, \qquad 0 < t \le T.$$

On the other hand, by Step 2,

$$A[U \diamond G] = (AU) \diamond G \in \mathcal{C}([0,T]; E) \qquad \text{a.s.}$$

Hence, it suffices to prove that

$$U \diamond G(t) + \int_0^t (AU) \diamond G(s)ds = \int_0^t G(s)dW(s), \qquad 0 \le t \le T. \tag{3.7}$$

To prove (3.7), we use the Fubini theorem. Before that, however, we observe from (2.5) and (3.3) that

$$\begin{aligned}
&\int_0^t \mathbb{E}\|G(s)\|^2_{\gamma(H;E)}ds \\
&\le \int_0^t \|A(s)^{-\delta}\|^2 \mathbb{E}\|A(s)^{\delta}G(s)\|^2_{\gamma(H;E)}ds \\
&\le 2\int_0^t \iota_\delta^2[\mathbb{E}\|A(0)^{\delta}G(0)\|^2_{\gamma(H;E)} + \mathbb{E}\zeta^2 s^{2\sigma}]ds \\
&= 2\iota_\delta^2 \mathbb{E}\|A(0)^{\delta}G(0)\|^2_{\gamma(H;E)}t + \frac{2\mathbb{E}\zeta^2\iota_\delta^2 t^{1+2\sigma}}{1+2\sigma} \le C^*, \qquad 0 \le t \le T.
\end{aligned}$$

Thereby, the stochastic integral $\int_0^t G(s)dW(s)$ is well-defined for $0 \le t \le T$.

On the other hand, with $\theta = 1$, (3.4) gives

$$\begin{aligned}
&\int_0^t \mathbb{E}\|A(t)U(t,s)G(s)\|^2_{\gamma(H;E)}ds \\
&\le \frac{2\kappa_1^2\mathbb{E}\|A(0)^{\delta}G(0)\|^2_{\gamma(H;E)}t^{2\delta-1}}{2\delta-1} + 2\mathbb{E}\zeta^2\kappa_1^2 B(1+2\sigma, 2\delta-1)t^{2(\delta+\sigma)-1} \\
&\le C^*, \qquad 0 \le t \le T.
\end{aligned} \tag{3.8}$$

Theorem 2.7 and (3.8) then yield

$$\begin{aligned}
&\mathbb{E}\Big(\int_0^t \|(AU)\diamond G(s)\|ds\Big)^2 \\
&\le t\int_0^t \mathbb{E}\|(AU)\diamond G(s)\|^2 ds \\
&= t\int_0^t \mathbb{E}\Big|\Big|\int_0^s A(s)U(s,u)G(u)dW(u)\Big|\Big|^2 ds \\
&\le c(E)t\int_0^t\int_0^s \mathbb{E}\|A(s)U(s,u)G(u)\|^2_{\gamma(H;E)}duds \\
&\le \frac{2c(E)\kappa_1^2\mathbb{E}\|A(0)^{\delta}G(0)\|^2_{\gamma(H;E)}t\int_0^t s^{2\delta-1}ds}{2\delta-1} \\
&\quad + 2c(E)\mathbb{E}\zeta^2\kappa_1^2 B(1+2\sigma,2\delta-1)t\int_0^t s^{2(\delta+\sigma)-1}ds \\
&= \frac{2c(E)\kappa_1^2\mathbb{E}\|A(0)^{\delta}G(0)\|^2_{\gamma(H;E)}t^{2\delta+1}}{2\delta(2\delta-1)} \\
&\quad + \frac{2c(E)\mathbb{E}\zeta^2\kappa_1^2 B(1+2\sigma,2\delta-1)t^{2(\delta+\sigma)+1}}{2(\delta+\sigma)} \\
&\le C^*, \qquad 0 \le t \le T.
\end{aligned}$$

The integral $\int_0^t \|(AU)\diamond G(s)\|ds$ is hence well-defined a.s. for $0 \le t \le T$.

Now we will use the Fubini theorem. We have

$$\begin{aligned}\int_0^t (AU)\diamond G(s)ds &= \int_0^t \int_0^s A(s)U(s,u)G(u)dW(u)ds \\ &= \int_0^t \int_u^t A(s)U(s,u)G(u)dsdW(u) \\ &= \int_0^t [G(u) - U(t,u)G(u)]dW(u) \\ &= \int_0^t G(u)dW(u) - \int_0^t U(t,u)G(u)dW(u) \\ &= \int_0^t G(u)dW(u) - U\diamond G(t), \qquad 0 \le t \le T.\end{aligned}$$

Thus, (3.7) has been verified.

Step 4. Let us verify that $A^\beta X \in \mathcal{C}([0,T];E)$ a.s. and that X satisfies (3.1) and (3.2).

For the first statement, it is seen that

$$A(t)^\beta X(t) = A(t)^{\beta-1}A(t)X(t), \qquad 0 < t \le T.$$

Since $A(\cdot)^{\beta-1} \in \mathcal{C}([0,T];L(E))$, the continuity of $A^\beta X$ on $(0,T]$ follows Step 3. In addition, thanks to Theorem 2.12 and Step 2, $A^\beta I_1$ and $A^\beta U\diamond G$ are continuous at $t=0$. Therefore, $A^\beta X \in \mathcal{C}([0,T];E)$ a.s.

For the second statement, due to (3.4) and (3.5), when $\beta \ge \delta$,

$$\begin{aligned}&\int_0^t \mathbb{E}\|A(t)^\beta U(t,s)G(s)\|^2_{\gamma(H;E)}ds \\ &\le \frac{2\kappa_\beta^2 \mathbb{E}\|A(0)^\delta G(0)\|^2_{\gamma(H;E)} t^{1-2(\beta-\delta)}}{1-2(\beta-\delta)} \\ &\quad + 2\mathbb{E}\zeta^2\kappa_\beta^2 B(1+2\sigma, 1-2(\beta-\delta))t^{1-2(\beta-\delta)+2\sigma} \\ &\le C^*, \qquad 0 \le t \le T. \end{aligned} \tag{3.9}$$

Meanwhile, when $\beta < \delta$,

$$\begin{aligned}&\int_0^t \mathbb{E}\|A(t)^\beta U(t,s)G(s)\|^2_{\gamma(H;E)}ds \\ &\le 2\kappa_\beta^2 \iota_{\delta-\beta}^2 \left[\mathbb{E}\|A(0)^\delta G(0)\|^2_{\gamma(H;E)} t + \frac{\mathbb{E}\zeta^2 t^{1+2\sigma}}{1+2\sigma}\right] \\ &\le C^*, \qquad 0 \le t \le T. \end{aligned} \tag{3.10}$$

On the other hand, since $X = I_1 + U\diamond G$,

$$\mathbb{E}\|A(t)^\beta X(t)\|^2 \le 2\mathbb{E}\|A(t)^\beta I_1(t)\|^2 + 2\mathbb{E}\|A(t)^\beta [U\diamond G](t)\|^2.$$

Using Theorem 2.12, it is easily seen that

$$\mathbb{E}\|A(t)^\beta I_1(t)\|^2 \le C[\mathbb{E}\|A(0)^\beta \xi\|^2 + \mathbb{E}\|F\|^2_{\mathcal{F}^{\beta,\sigma}}],$$

where C is some constant depending on the exponents. In the meantime, using Theorem 2.7,

$$\mathbb{E}\|A(t)^{\beta}[U \diamond G](t)\|^2 \le c(E)\int_0^t \mathbb{E}\|A(t)^{\beta}U(t,s)G(s)\|^2_{\gamma(H;E)}ds.$$

Thus,

$$\mathbb{E}\|A(t)^{\beta}X(t)\|^2 \le C\Big[\mathbb{E}\|A(0)^{\beta}\xi\|^2 + \mathbb{E}\|F\|^2_{\mathcal{F}^{\beta,\sigma}} + \int_0^t \mathbb{E}\|A(t)^{\beta}U(t,s)G(s)\|^2_{\gamma(H;E)}ds\Big]. \tag{3.11}$$

Using (3.9), (3.10) and (3.11), we arrive at (3.1) and (3.2).

Step 5. Let us show that for any $0 < \gamma_1 \le \min\{\beta, \frac{1}{2}\}$ and $\gamma_1 \ne \frac{1}{2}$,

$$X \in \mathcal{C}^{\gamma_1}([0,T];E) \qquad \text{a.s.},$$

and that if $G(0) = 0$, then for any $0 < \gamma_1 \le \min\{\frac{1+\sigma}{2}, \delta, \beta\}$, $\gamma_1 \notin \{\frac{1+\sigma}{2}, \delta\}$,

$$X \in \mathcal{C}^{\gamma_1}([0,T];E) \qquad \text{a.s.}$$

First, we prove that I_1 is β-Hölder continuous a.s. on $[0,T]$. According to Theorem 2.12, $\frac{dI_1}{dt} \in \mathcal{F}^{\beta,\sigma}((0,T];E)$ a.s. Then, (2.3) gives that

$$\begin{aligned}\|I_1(t) - I_1(s)\| &= \Big\|\int_s^t \frac{dI_1(u)}{du}du\Big\| \le \int_s^t \Big\|\frac{dI_1(u)}{du}\Big\|du \\ &\le \int_s^t \Big\|\frac{dI_1}{du}\Big\|_{\mathcal{F}^{\beta,\sigma}} u^{\beta-1}du \le \Big\|\frac{dI_1}{du}\Big\|_{\mathcal{F}^{\beta,\sigma}} \frac{t^{\beta}-s^{\beta}}{\beta} \\ &\le \Big\|\frac{dI_1}{du}\Big\|_{\mathcal{F}^{\beta,\sigma}} \frac{(t-s)^{\beta}}{\beta}, \qquad 0 \le s < t \le T.\end{aligned}$$

This means that $I_1 \in \mathcal{C}^{\beta}([0,T];E)$ a.s.

Next, we verify that for any $0 < \rho < \frac{1}{2}$,

$$U \diamond G \in \mathcal{C}^{\rho}([0,T];E) \qquad \text{a.s.}$$

Let $0 \le s < t \le T$. Then

$$\begin{aligned}&U \diamond G(t) - U \diamond G(s) \\ &\quad = \int_s^t U(t,r)G(r)dW(r) + \int_0^s [U(t,r) - U(s,r)]G(r)dW(r).\end{aligned}$$

Since the integrals in the right-hand side are independent stochastic functions which have zero expectation, we have

$$
\begin{aligned}
&\mathbb{E}\|U\diamond G(t)-U\diamond G(s)\|^2\\
&\quad=\mathbb{E}\Big|\Big|\int_s^t U(t,r)G(r)dW(r)\Big|\Big|^2+\mathbb{E}\Big|\Big|\int_0^s [U(t,r)-U(s,r)]G(r)dW(r)\Big|\Big|^2\\
&\quad\le c(E)\int_s^t \mathbb{E}\|U(t,r)A(r)^{-\delta}A(r)^{\delta}G(r)\|^2_{\gamma(H;E)}dr\\
&\qquad+c(E)\int_0^s \mathbb{E}\|[U(t,r)-U(s,r)]G(r)\|^2_{\gamma(H;E)}dr\\
&\quad\le c(E)\int_s^t \mathbb{E}\|U(t,r)\|^2\|A(r)^{-\delta}\|^2\|A(r)^{\delta}G(r)\|^2_{\gamma(H;E)}dr\\
&\qquad+c(E)\int_0^s \|[U(t,r)-U(s,r)]A(r)^{-\delta}\|^2\mathbb{E}\|A(r)^{\delta}G(r)\|^2_{\gamma(H;E)}dr.
\end{aligned}
$$

By using (2.4), (2.5) and (3.3),

$$
\begin{aligned}
&\mathbb{E}\|U\diamond G(t)-U\diamond G(s)\|^2\\
&\quad\le c(E)\iota_0^2\iota_\delta^2\int_s^t \mathbb{E}[\|A(0)^{\delta}G(0)\|_{\gamma(H;E)}+\zeta r^{\sigma}]^2dr\\
&\qquad+c(E)\int_0^s \|[U(t,r)-U(s,r)]A(r)^{-\delta}\|^2\mathbb{E}[\|A(0)^{\delta}G(0)\|_{\gamma(H;E)}+\zeta r^{\sigma}]^2dr.
\end{aligned}
$$

The first term on the right-hand side of the latter inequality can be estimated as

$$
\begin{aligned}
&c(E)\iota_0^2\iota_\delta^2\int_s^t \mathbb{E}[\|A(0)^{\delta}G(0)\|_{\gamma(H;E)}+\zeta r^{\sigma}]^2dr\\
&\quad\le 2c(E)\iota_0^2\iota_\delta^2\int_s^t [\mathbb{E}\|A(0)^{\delta}G(0)\|^2_{\gamma(H;E)}+\mathbb{E}\zeta^2 r^{2\sigma}]dr\\
&\quad=2c(E)\iota_0^2\iota_\delta^2\mathbb{E}\|A(0)^{\delta}G(0)\|^2_{\gamma(H;E)}(t-s)+\frac{2c(E)\mathbb{E}\zeta^2\iota_0^2\iota_\delta^2(t^{1+2\sigma}-s^{1+2\sigma})}{1+2\sigma}\\
&\quad\le 2c(E)\iota_0^2\iota_\delta^2\mathbb{E}\|A(0)^{\delta}G(0)\|^2_{\gamma(H;E)}(t-s)+\frac{2c(E)\mathbb{E}\zeta^2\iota_0^2\iota_\delta^2(t-s)^{1+2\sigma}}{1+2\sigma}.
\end{aligned}
$$

In the meantime, since

$$
\begin{aligned}
&\|[U(t,r)-U(s,r)]A(r)^{-\delta}\|^2\\
&\quad=\Big|\Big|\int_s^t A(u)S(u,r)A(r)^{-\delta}du\Big|\Big|^2\\
&\quad\le\kappa_1^2\left[\int_s^t (u-r)^{\delta-1}du\right]^2=\frac{\kappa_1^2}{\delta^2}[(t-r)^{\delta}-(s-r)^{\delta}]^2\\
&\quad\le\frac{\kappa_1^2}{\delta^2}(t-s)^{2\delta},
\end{aligned}
$$

the second term is evaluated as

$$
\begin{aligned}
&c(E)\int_0^s \|[U(t,r)-U(s,r)]A(r)^{-\delta}\|^2\mathbb{E}[\|A(0)^\delta G(0)\|_{\gamma(H;E)}+\zeta r^\sigma]^2dr\\
&\le \frac{2c(E)\kappa_1^2}{\delta^2}(t-s)^{2\delta}\int_0^s [\mathbb{E}\|A(0)^\delta G(0)\|^2_{\gamma(H;E)}+\mathbb{E}\zeta^2 r^{2\sigma}]dr\\
&= \left[\frac{2c(E)\kappa_1^2\mathbb{E}\|A(0)^\delta G(0)\|^2_{\gamma(H;E)}s}{\delta^2}+\frac{2c(E)\mathbb{E}\zeta^2\kappa_1^2 s^{1+2\sigma}}{(1+2\sigma)\delta^2}\right](t-s)^{2\delta}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\mathbb{E}\|U\diamond G(t)-U\diamond G(s)\|^2\\
&\le 2c(E)\iota_0^2\iota_\delta^2\mathbb{E}\|A(0)^\delta G(0)\|^2_{\gamma(H;E)}(t-s)+\frac{2c(E)\mathbb{E}\zeta^2\iota_0^2\iota_\delta^2(t-s)^{1+2\sigma}}{1+2\sigma}\\
&\quad+\left[\frac{2c(E)\kappa_1^2\mathbb{E}\|A(0)^\delta G(0)\|^2_{\gamma(H;E)}s}{\delta^2}+\frac{2c(E)\mathbb{E}\zeta^2\kappa_1^2 s^{1+2\sigma}}{(1+2\sigma)\delta^2}\right](t-s)^{2\delta},\\
&\qquad\qquad 0\le s<t\le T.
\end{aligned}
$$

On the other hand, by the definition of stochastic integrals, $U\diamond G$ is a Gaussian process on $[0,T]$. We then apply Theorem 2.10 to $U\diamond G$ to obtain that for any $0<\rho<\frac{1}{2}$,

$$U\diamond G\in\mathcal{C}^\rho([0,T];E)\qquad\text{a.s.}$$

In addition, if $G(0)=0$, then for any $0<\rho<\min\{\frac{1+\sigma}{2},\delta\}$,

$$U\diamond G\in\mathcal{C}^\rho([0,T];E)\qquad\text{a.s.}$$

In this way, we conclude that for any $0<\gamma_1\le\min\{\beta,\frac{1}{2}\}$, $\gamma_1\ne\frac{1}{2}$,

$$X=I_1+U\diamond G\in\mathcal{C}^{\gamma_1}([0,T];E)\qquad\text{a.s.}$$

Furthermore, when $G(0)=0$, for any $0<\gamma_1\le\min\{\frac{1+\sigma}{2},\delta,\beta\}$, $\gamma_1\ne\{\frac{1+\sigma}{2},\delta\}$,

$$X\in\mathcal{C}^{\gamma_1}([0,T];E)\qquad\text{a.s.}$$

Step 6. Fix $0<\gamma_2<\min\{\delta-\frac{1}{2},\sigma\}$ and $0<\epsilon\le T$. Let us prove that

$$AX\in\mathcal{C}^{\gamma_2}([\epsilon,T];E)\qquad\text{a.s.}$$

Theorem 2.12 and Step 2 provide that

$$AI_1\in\mathcal{F}^{\beta,\sigma}((0,T];E)\subset\mathcal{C}^\sigma([\epsilon,T];E)$$

and

$$(AU)\diamond G\in\mathcal{C}^{\gamma_2}([0,T];E)\qquad\text{a.s.}$$

Therefore, $AX=AI_1+(AU)\diamond G\in\mathcal{C}^{\gamma_2}([\epsilon,T];E)$ a.s.

By Steps 1–6, the proof of the theorem is now complete. □

3.2 *Case where* $\nu = 1$

Consider the favorable case where $\nu = 1$. In other words, the domain $\mathcal{D}(A(t))$ does not depend on time t. We show that the condition (G) can be naturally replaced by the condition:

(G′) There exist $\delta_1 > \frac{1}{2}$ and a square integrable random variable $\bar{\zeta}$ such that

$$G(t) \in \mathcal{D}(A(0)^{\delta_1}) \qquad \text{a.s., } 0 \le t \le T$$

and

$$\|A(0)^{\delta_1}[G(t) - G(s)]\|_{\gamma(H;E)} \le \bar{\zeta}|t - s|^{\sigma} \qquad \text{a.s., } 0 \le s \le t \le T.$$

In addition,

$$\mathbb{E}\|A(0)^{\delta_1} G(0)\|^2_{\gamma(H;E)} < \infty.$$

PROPOSITION 3.3

Let (A1), (A2) *and* (A3) *with* $\nu = 1$ *be satisfied. If* (G′) *takes place, then so does* (G).

For the proof, we notice the following lemma.

Lemma 3.4. *For any* $0 < \theta_1 < \theta_2 < 1$, *there exists* $\alpha_{\theta_1,\theta_2} > 0$ *such that*

$$\|[A(t)^{\theta_1} - A(s)^{\theta_1}]A(s)^{-\theta_2}\| \le \alpha_{\theta_1,\theta_2}|t - s|^{\mu}, \qquad 0 \le s \le t \le T.$$

For the proof of Lemma 3.4, see [14, (3.92)] with $\tau = 0$.

Proof of Proposition 3.3. Let $\frac{1}{2} < \delta < \delta' < \delta_1$. For $0 \le s \le t \le T$,

$$\begin{aligned}
&\|A(t)^{\delta} G(t) - A(s)^{\delta} G(s)\|_{\gamma(H;E)} \\
&\quad = \|[A(t)^{\delta}[G(t) - G(s)] + [A(t)^{\delta} - A(s)^{\delta}]G(s)\|_{\gamma(H;E)} \\
&\quad \le \|[A(t)^{\delta} - A(0)^{\delta} + A(0)^{\delta}]A(0)^{-\delta_1}\| \|A(0)^{\delta_1}[G(t) - G(s)]\|_{\gamma(H;E)} \\
&\qquad + \|[A(t)^{\delta} - A(s)^{\delta}]A(s)^{-\delta'}\| \|[A(s)^{\delta'} - A(0)^{\delta'} + A(0)^{\delta'}]A(0)^{-\delta_1}\| \\
&\qquad \times \|A(0)^{\delta_1} G(s)\|_{\gamma(H;E)} \\
&\quad \le \|[A(t)^{\delta} - A(0)^{\delta}]A(0)^{-\delta_1}\| \|A(0)^{\delta_1}[G(t) - G(s)]\|_{\gamma(H;E)} \\
&\qquad + \|A(0)^{\delta-\delta_1}\| \|A(0)^{\delta_1}[G(t) - G(s)]\|_{\gamma(H;E)} \\
&\qquad + \|[A(t)^{\delta} - A(s)^{\delta}]A(s)^{-\delta'}\| \|[A(s)^{\delta'} - A(0)^{\delta'}]A(0)^{-\delta_1}\| \\
&\qquad \times \|A(0)^{\delta_1} G(s)\|_{\gamma(H;E)} \\
&\qquad + \|[A(t)^{\delta} - A(s)^{\delta}]A(s)^{-\delta'}\| \|A(0)^{\delta'-\delta_1}\| \|A(0)^{\delta_1} G(s)\|_{\gamma(H;E)}.
\end{aligned}$$

By using Lemma 3.4 and (G′), it is easily seen that

$$\begin{aligned}
&\|A(t)^{\delta} G(t) - A(s)^{\delta} G(s)\|_{\gamma(H;E)} \\
&\quad \le [\alpha_{\delta,\delta_1} t^{\mu}(t - s)^{\sigma} + \|A(0)^{-1}\|^{\delta_1-\delta}]\bar{\zeta}(t - s)^{\sigma}
\end{aligned}$$

$$+\alpha_{\delta,\delta'}[\alpha_{\delta',\delta_1}s^{\mu}+\|A(0)^{-1}\|^{\delta_1-\delta'}]\|A(0)^{\delta_1}G(s)\|_{\gamma(H;E)}(t-s)^{\mu},$$
$$0\le s\le t\le T.$$

Note that $0<\sigma<\mu$ and

$$\begin{aligned}\mathbb{E}\|A(0)^{\delta_1}G(s)\|^2_{\gamma(H;E)} &\le \mathbb{E}[\|A(0)^{\delta_1}G(0)\|_{\gamma(H;E)}+\bar{\zeta}s^{\sigma}]^2\\ &\le 2\mathbb{E}\|A(0)^{\delta_1}G(0)\|^2_{\gamma(H;E)}+2\mathbb{E}\bar{\zeta}^2s^{2\sigma}.\end{aligned}$$

Thus, it is easily seen that (G) holds true. □

4. An example

Consider a stochastic partial differential equation

$$\begin{cases}\dfrac{\partial X}{\partial t}(x,t)=\displaystyle\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left[a_{ij}(x,t)\frac{\partial}{\partial x_i}X\right]-b(x,t)X+\varphi_1(x)f(t)\\ \qquad\qquad +\varphi_2(x)g(t)\dfrac{\partial W}{\partial t}(x,t)\quad \text{in } \mathcal{O}\times(0,T),\\ X=0\quad \text{on } \partial\mathcal{O}\times(0,T),\\ X(x,0)=X_0(x)\quad \text{in } \mathcal{O}\end{cases}\tag{4.1}$$

in a bounded domain $\mathcal{O}\subset\mathbb{R}^n$ $(n=1,2,\dots)$ with $\mathcal{C}^2$ boundary $\partial\mathcal{O}$. Here, $a_{ij}(\cdot,\cdot)$, $1\le i,j\le n$ and $b(\cdot,\cdot)$ are real-valued functions in $\mathcal{O}\times[0,T]$; $\varphi_i(i=1,2)$ are non-random real-valued functions in $\mathcal{O}$; f is a real-valued stochastic process on $[0,T]$ and is defined on a complete filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge0},\mathbb{P})$. The term $\frac{\partial W}{\partial t}(\cdot,\cdot)$ is a space-time white noise with intensity $\varphi_2(x)g(t)$ at (x,t).

Let us precisely say what we mean by $\frac{\partial W}{\partial t}(x,t)$. Since the process $W(x,t)$ depends on both position x and time t, it is often chosen of the form

$$W(x,t)=\sum_{j=1}^{\infty}e_j(x)B_j(t),$$

where $\{e_j\}_{j=1}^{\infty}$ is an orthonormal and complete basis of some Hilbert space, say H_0, and $\{B_j\}_{j=1}^{\infty}$ is a family of independent real-valued standard Wiener processes on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge0},\mathbb{P})$.

It is known that (see, e.g., [3]) the series $\sum_{j=1}^{\infty}e_jB_j(t)$ converges to a cylindrical Wiener process on a separable Hilbert space $H\supseteq H_0$. (The embedding of H_0 into H is a Hilbert–Schmidt operator.) We still denote the cylindrical Wiener process by $\{W(t),t\in[0,T]\}$. The noise term $g(t)\frac{\partial W}{\partial t}(x,t)$ in (4.1) is hence considered as $g(t)\frac{\partial W}{\partial t}(t)$, where W is the cylindrical Wiener process on H and $g(t)$, $0\le t\le T$ are real-valued linear operators on H.

Let us next assume that

(i) There exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x,t) z_i z_j \geq a_0 \|z\|_{\mathbb{R}^n}^2$$

for $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$, a.e. $x \in \mathcal{O}$ and $0 \leq t \leq T$.

(ii) There exists a constant $b_0 > 0$ such that $b(x,t) \geq b_0$ for a.e. $x \in \mathcal{O}$ and $0 \leq t \leq T$.

(iii) $a_{ij}, b \in \mathcal{C}^{\mu}([0,T]; L_\infty(\mathcal{O}))$ for some $\frac{1}{2} < \mu \leq 1$.

We formulate (4.1) as a problem of the form (1.1). Set the underlying space $E = H^{-1}(\mathcal{O}) = [\mathring{H}^1(\mathcal{O})]'$, where

$$\mathring{H}^1(\mathcal{O}) = \{u \in H^1(\mathcal{O}); u|_{\partial\mathcal{O}} = 0\},$$

and $H^1(\mathcal{O})$ is the space of all $L^2(\mathcal{O})$ functions u whose first derivatives in the distribution sense belong to $L^2(\mathcal{O})$. The norm in E is denoted by $\|\cdot\|$. Since E is the dual space of the Hilbert space $\mathring{H}^1(\mathcal{O})$, E is a Hilbert space. So, E is a UMD Banach space of type 2.

Let $A(t)$ be a realization of the differential operator

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left[a_{ij}(x,t)\frac{\partial}{\partial x_i}\right] + b(x,t)$$

in E under the Dirichlet boundary conditions. According to [14], it is known that

- $\mathcal{D}(A(t)) = \mathring{H}^1(\mathcal{O})$ for $t \geq 0$, i.e. $\{A(t), 0 \leq t \leq T\}$ fulfills (A2) with $\nu = 1$;
- $\{A(t), 0 \leq t \leq T\}$ satisfies (A1) and (A3).

We further assume the conditions:

(iv) For some $0 < \beta \leq 1$ and $0 < \sigma < \min\{\beta, \mu\}$,

$$f \in \mathcal{F}^{\beta,\sigma}((0,T]; \mathbb{R}) \qquad \text{a.s.}$$

(v) $g(t) \in L_2(H; R)$, the space of Hilbert–Schmidt operators from H to $\mathbb{R}$. Furthermore, $\mathbb{E}\|g(0)\|_{L_2(H;\mathbb{R})}^2 < \infty$, and for the σ as above, there exists a square integrable real-valued random variable $\bar{\zeta}$ such that

$$\|g(t) - g(s)\|_{L_2(H;\mathbb{R})} \leq \bar{\zeta}(t-s)^{\sigma} \qquad \text{a.s., } 0 \leq s \leq t \leq T.$$

(vi) $\varphi_1 \in E$ and there exists $\frac{1}{2} < \delta_1 < 1$ such that $\varphi_2 \in \mathcal{D}(A(0)^{\delta_1})$.

(vii) X_0 takes values in $\mathcal{D}(A(0)^{\beta})$ a.s. and satisfies

$$\mathbb{E}\|A(0)^{\beta} X_0\|^2 < \infty.$$

Let

$$F(t) = f(t)\varphi_1(x), \quad G(t) = g(t)\varphi_2(x), \quad \xi = X_0.$$

By using $A(t)$, F and G, (4.1) has been formulated as a problem of the form (1.1). Furthermore, it is obvious that (F) and (G′) hold true.

Let $\frac{1}{2} < \delta < \delta_1$, then Proposition 3.3 implies (G) with some square integrable real-valued random variable ζ.

Since all the structural assumptions (A1), (A2), (A3), (F) and (G) are fulfilled, Theorem 3.2 is available to (4.1).

Theorem 4.1. *Under* (i)–(vii), *there exists a unique strict solution of* (4.1) *in* $H^{-1}(\mathcal{O})$ *possessing the regularities*

$$X \in \mathcal{C}^{\gamma_1}([0,T];H^{-1}(\mathcal{O})), \quad A^{\beta}X \in \mathcal{C}([0,T];H^{-1}(\mathcal{O})) \qquad a.s.$$

and

$$AX \in \mathcal{C}^{\gamma_2}([\epsilon,T];H^{-1}(\mathcal{O})) \qquad a.s.$$

for any $0<\gamma_1<\min\{\beta,\frac{1}{2}\}$, $0<\gamma_2<\min\{\delta-\frac{1}{2},\sigma\}$ *and* $0<\epsilon\le T$. *Furthermore*,

(*a*) *when* $\beta\ge\delta$,

$$\begin{aligned}&\mathbb{E}\|A^{\beta}X(t)\|^2\\&\quad\le C[\mathbb{E}\|A(0)^{\beta}X_0\|^2+\mathbb{E}\|f\|^2_{\mathcal{F}^{\beta,\sigma}(\mathbb{R})}\|\varphi_1\|^2\\&\qquad+\|A(0)^{\delta}\varphi_2\|^2\mathbb{E}\|g(0)\|^2_{L_2(H;\mathbb{R})}t^{1-2(\beta-\delta)}+\mathbb{E}\|\zeta\|^2t^{1-2(\beta-\delta)+2\sigma}],\\&\qquad0\le t\le T;\end{aligned}$$

(*b*) *when* $\beta<\delta$,

$$\begin{aligned}&\mathbb{E}\|A^{\beta}X(t)\|^2\\&\quad\le C[\mathbb{E}\|A(0)^{\beta}X_0\|^2+\mathbb{E}\|f\|^2_{\mathcal{F}^{\beta,\sigma}(\mathbb{R})}\|\varphi_1\|^2\\&\qquad+\mathbb{E}\|A(0)^{\delta}\varphi_2\|^2\|g(0)\|^2_{L_2(H;\mathbb{R})}t+\mathbb{E}\|\zeta\|^2t^{1+2\sigma}],\qquad0\le t\le T.\end{aligned}$$

Here, C *is some constant depending on the exponents.*

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