Pramana – J. Phys. (2019) 92:97
https://doi.org/10.1007/s12043-019-1756-9

Parameter estimation of chaotic systems based on extreme value points

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MS received 3 February 2018; revised 19 September 2018; accepted 4 December 2018;
published online 11 April 2019

Abstract. Parameter estimation and synchronisation of chaotic systems are one of the hottest topics in the field of nonlinear science. In this paper, we addressed how to utilise the obtained experimental time series to estimate multiple parameters in chaotic systems. On the basis of relations of critical points and extreme value points, as well as the least squares estimation, we deduced a novel statistical parameter estimation corollary method to evaluate the unknown parameters in chaotic systems. In order to illustrate the feasibility and effectiveness of the proposed method, three numerical simulation results are presented, where the validity of the proposed method is verified in detail. Furthermore, we also investigated the effects of time-series noise and system disturbances for the proposed method, and the results showed that the proposed method is robust to uncertainties.

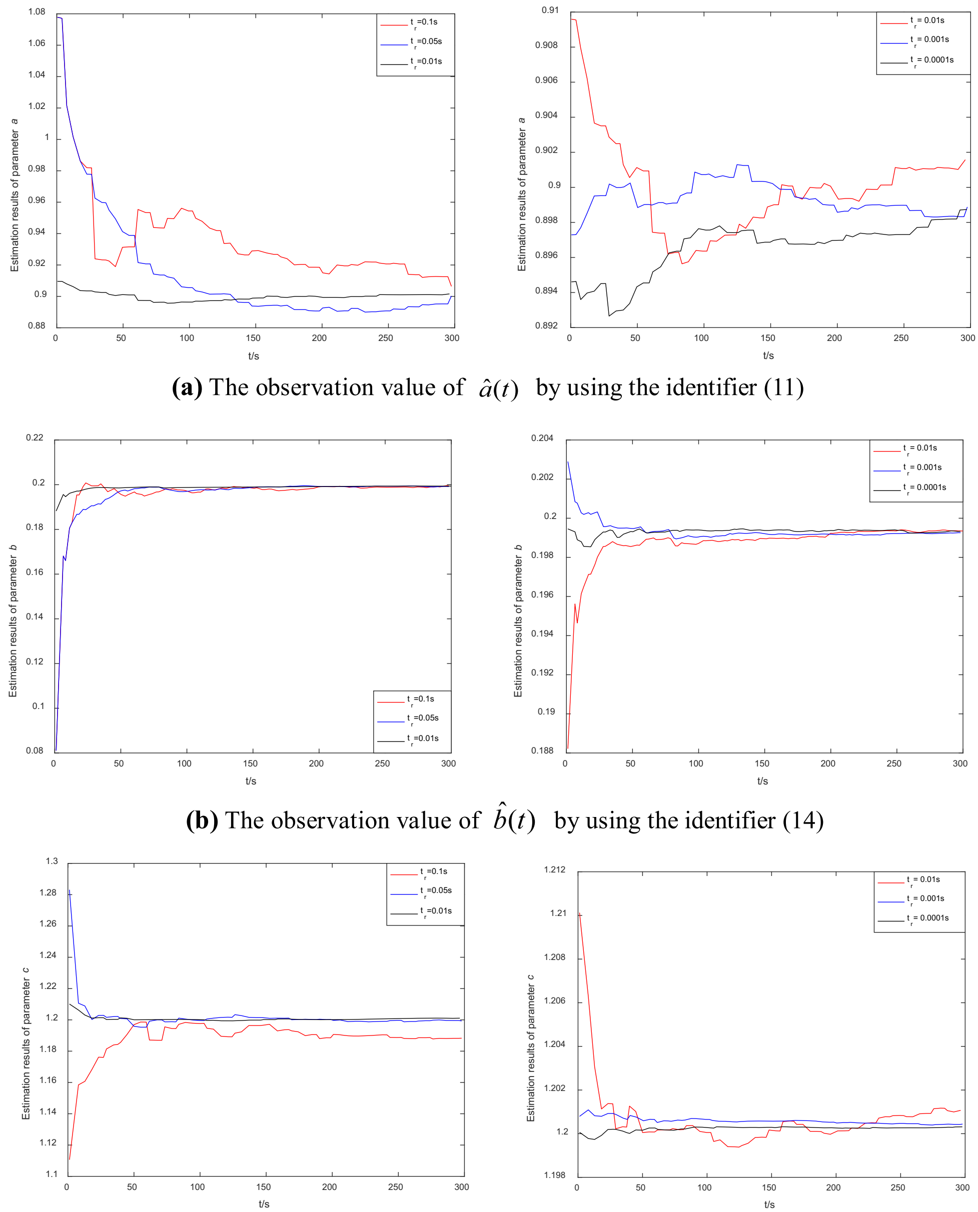
Keywords. Parameter estimation; chaotic system; time series; least squares estimation; noise.

PACS Nos 05.45.−a; 05.40.Ca

1. Introduction

Parameter estimation of chaotic systems from its experimental time series is an active subject in many disciplines in the field of natural sciences. In most cases, the chaotic system can be described by a set of differential equations which governs the orbits of all state variables in the system [1,2]. Usually, many chaotic systems contain some unknown or immeasurable parameters, which one expects to evaluate accurately through some effective strategies. Many approaches, such as synchronisation-based methods [3–7], Kalman filter [8], integrator theory [9], statistical analysis method [10] and intelligence algorithms [11–15], have been developed for the parameter identification of various chaotic systems in recent years. However, majority of the abovementioned methods need to construct one or more differential equations with respect to the original chaotic system, which is hard to realise in practice for some special situations. In order to address this problem, a new off-line estimation method of chaotic systems has been proposed based on its time series.

From off-line estimation methods, the least squares method and its variant are the most popular approaches for the parameter estimation. The principle of this method is simple and the method can be easily implemented, has high accuracy and the performance is good [16]. In this study, the classical least squares approach was adopted, and a statistical procedure based on the measured extreme value points was proposed to estimate the unknown parameters of the chaotic systems. By a simple combination of central limit theorem and least squares estimation, it is proved that all the unknown parameters in different chaotic systems can be estimated exactly from its time series. Although a systematic proof cannot be given at present, some representative examples are used to show the



(a) The observation value of $\hat{a}(t)$ by using the identifier (11)

(b) The observation value of $\hat{b}(t)$ by using the identifier (14)

(c) The observation value of $\hat{c}(t)$ by using the identifier (17)

Figure 1. Parameter estimation results in chaotic finance system.

effectiveness of this method and offer some new and interesting results. These results indicate that the proposed method is nonlinearly stable, robust enough against time-series noise and system disturbances. As the proposed method only requires a time series, they are more applicable in practice, especially for the chaotic communication where the parameters of chaotic systems should be known prior to the implementation of a complete communication protocol. From the theoretical perspective, our approach provides a feasible mathematical principle for the parameter estimation in chaotic communication just by adopting sufficient time-series data, thus strengthening the security of chaotic modulation process. From the engineering perspective, the ability to evaluate unknown parameters for a majority of chaotic systems by a simple statistical mathematical analysis, instead of establishing differential equations, simplifies the design and implementation of parameter observers in chaotic communication.

The rest of this paper is organised as follows. The principle of the proposed method is introduced in detail in §2, and examples of three typical chaotic systems are used to demonstrate the effectiveness of the proposed

Table 1. The results of parameter estimation in chaotic finance system.

Parameters	a	b	c
	Sample time interval $t_r = 0.1$ s		
Real values	0.9	0.2	1.2
Estimation values (100 s)	0.9659 (7.3%)	0.1915 (4.3%)	1.1836 (1.4%)
Estimation values (150 s)	0.9489 (5.4%)	0.1948 (2.6%)	1.1881 (1.0%)
Estimation values (200 s)	0.9383 (4.3%)	0.1962 (1.9%)	1.1884 (1.0%)
	Sample time interval $t_r = 0.05$ s		
Real values	0.9	0.2	1.2
Estimation values (100 s)	0.9593 (6.6%)	0.1898 (5.1%)	1.2041 (3.4‰)
Estimation values (150 s)	0.9291(3.2%)	0.1943 (2.9%)	1.2026 (2.2‰)
Estimation values (200 s)	0.9170 (1.9%)	0.1959 (2.1%)	1.2015 (1.3‰)
	Sample time interval $t_r = 0.01$ s		
Real values	0.9	0.2	1.2
Estimation values (100 s)	0.9012 (1.3‰)	0.1980 (1.0%)	1.2012 (1.0‰)
Estimation values (150 s)	0.9002 (0.2‰)	0.1985 (7.5‰)	1.2006 (0.5‰)
Estimation values (300 s)	0.9000 (0‰)	0.1988 (6.0‰)	1.2006 (0.5‰)

Note: The numbers in parenthesis are the parameter estimation accuracies measured as PE $= |\theta_i - \hat{\theta}_i|/\theta_i$, where θ_i are the parameters to be identified, i.e. a, b and c.

method in §3. Section 4 gives discussion, conclusions are summarised in §5 and the acknowledgments are given in the last part.

2. Principle of the proposed method

Consider the following dynamic system:

$$\dot{X} = F(X, \theta), \tag{1}$$

where $X = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in R^n$ is the observed system vector, $\theta = (\theta_1, \theta_2, \ldots, \theta_m)^{\mathrm{T}} \in R^m$ is the unknown system parameter and $F \in C[R^n \times R^m \to R^n]$ is the function vector of X and θ.

Assuming that the unknown parameters are located in the ith equation of (1), according to the theory of mathematic calculus, the critical point of a differential function of one variable is a point on the graph of the function where the function's derivative is zero. Thus $\dot{X}_i = 0$ at the critical point of the ith variable, and then the following formula can be obtained:

$$F_i(x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)}, \theta_1, \theta_2, \ldots, \theta_m) = 0, \tag{2}$$

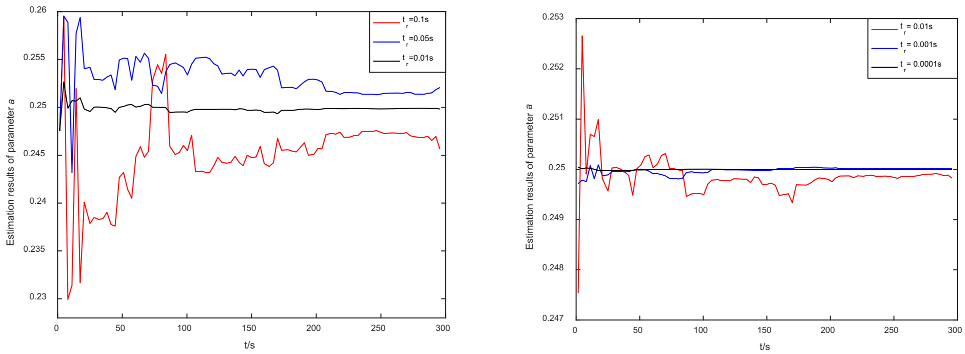
where $x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)}$ represent the values of the variables of the system when the ith variable is located in the critical point. In theory, we can get the following m equations that are similar to eq. (2) from different critical points of X_i:

$$\begin{cases} F_i(x_1^{(i)}(1), x_2^{(i)}(1), \ldots, x_n^{(i)}(1), \theta_1, \theta_2, \ldots, \theta_m) = 0, \\ F_i(x_1^{(i)}(2), x_2^{(i)}(2), \ldots, x_n^{(i)}(2), \theta_1, \theta_2, \ldots, \theta_m) = 0, \\ \vdots \\ F_i(x_1^{(i)}(m), x_2^{(i)}(m), \ldots, x_n^{(i)}(m), \theta_1, \theta_2, \ldots, \theta_m) = 0. \end{cases} \tag{3}$$

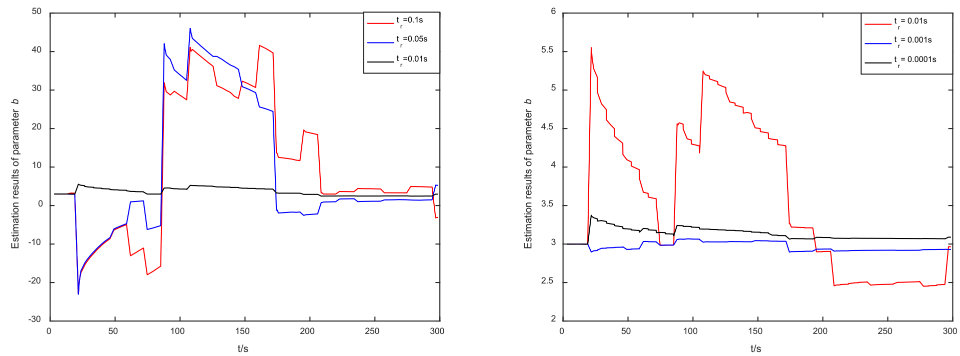
The unknown parameters θ_i $(i = 1, 2, \ldots, m)$ can be obtained by solving eq. (3), i.e. we can identify the unknown parameters in accordance with the critical points in eq. (2).

However, in practical engineering, the measured data depend on the adopted sample points, and in most cases, the extreme value points of a differential function of one variable can be obtained based on these sample points. But they are not always the same as the critical points of a differential function of one variable, and hence, the derivatives of extreme value points of one variable are not always strictly equal to zero, i.e. a deviation from zero is inevitable for the derivatives of the extreme value points. Therefore, it is difficult to directly apply eq. (3) for estimating the unknown parameters in the chaotic systems.

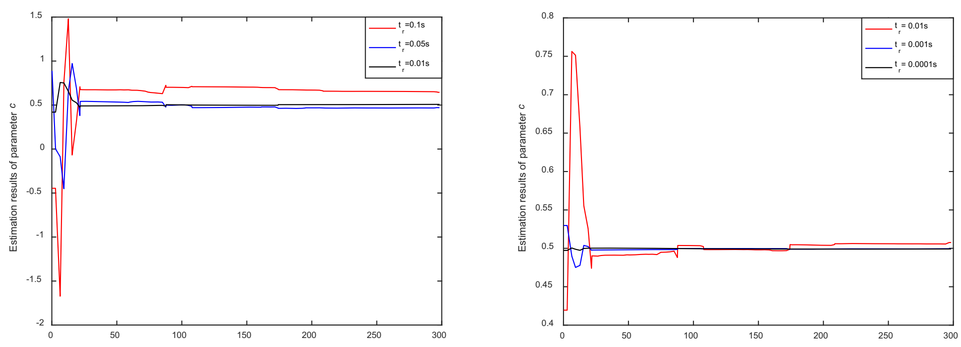
To address this issue, the central limit theorem is introduced. It is well known that the sampling process of the measured data is repeated in equal intervals of time. When the sampling interval is very small, the size of the adopted data is sufficiently large, which satisfies the first precondition of the central limit theorem. Furthermore, when the sampling interval is very



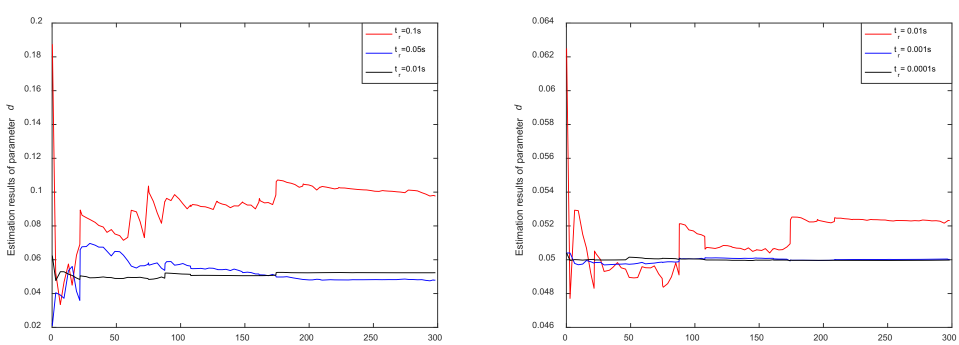
(a) The observation value of $\hat{a}(t)$ by using the identifier (20)



(b) The observation value of $\hat{b}(t)$ by using the identifier (22)



(c) The observation value of $\hat{c}(t)$ by using the identifier (27)



(d) The observation value of $\hat{d}(t)$ by using the identifier (28)

Figure 2. Parameter estimation results in hyperchaotic Rossler system.

Table 2. The results of parameter estimation in hyperchaotic Rossler system.

Parameters	a	b	c	d
		Sample time interval $t_r = 0.1$ s		
Real values	0.25	3	0.5	0.05
Estimation values (100 s)	0.2436 (2.6%)	−2.3864 (1.79)	0.5439 (8.8%)	0.0828 (0.66)
Estimation values (200 s)	0.2442 (2.3%)	13.259 (3.42)	0.6245 (0.25)	0.0898 (0.80)
Estimation values (300 s)	0.2451 (2.0%)	10.097 (2.37)	0.6347 (0.27)	0.0935 (0.87)
		Sample time interval $t_r = 0.05$ s		
Real values	0.25	3	0.5	0.05
Estimation values (100 s)	0.2539 (1.6%)	2.0215 (0.33)	0.4964 (7.2‰)	0.0565 (0.13)
Estimation values (200 s)	0.2537 (1.5%)	12.732 (3.24)	0.4838 (3.2%)	0.0525 (5.0%)
Estimation values (300 s)	0.2530 (1.2%)	8.8288 (1.94)	0.4782 (4.4%)	0.0504 (8.0‰)
		Sample time interval $t_r = 0.01$ s		
Real values	0.25	3	0.5	0.05
Estimation values (100 s)	0.2499 (0.4‰)	3.7310 (0.24)	0.5105 (2.1%)	0.0503 (6.0‰)
Estimation values (200 s)	0.2499 (0.4‰)	3.9828 (0.33)	0.5100 (2.0%)	0.0507 (1.4‰)
Estimation values (300 s)	0.2499 (0.4‰)	3.6285 (0.21)	0.5052 (1.0%)	0.0512 (2.8‰)

Note: The numbers in parenthesis are the parameter estimation accuracies measured as PE $= |\theta_i - \hat{\theta}_i|/\theta_i$, where θ_i are the parameters to be identified, i.e. a, b, c and d.

small, the deviations between the derivatives of extreme value points and the derivatives of critical points are minimal, and considering that the positions of all critical points in the chaotic systems are uniformly randomly distributed, which satisfy the second condition of the central limit theorem, we get the following corollary:

COROLLARY 1

If the sample of extreme value points is enough, then the deviations between the derivatives of extreme value points and derivatives of critical points (*i.e.* $\dot{X}_i = 0$) *tend towards a norm distribution, whose arithmetic mean is zero. In other words, for the extreme value points of a differential function of the ith variable, eq.* (3) *can be modified as follows*:

$$F_i(x_1^{(i)}(k), x_2^{(i)}(k), \ldots, x_n^{(i)}(k), \theta_1, \theta_2, \ldots, \theta_m) = \varepsilon_i(k) \quad (k = 1, 2, \ldots, N), \tag{4}$$

where ε_i *is the residual error set that follows the norm distribution and N is the number of extreme value points obtained from the measured data.*

Least squares method is one of the most popular approaches for parameter estimation. In order to evaluate the precise values of unknown parameters in the systems, the least squares method is introduced in eq. (4). Assuming that $\hat{\theta}$ is the observed value of θ, we get

$$\begin{cases} F_i(x_1^{(i)}(1), x_2^{(i)}(1), \ldots, x_n^{(i)}(1), \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m) = 0, \\ F_i(x_1^{(i)}(2), x_2^{(i)}(2), \ldots, x_n^{(i)}(2), \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m) = 0, \\ \vdots \\ F_i(x_1^{(i)}(m), x_2^{(i)}(m), \ldots, x_n^{(i)}(m), \hat{\theta}_1, \\ \quad \hat{\theta}_2, \ldots, \hat{\theta}_m) = 0. \end{cases} \tag{5}$$

Choosing the following criterion:

$$J = \mathrm{e}^{\mathrm{T}}\mathrm{e} = \sum_{k=1}^{N} [\dot{X}_i^{(i)}(k) - F_i(X^{(i)}(k),\ \hat{\theta})]^2 = \sum_{k=1}^{N} (F_i(X^{(i)}(k),\ \hat{\theta}))^2 \tag{6}$$

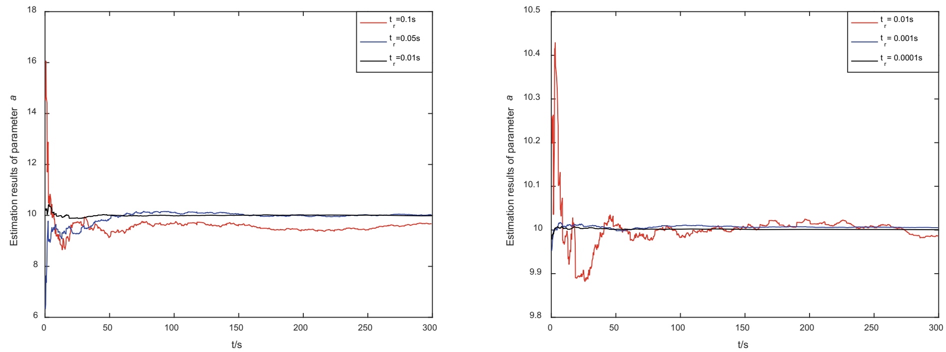
and the best selected value of $\hat{\theta}$ will result in J having the smallest value, i.e.

$$\partial J/\partial\hat{\theta} = 0 \quad (j = 1, 2, \ldots, m). \tag{7}$$

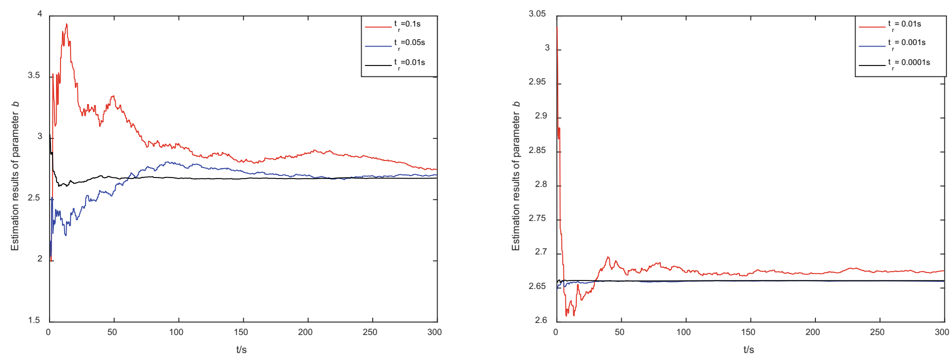
Therefore, we obtain the estimation value of $\hat{\theta}_j$ from eqs (6) and (7).

3. Simulation experiments

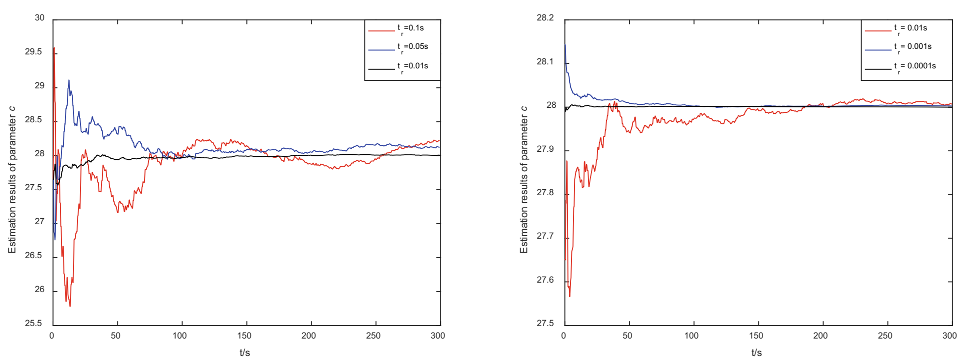
In this section, numerical simulation and comparison are carried out based on several typical chaotic systems including chaotic finance system, hyperchaotic Rossler system and classical Lorenz system. Furthermore, in



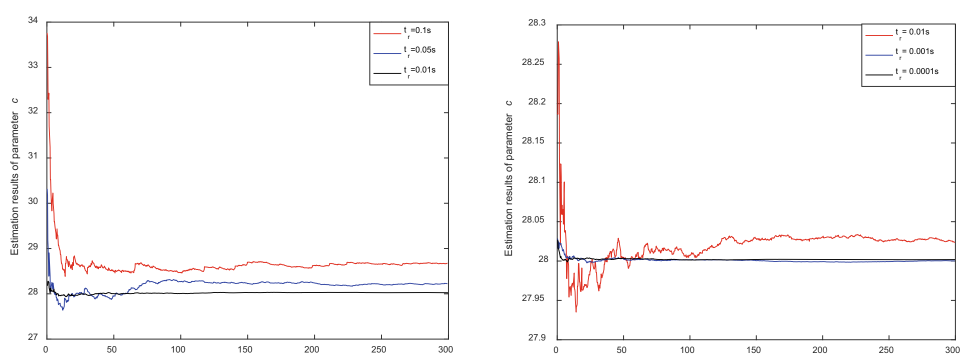
(a) The observation value of $\hat{a}(t)$ by using the identifier (32)



(b) The observation value of $\hat{b}(t)$ by using the identifier (35)



(c) The observation value of $\hat{c}(t)$ by using the identifier (33)



(d) The observation value of $\hat{c}(t)$ by using the identifier (37)

Figure 3. Parameter estimation results in classical Lorenz system.

Table 3. The results of parameter estimation in classical Lorenz system.

Parameters	a	b	c_1	c_2
		Sample time interval $t_r = 0.1$ s		
Real values	10	8/3	28	28
Estimation values (100 s)	9.6405 (3.6%)	3.1775 (1.9%)	27.519 (1.7%)	28.753 (2.7%)
Estimation values (200 s)	9.5802 (4.2%)	3.0134 (1.3%)	27.792 (7.4‰)	28.625 (2.2%)
Estimation values (300 s)	9.5813 (4.2%)	2.9523 (1.1%)	27.862 (4.9‰)	28.672 (2.4%)
		Sample time interval $t_r = 0.05$ s		
Real values	10	8/3	28	28
Estimation values (100 s)	9.7422 (2.6%)	2.5796 (3.2%)	28.353 (1.3%)	28.097 (3.5‰)
Estimation values (200 s)	9.8922 (1.1%)	2.6566 (3.7‰)	28.131 (4.7‰)	28.249 (8.9‰)
Estimation values (300 s)	9.9285 (7.2‰)	2.6672 (0.2‰)	28.142 (5.4‰)	28.184 (6.6‰)
		Sample time interval $t_r = 0.01$ s		
Real values	10	8/3	28	28
Estimation values (100 s)	9.9975 (2.5‰)	2.6791 (4.6‰)	27.923 (2.8‰)	28.007 (0.3‰)
Estimation values (200 s)	10.005 (5.3‰)	2.6734 (2.5‰)	28.008 (0.3‰)	28.016 (0.6‰)
Estimation values (300 s)	10.002 (2.4‰)	2.6738 (2.6‰)	27.973 (1.0‰)	28.019 (0.7‰)

Note: The numbers in parenthesis are the parameter estimation accuracies measured as $\mathrm{PE} = |\theta_i - \hat{\theta}_i|/\theta_i$, where θ_i are the parameters to be identified, i.e. a, b, c_1 (obtained by the identifier (33)) and c_2 (obtained by the identifier (37)).

order to test the robustness of the proposed parameter estimation method, different amplitude-based random noise and disturbances are added to the measured time-series and chaotic systems, respectively, and the effect of parameter estimation results has been investigated.

3.1 *Simulation results of noiseless time series*

Example 1. Parameter estimation of the chaotic finance system.

The first example is the chaotic finance system, which is defined by

$$\begin{cases} \dot{x} = z + (y - a)x, \\ \dot{y} = 1 - by - x^2, \\ \dot{z} = -x - cz, \end{cases} \tag{8}$$

where a, b and c are unknown parameters. The finance system is in the chaotic state when parameters $a = 0.9$, $b = 0.2$ and $c = 1.2$.

For the extreme value point set of system (8) of variable x, the least squares criterion method can be calculated by the following formula according to eq. (6):

$$J_1 = \sum_{k=1}^{N_1} [z^{(1)}(k) + (y^{(1)}(k) - \hat{a})x^{(1)}(k)]^2, \tag{9}$$

where $x^{(1)}$, $y^{(1)}$ and $z^{(1)}$ are the state variable sets when the variable x is located in positions of extreme value points and N_1 is the number of extreme value points of the variable x, which are obtained from the measured data. In accordance with eq. (7), we get

$$\frac{\partial J_1}{\partial \hat{a}} = 2\sum_{k=1}^{N_1} [\hat{a}(x^{(1)}(k))^2 - x^{(1)}(k)z^{(1)}(k) - (x^{(1)}(k))^2 y^{(1)}(k)] = 0. \tag{10}$$

Then $\hat{a}$ is deduced as

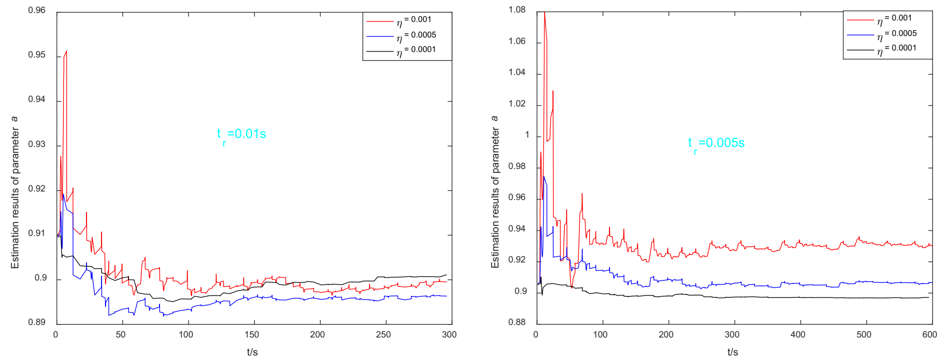
$$\hat{a} = \frac{\sum_{k=1}^{N_1} x^{(1)}(k)z^{(1)}(k) + (x^{(1)}(k))^2 y^{(1)}(k)}{\sum_{k=1}^{N_1} (x^{(1)}(k))^2}. \tag{11}$$

For the extreme value point set of system (8) of variable y, the least squares method can be calculated by the following formula according to eq. (6):

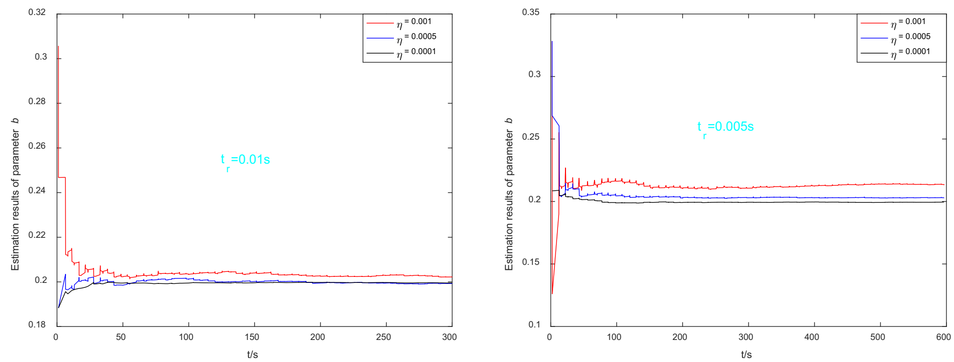
$$J_2 = \sum_{k=1}^{N_2} [1 - (x^{(2)}(k))^2 - \hat{b}y^{(2)}(k)]^2, \tag{12}$$

where $x^{(2)}$ and $y^{(2)}$ are the state variable sets when the variable y is located in positions of extreme value points and N_2 is the number of extreme value points of the variable y, which are obtained from the measured data. In accordance with eq. (7), we get

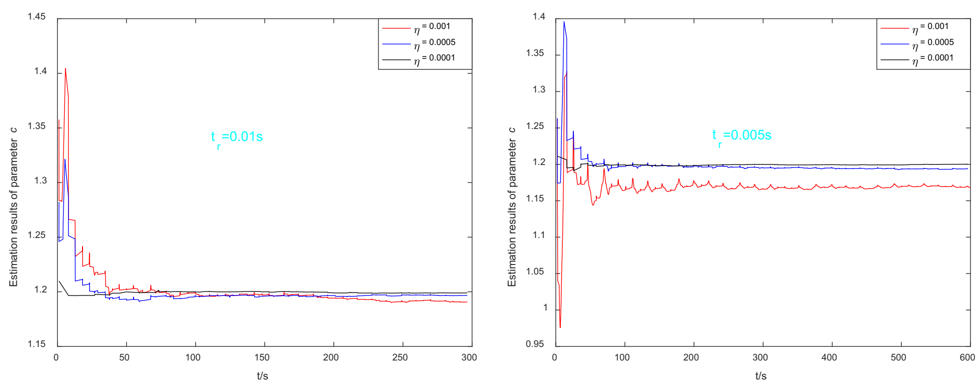
$$\frac{\partial J_2}{\partial \hat{b}} = 2\sum_{k=1}^{N_1} [\hat{b}(y^{(2)}(k))^2 - y^{(2)}(k) + (x^{(2)}(k))^2 y^{(2)}(k)] = 0. \tag{13}$$



(a) The observation value of $\hat{a}(t)$ by using the identifier (11) in the presence of time series noise



(b) The observation value of $\hat{b}(t)$ by using the identifier (14) in the presence of time series noise



(c) The observation value of $\hat{c}(t)$ by using the identifier (17) in the presence of time series noise

Figure 4. Parameter estimation results in noisy time-series chaotic finance system.

Then $\hat{b}$ is deduced as

$$\hat{b} = \frac{\sum_{k=1}^{N_2} y^{(2)}(k) - (x^{(2)}(k))^2 y^{(2)}(k)}{\sum_{k=1}^{N_2} (y^{(2)}(k))^2}. \tag{14}$$

Similarly, for the extreme value points set of system (8) of variable z, we get

$$J_3 = \sum_{k=1}^{N_3} [-x^{(3)}(k) - \hat{c} z^{(3)}(k)]^2, \tag{15}$$

where $x^{(3)}$ and $z^{(3)}$ are the state variable sets when the variable z is located in positions of extreme value points and N_3 is the number of extreme value points of the variable z, which are obtained from the measured data. In accordance with eq. (7), we get

$$\frac{\partial J_3}{\partial \hat{c}} = 2 \sum_{k=1}^{N_1} [\hat{c}(z^{(3)}(k))^2 + x^{(3)}(k) z^{(3)}(k)] = 0. \tag{16}$$

Then $\hat{c}$ is deduced as

$$\hat{c} = \frac{\sum_{k=1}^{N_3} -x^{(3)}(k) z^{(3)}(k)}{\sum_{k=1}^{N_3} (z^{(3)}(k))^2}. \tag{17}$$

Table 4. The results of parameter estimation in noisy time-series chaotic finance system.

Parameters	a	b	c
	Sample time interval $t_r = 0.01$ s		
Real values	0.9	0.2	1.2
Noise item $\eta = 0.001$	0.9023 (2.6‰)	0.2041 (2.1%)	1.2076 (6.3‰)
Noise item $\eta = 0.0005$	0.8982 (2.0‰)	0.2000 (0%)	1.2011 (0.9‰)
Noise item $\eta = 0.0001$	0.9013 (1.4‰)	0.1994 (3.0‰)	1.1993 (0.6‰)
	Sample time interval $t_r = 0.005$ s		
Real values	0.9	0.2	1.2
Noise item $\eta = 0.001$	0.9316 (3.5%)	0.2134 (6.7%)	1.1666 (2.8%)
Noise item $\eta = 0.0005$	0.9097 (1.1%)	0.2042 (2.1%)	1.2013 (1.1‰)
Noise item $\eta = 0.0001$	0.8995 (0.6‰)	0.1998 (1.0‰)	1.1994 (0.5‰)

Note: All the simulation results are based on the statistical average results of extreme point values in the time range [0 s, 300 s].

In simulations, fourth-order Runge–Kutta method is used to solve eq. (8), and the initial state of this system is set as $(x(0), y(0), z(0)) = (1, 3, 2)$. Figure 1 shows the estimation results of parameters a, b and c at different sample time interval t_r. The numbers of extreme value points of the variables x, y and z, which are obtained from the measured time series ($t_r = 0.01$ s), are $N_1 = 75$, $N_2 = 112$ and $N_3 = 73$. Table 1 lists the statistical results of parameters a, b and c at different sample time interval t_r. From figure 1 and table 1, it can be seen that the relative estimation errors of all the unknown parameters in chaotic finance system are very small, which demonstrate the effectiveness of the proposed parameter estimation method.

Example 2. Parameter estimation of the Rossler system.

The second example is the hyperchaotic Rossler system [17], which is defined by

$$\begin{cases} \dot{x}_1 = -x_2 - x_3, \\ \dot{x}_2 = x_1 + ax_2 + x_4, \\ \dot{x}_3 = x_1x_3 + b, \\ \dot{x}_4 = -cx_3 + dx_4, \end{cases} \tag{18}$$

where a, b, c and d are unknown parameters. The Rossler system is in the hyperchaotic state when the parameters are set as $a = 0.25$, $b = 3$, $c = 0.5$ and $d = 0.05$.

For the extreme value point set of system (18) of variable x_2, the least squares method can be calculated by the following formula according to eq. (6):

$$J_4 = \sum_{k=1}^{N_4} [x_1^{(2)}(k) + x_4^{(2)}(k) + \hat{a}x_2^{(2)}(k)]^2, \tag{19}$$

where $x_1^{(2)}$, $x_2^{(2)}$ and $x_4^{(2)}$ are the state variable sets when the variable x_2 is located in positions of extreme value points and N_4 is the number of extreme value points of the variable x_2, which are obtained from the measured data. In accordance with eq. (7), we get

$$\hat{a} = \frac{\sum_{k=1}^{N_4} -x_1^{(2)}(k)x_2^{(2)}(k) - x_2^{(2)}(k)x_4^{(2)}(k)}{\sum_{k=1}^{N_4} (x_2^{(2)}(k))^2}. \tag{20}$$

For the extreme value point set of system (18) of variable x_3, the least squares method can be calculated by the following formula according to eq. (6):

$$J_5 = \sum_{k=1}^{N_5} [x_1^{(3)}(k)x_3^{(3)}(k) + \hat{b}]^2, \tag{21}$$

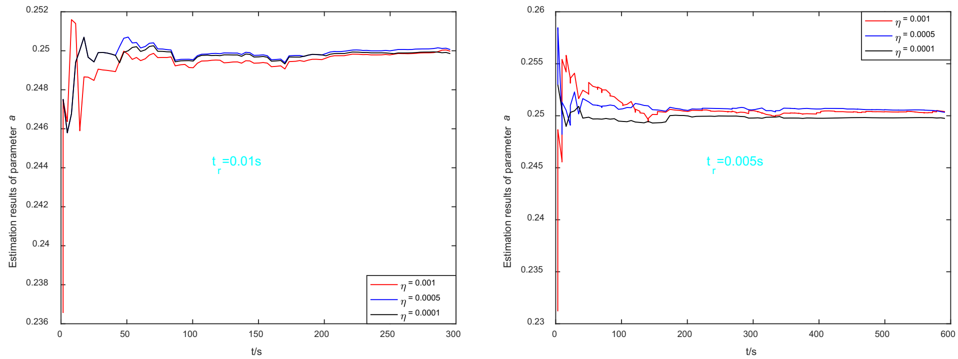
where $x_1^{(3)}$ and $x_3^{(3)}$ are the state variable sets when the variable x_3 is located in positions of extreme value points and N_5 is the number of the extreme value points of the variable x_3, which are obtained from the measured time series. In accordance with eq. (7), we get

$$\hat{b} = -\frac{1}{k}\sum_{k=1}^{N_5} x_1^{(3)}(k)x_3^{(3)}(k). \tag{22}$$

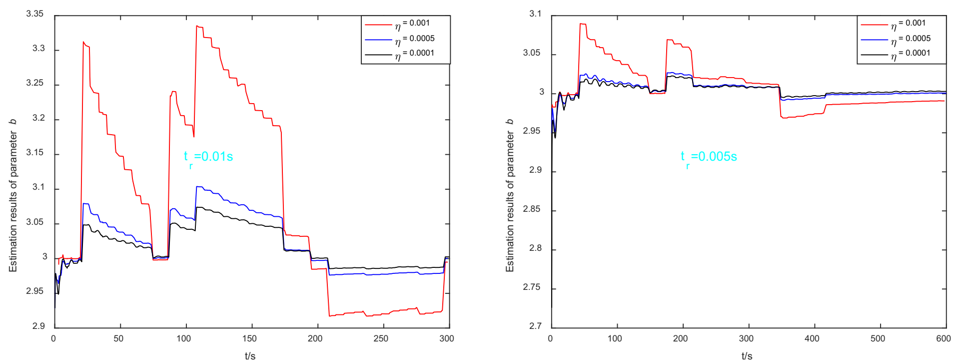
For the extreme value point set of system (18) of variable x_4, the least squares method can be calculated by the following formula according to eq. (6):

$$J_6 = \sum_{k=1}^{N_6} (-\hat{c}x_3^{(4)}(k) + \hat{d}x_4^{(4)}(k))^2, \tag{23}$$

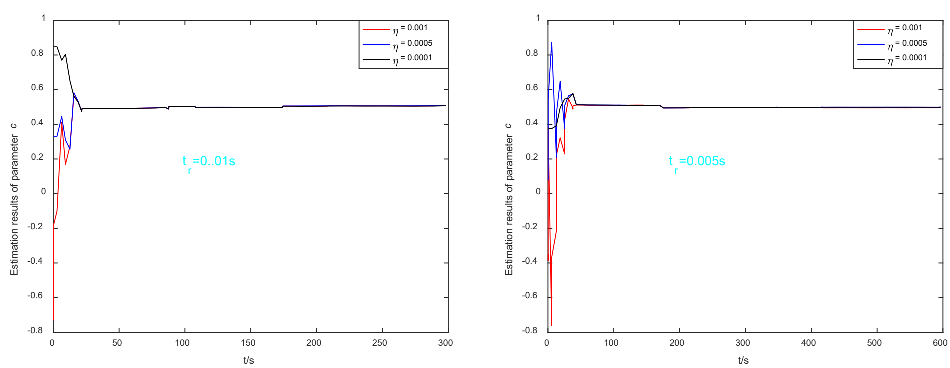
where $x_3^{(4)}$ and $x_4^{(4)}$ are the state variable sets when the variable x_4 is located in positions of extreme value points and N_6 is the number of extreme value points of the variable x_4, which are obtained from the measured time series. In accordance with eq. (7), we get



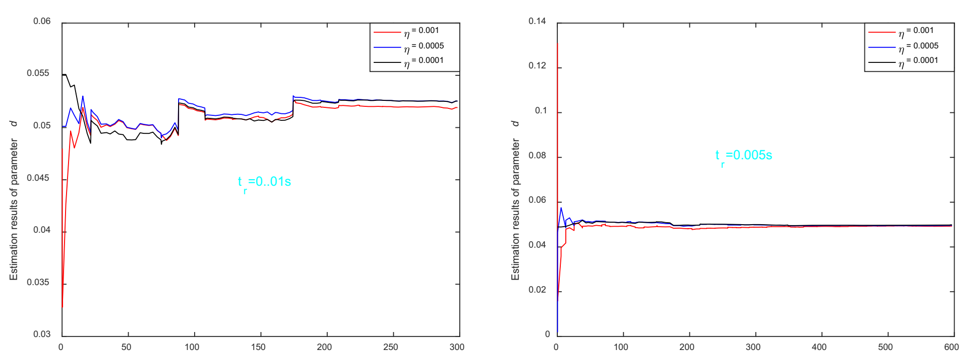
(a) The observation value of $\hat{a}(t)$ by using the identifier (20) in the presence of time series noise



(b) The observation value of $\hat{b}(t)$ by using the identifier (22) in the presence of time series noise



(c) The observation value of $\hat{c}(t)$ by using the identifier (27) in the presence of time series noise



(d) The observation value of $\hat{d}(t)$ by using the identifier (28) in the presence of time series noise

Figure 5. Parameter estimation results in noisy time-series hyperchaotic Rossler system.

Table 5. The results of parameter estimation in noisy time-series hyperchaotic Rossler system.

Parameters	a	b	c	d
		Sample time interval $t_r = 0.01$ s		
Real values	0.25	3	0.5	0.05
Noise item $\eta = 0.001$	0.2493 (2.8‰)	3.0172 (5.7‰)	0.4678 (6.4%)	0.0510 (2.0%)
Noise item $\eta = 0.0005$	0.2498 (0.8‰)	3.0246 (8.2‰)	0.4949 (1.0%)	0.0517 (3.4%)
Noise item $\eta = 0.0001$	0.2497 (1.2‰)	3.0849 (2.8%)	0.5130 (2.6%)	0.0514 (2.8%)
		Sample time interval $t_r = 0.005$ s		
Real values	0.25	3	0.5	0.05
Noise item $\eta = 0.001$	0.2505 (2.0‰)	3.0047 (1.6‰)	0.4686 (6.3%)	0.0491 (1.8%)
Noise item $\eta = 0.0005$	0.2507 (2.8‰)	3.0053 (1.8‰)	0.4974 (5.2‰)	0.0497 (6.0‰)
Noise item $\eta = 0.0001$	0.2498 (0.8‰)	3.0103 (3.4‰)	0.4989 (2.2‰)	0.0500 (0%)

Note: All the simulation results are based on the statistical average results of extreme point values in the time range [0 s, 300 s].

$$\hat{c} = \frac{\left\{\begin{array}{c}\sum_{i=1}^{N_7} x_2^{(5)}(k)x_4^{(5)}(k)\sum_{i=1}^{N_7} x_3^{(5)}(k)x_4^{(5)}(k) + \left(\sum_{i=1}^{N_7} x_3^{(5)}(k)x_4^{(5)}(k)\right)^2 \\ -\left(\sum_{i=1}^{N_7} x_2^{(5)}(k)x_3^{(5)}(k) + \sum_{i=1}^{N_7}(x_3^{(5)}(k))^2\right)\sum_{i=1}^{N_7}(x_4^{(5)}(k))^2\end{array}\right\}}{\left(\sum_{i=1}^{N_7} x_3^{(5)}(k)x_4^{(5)}(k)\right)^2 - \sum_{i=1}^{N_7}(x_3^{(5)}(k))^2\sum_{i=1}^{N_7}(x_4^{(5)}(k))^2}, \tag{27}$$

$$\hat{d} = \frac{\sum_{i=1}^{N_7} x_2^{(5)}(k)x_4^{(5)}(k)\sum_{i=1}^{N_7}(x_3^{(5)}(k))^2 - \sum_{i=1}^{N_7} x_2^{(5)}(k)x_3^{(5)}(k)\sum_{i=1}^{N_7} x_3^{(5)}(k)x_4^{(5)}(k)}{\left(\sum_{i=1}^{N_7} x_3^{(5)}(k)x_4^{(5)}(k)\right)^2 - \sum_{i=1}^{N_7}(x_3^{(5)}(k))^2\sum_{i=1}^{N_7}(x_4^{(5)}(k))^2}. \tag{28}$$

$$\hat{d} = \frac{\sum_{k=1}^{N_6} x_3^{(4)}(k)x_4^{(4)}(k)}{\sum_{k=1}^{N_6}(x_4^{(4)}(k))^2}\hat{c},$$
$$\hat{c} = \frac{\sum_{k=1}^{N_6} x_3^{(4)}(k)x_4^{(4)}(k)}{\sum_{k=1}^{N_6}(x_3^{(4)}(k))^2}\hat{d}. \tag{24}$$

It is obvious that we cannot evaluate the values of the parameters c and d with expression (24). In order to solve this problem, an auxiliary variable is introduced. Assume that $u = x_4 - x_1$, then we have:

$$\dot{u} = (1-c)x_3 + dx_4 + x_2. \tag{25}$$

For the extreme value point set of system (18) of variable u, the least squares method can be calculated by the following formula according to eq. (6):

$$J_7 = \sum_{k=1}^{N_7}[(1-\hat{c})x_3^{(5)}(k) + \hat{d}x_4^{(5)}(k) + x_2^{(5)}(k)]^2, \tag{26}$$

where $x_2^{(5)}$, $x_3^{(5)}$ and $x_4^{(5)}$ are the state variable sets when the variable u is located in positions of extreme value points and N_7 is the number of extreme value points of the variable u that is obtained from the measured time series. In accordance with eq. (7), we get

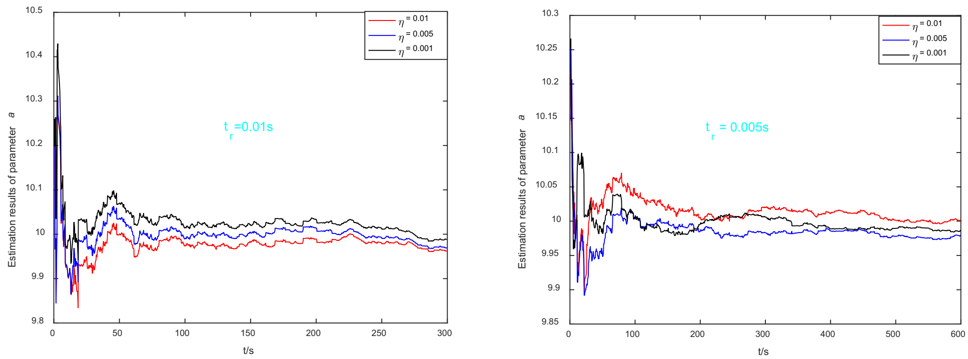
In simulations, the initial state of this system is set as $(x_1(0), x_2(0), x_3(0), x_4(0)) = (-20, 0, 0, 15)$. Figure 2 shows the estimation results of parameters a, b, c and d at different sample time interval t_r. The numbers of extreme value points of the variables x_1, x_2 and u, which are obtained from the measured time series ($t_r = 0.01$s), are $N_4 = 89$, $N_5 = 289$ and $N_7 = 118$. Table 2 lists the statistical results of the parameters a, b, c and d at different sample time interval t_r. From figure 2 and table 2, it is seen that the estimated values obtained by the proposed method are very close to the real values, implying that the smaller is the sample time interval, the better is the parameter estimation result.

Example 3. Parameter estimation of the Lorenz system.

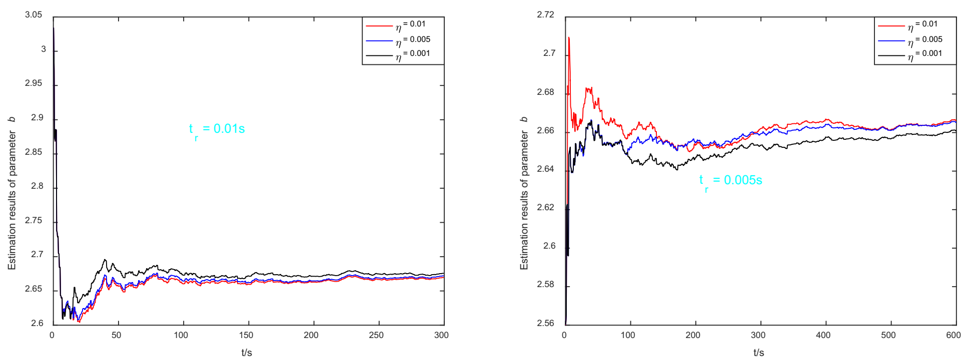
The third example is the classical Lorenz system [18], which is defined by

$$\begin{cases}\dot{x}_1 = a(x_2 - x_1),\\ \dot{x}_2 = cx_1 - x_1x_3 - x_2,\\ \dot{x}_3 = x_1x_2 - bx_3,\end{cases} \tag{29}$$

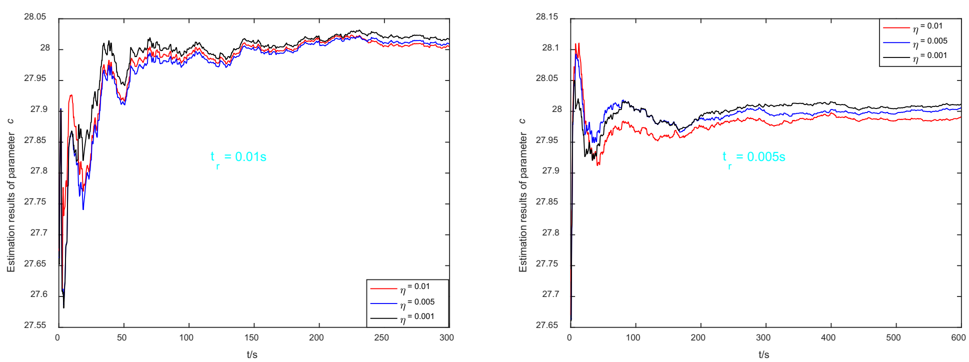
where a, b and c are the unknown parameters. The Lorenz system is in the chaotic state when the parameters are set as $a = 10$, $b = 8/3$ and $c = 28$.



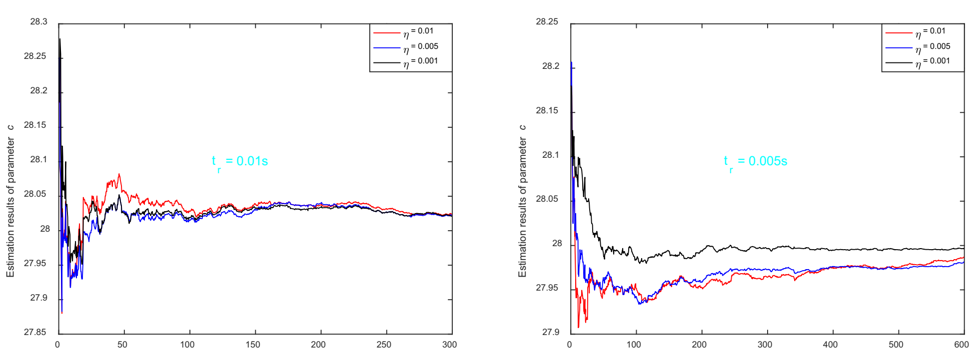
(a) The observation value of $\hat{a}(t)$ by using the identifier (32) in the presence of time series noise



(b) The observation value of $\hat{b}(t)$ by using the identifier (35) in the presence of time series noise



(c) The observation value of $\hat{c}(t)$ by using the identifier (33) in the presence of time series noise



(d) The observation value of $\hat{c}(t)$ by using the identifier (37) in the presence of time series noise

Figure 6. Parameter estimation results in noisy time-series classical Lorenz system.

Table 6. The results of parameter estimation in noisy time-series classical Lorenz system.

Parameters	a	b	c_1	c_2
		Sample time interval $t_r = 0.01$ s		
Real values	10	8/3	28	28
Noise item $\eta = 0.001$	9.9773 (2.3‰)	2.6622 (1.7‰)	27.979 (0.8‰)	28.033 (1.2‰)
Noise item $\eta = 0.0005$	10.000 (0%)	2.6649 (0.7‰)	27.972 (1.0‰)	28.023 (0.8‰)
Noise item $\eta = 0.0001$	10.028 (2.8‰)	2.6738 (2.9‰)	27.989 (0.4‰)	28.028 (1.0‰)
		Sample time interval $t_r = 0.005$ s		
Real values	10	8/3	28	28
Noise item $\eta = 0.001$	10.014 (1.4‰)	2.6620 (0.5‰)	27.979 (0.8‰)	27.966 (1.2‰)
Noise item $\eta = 0.0005$	9.9847 (1.5‰)	2.6582 (3.2‰)	27.994 (0.2‰)	27.969 (1.1‰)
Noise item $\eta = 0.0001$	9.9989 (0.1‰)	2.6526 (5.3‰)	27.993 (0.3‰)	27.999 (0‰)

Note: All the simulation results are based on the statistical average results of extreme point values in the time range [0 s, 300 s].

For the extreme value point set of system (29) of variable x_1, it is hard to evaluate the value of the parameter a with eqs (6) and (7). Similarly, assuming that $u = x_2 - x_1$, then we have

$$\dot{u} = cx_1 - x_2 - x_1x_3 - au. \tag{30}$$

For the extreme value point set of system (29) of variable u, the least squares method can be calculated by the following formula according to eq. (6):

$$J_8 = \sum_{k=1}^{N_8}[\hat{c}x_1^{(4)}(k) - x_1^{(4)}(k)x_3^{(4)}(k) - x_2^{(4)}(k) - au^{(4)}(k)]^2, \tag{31}$$

where $x_1^{(4)}$, $x_2^{(4)}$, $x_3^{(4)}$ and $u^{(4)}$ are the state variable sets when the variable u is located in positions of extreme value points and N_8 is the number of extreme value points of the variable u, which are obtained from the measured data. In accordance with eq. (7), we get

For the extreme value point set of system (29) of variable x_3, the least squares method can be calculated by the following formula according to eq. (6):

$$J_9 = \sum_{k=1}^{N_9}[x_1^{(3)}(k)x_2^{(3)}(k) - \hat{b}x_3^{(3)}(k)]^2, \tag{34}$$

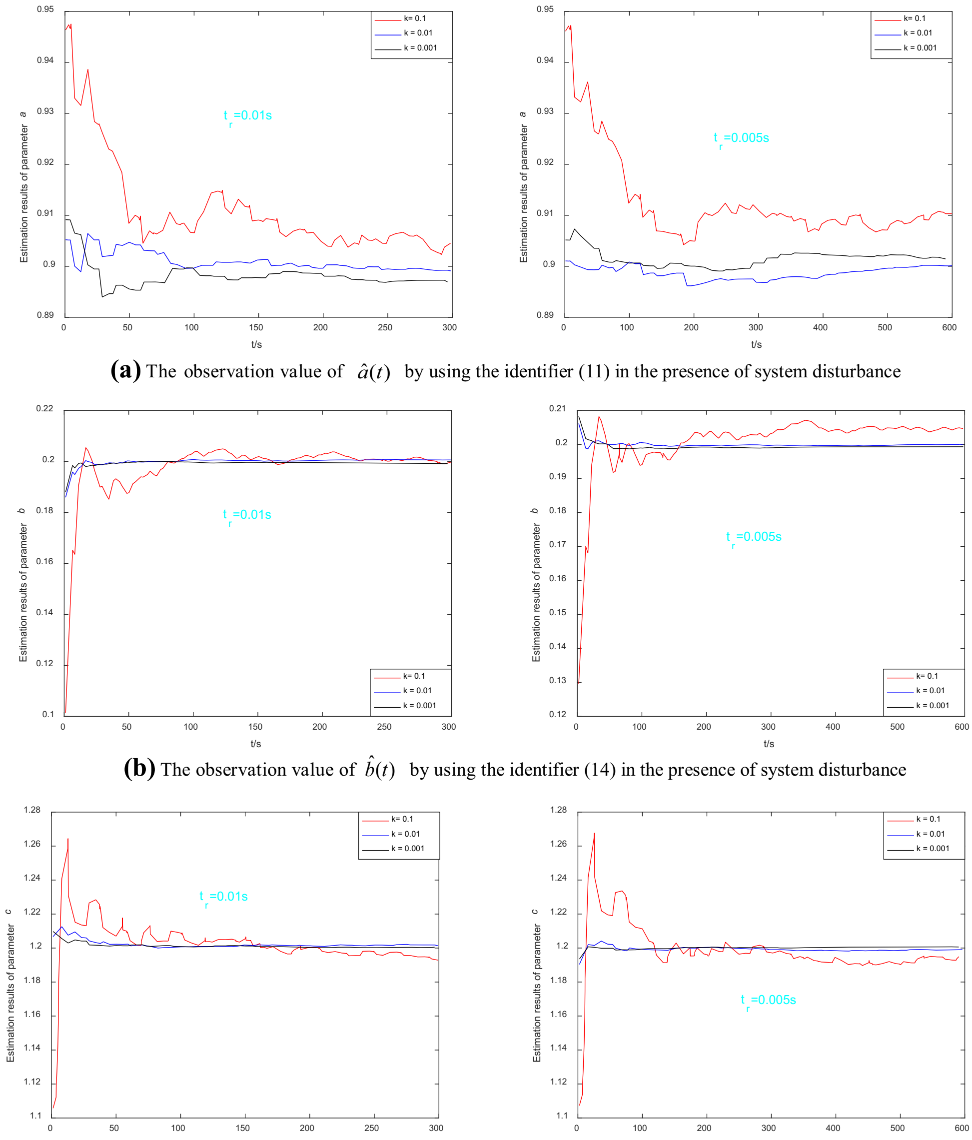
where $x_1^{(3)}$, $x_2^{(3)}$ and $x_3^{(3)}$ are the state variable sets when the variable x_3 is located in positions of extreme value points and N_9 is the number of extreme value points of the variable x_2, which are obtained from the measured data. In accordance with eq. (7), we get

$$\hat{b} = \frac{\sum_{k=1}^{N_9} x_1^{(3)}(k)x_2^{(3)}(k)x_3^{(2)}(k)}{\sum_{k=1}^{N_9}(x_3^{(3)}(k))^2}. \tag{35}$$

Here, we get the estimation values of all the unknown parameters a, b and c. To add more discussion, another

$$\hat{a} = \frac{\left\{\begin{array}{l} \sum_{i=1}^{N_8}[x_1^{(4)}(k)x_3^{(4)}(k) + x_2^{(4)}(k)]x_1^{(4)}(k)\sum_{i=1}^{N_8} x_1^{(4)}(k)u^{(4)}(k) \\ \quad - \sum_{i=1}^{N_8}[x_1^{(4)}(k)x_3^{(4)}(k)+x_2^{(4)}(k)]u^{(4)}(k)\sum_{i=1}^{N_8}(x_1^{(4)}(k))^2 \end{array}\right\}}{\sum_{i=1}^{N_8}(u^{(4)}(k))^2\sum_{i=1}^{N_8}(x_1^{(4)}(k))^2 - \left(\sum_{i=1}^{N_8} x_1^{(4)}(k)u^{(4)}(k)\right)^2}, \tag{32}$$

$$\hat{c} = \frac{\left\{\begin{array}{l} \sum_{i=1}^{N_8}[x_1^{(4)}(k)x_3^{(4)}(k) + x_2^{(4)}(k)]x_1^{(4)}(k)\sum_{i=1}^{N_8}(u^{(4)}(k))^2 \\ \quad - \sum_{i=1}^{N_8}[x_1^{(4)}(k)x_3^{(4)}(k)+x_2^{(4)}(k)]u^{(4)}(k)\sum_{i=1}^{N_8} x_1^{(4)}(k)u^{(4)}(k) \end{array}\right\}}{\sum_{i=1}^{N_8}(u^{(4)}(k))^2\sum_{i=1}^{N_8}(x_1^{(4)}(k))^2 - \left(\sum_{i=1}^{N_8} x_1^{(4)}(k)u^{(4)}(k)\right)^2}. \tag{33}$$



(a) The observation value of $\hat{a}(t)$ by using the identifier (11) in the presence of system disturbance

(b) The observation value of $\hat{b}(t)$ by using the identifier (14) in the presence of system disturbance

(c) The observation value of $\hat{c}(t)$ by using the identifier (17) in the presence of system disturbance

Figure 7. Parameter estimation results in disturbance-based chaotic finance system.

parameter estimation approach with respect to the parameter c is proposed. For the extreme value point set of system (29) of variable x_2, the least squares criterion method can be calculated by the following formula according to eq. (6):

$$J_{10} = \sum_{k=1}^{N_{10}} [\hat{c}x_1^{(2)}(k) - x_1^{(2)}(k)x_3^{(2)}(k) - x_2^{(2)}(k)]^2, \tag{36}$$

where $x_1^{(2)}$, $x_2^{(2)}$ and $x_3^{(2)}$ are the state variable sets when the variable x_2 is located in positions of extreme value points and N_{10} is the number of extreme value points of the variable x_2, which are obtained from the measured data. In accordance with eq. (7), we get

$$\hat{c} = \frac{\sum_{k=1}^{N_{10}} (x_1^{(2)}(k))^2 x_3^{(2)}(k) + x_1^{(2)}(k)x_2^{(2)}(k)}{\sum_{k=1}^{N_{10}} (x_1^{(2)}(k))^2}. \tag{37}$$

In simulations, the initial state of this system is set as $(x_1(0), x_2(0), x_3(0)) = (10, 10, 10)$. Figure 3 shows the estimation results of parameters a, b and c at

Table 7. The results of parameter estimation in disturbance-based chaotic finance system.

Parameters	a	b	c
	Sample time interval $t_r = 0.01$ s		
Real values	0.9	0.2	1.2
Disturbance item $\eta = 0.1$	0.9130 (1.4%)	0.1982 (0.9%)	1.2000 (0%)
Disturbance item $\eta = 0.01$	0.9010 (1.1‰)	0.2001 (0.5‰)	1.2021 (1.8‰)
Disturbance item $\eta = 0.001$	0.8983 (1.9‰)	0.1993 (3.5‰)	1.2012(1.0‰)
	Sample time interval $t_r = 0.005$ s		
Real values	0.9	0.2	1.2
Disturbance item $\eta = 0.1$	0.9138 (1.5%)	0.2012 (0.6%)	1.1955 (3.8‰)
Disturbance item $\eta = 0.01$	0.8987 (1.4‰)	0.1999 (0.5‰)	1.1992 (0.7‰)
Disturbance item $\eta = 0.001$	0.9016 (1.8‰)	0.1993 (3.5‰)	1.2001 (0.1‰)

Note: All the simulation results are based on the statistical average results of extreme point values in the time range [0 s, 300 s].

different sample time interval t_r. The numbers of extreme value points of the variables x_1, x_2 and u, which are obtained from the measured time series ($t_r = 0.01$s), are $N_8 = 979$, $N_9 = 797$ and $N_{10} = 979$. Table 3 lists the statistical results of parameters a, b and c at different sample time intervals t_r. From figure 3 and table 3, it is seen that for the parameter c, whether we adopt the identifier equation (33) or the identifier equation (37), the errors between the observation values and the real values are trivial, which verify the effectiveness of the proposed estimation method again.

3.2 *Simulation results in the presence of time-series noise*

Here, we shall analyse the influence of time-series noise on the estimation performance of the proposed scheme in chaotic systems. To be concrete, the noise term is assumed as Gaussian white noise, which is independently and identically distributed with zero mean, and suppose that the measured time-series output of n-dimensional chaotic system is described as

$$x_i^{\text{real}} = x_i + \eta_i, \quad i = 1, 2, \ldots, n, \tag{38}$$

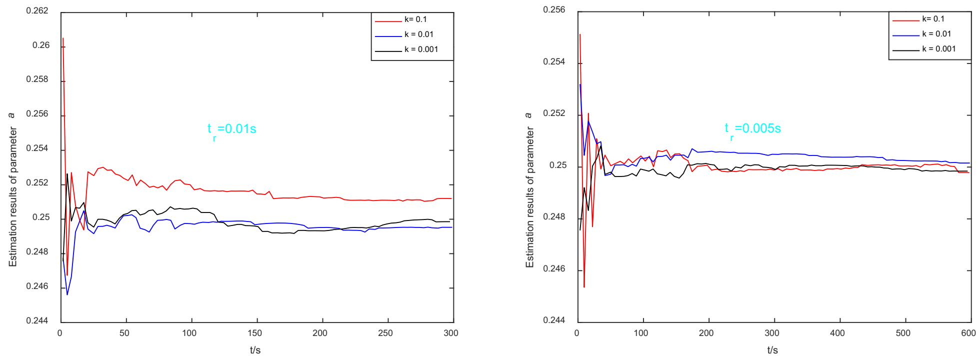
where x_i^{real} is the real measured output, x_i is the evaluation output in theory as shown in eq. (1) and η_i is the uncertain time-series noise with different amplitudes.

For Example 1, we keep all the conditions invariant and add the noise to the measured time series of chaotic finance system. Let the amplitudes of random noises η_i ($i = 1, 2, 3$) be 0.01, 0.005 and 0.001, and use the identifier equations (11), (14) and (17) to estimate the unknown parameters. Figure 4 shows the results of unknown parameter estimation in the presence of noise with zero mean, sample time 0.01 and 0.005 s, respectively. Table 4 lists the corresponding statistical indicators with respect to figure 4.

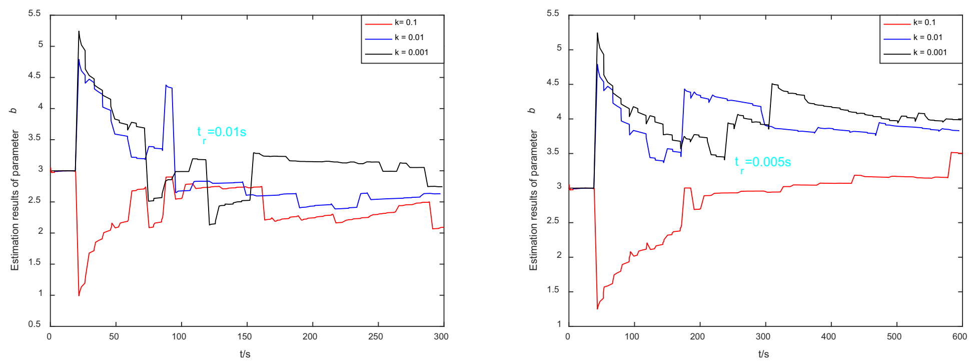
For Example 2, we keep all the conditions invariant and add the noise to the measured time series of hyperchaotic Rossler system. Let the amplitudes of random noises η_i ($i = 1, 2, 3$) be 0.01, 0.005 and 0.001, and use the identifier equations (20), (22), (27) and (28) to estimate the unknown parameters. Figure 5 shows the results of unknown parameter estimation in the presence of noise with sample time 0.01 and 0.005 s, respectively. Table 5 lists the corresponding results with respect to figure 5.

For Example 3, we keep all the conditions invariant and add the noise to the measured time series of classical Lorenz system. Let the amplitudes of random noises η_i ($i = 1, 2, 3$) be 0.01, 0.005 and 0.001. We still use the identifier equations (32), (33), (35) and (37) to estimate the unknown parameters. Figure 6 shows the results of unknown parameter estimation in the presence of noise with zero mean, sample time 0.01 and 0.005 s, respectively. Table 6 lists the corresponding results with respect to figure 6.

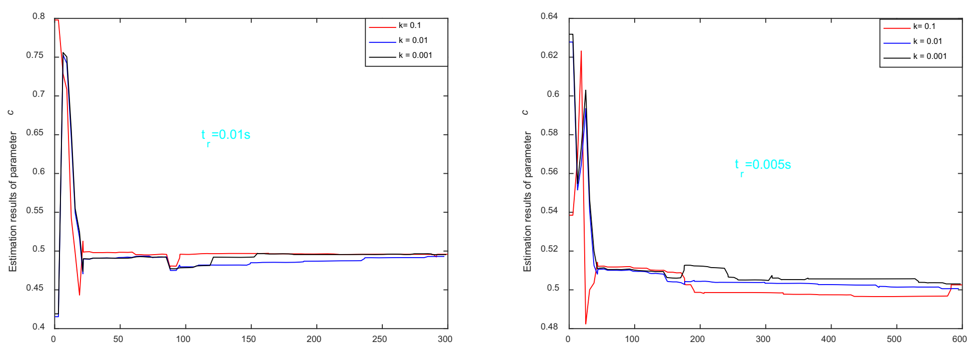
From figures 4–6, we can see that the estimation curves of the proposed method are a little oscillatory around the true parameter values for the chaotic systems, and from tables 4–6, we can see that when the amplitudes of time-series noise are lower or when the frequency of time-series noise is higher, the results



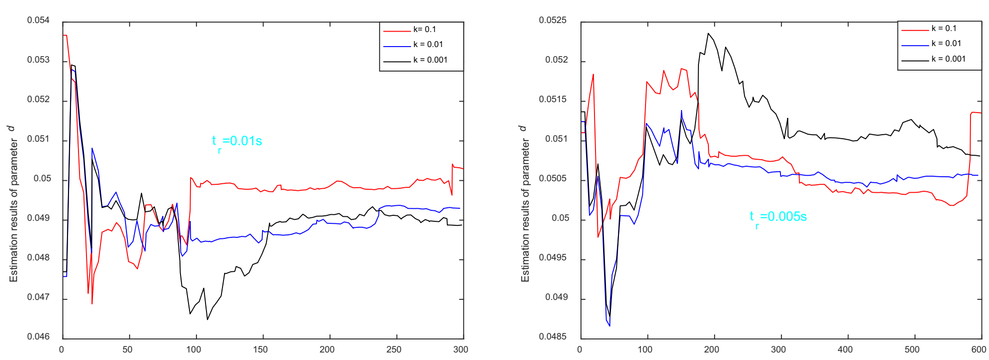
(a) The observation value of $\hat{a}(t)$ by using the identifier (20) in the presence of system disturbance



(b) The observation value of $\hat{b}(t)$ by using the identifier (22) in the presence of system disturbance



(c) The observation value of $\hat{c}(t)$ by using the identifier (27) in the presence of system disturbance



(d) The observation value of $\hat{d}(t)$ by using the identifier (28) in the presence of system disturbance

Figure 8. Parameter estimation results in disturbance-based hyperchaotic Rossler system.

Table 8. The results of parameter estimation in disturbance-based hyperchaotic Rossler system.

Parameters	a	b	c	d
		Sample time interval $t_r = 0.01$ s		
Real values	0.25	3	0.5	0.05
Disturbance item $\eta = 0.1$	0.2516 (6.4‰)	2.4132 (0.20)	0.5050 (1.0%)	0.0497 (0.6%)
Disturbance item $\eta = 0.01$	0.2496 (1.6‰)	3.0575 (1.9%)	0.4928 (1.4%)	0.0490 (2.0%)
Disturbance item $\eta = 0.001$	0.2499 (0.4‰)	3.0895 (3.0%)	0.4978 (4.4‰)	0.0488 (2.4%)
		Sample time interval $t_r = 0.005$ s		
Real values	0.25	3	0.5	0.05
Disturbance item $\eta = 0.1$	0.2501 (0.4‰)	2.8313 (5.6%)	0.5030 (0.6%)	0.0507 (1.4%)
Disturbance item $\eta = 0.01$	0.2504 (1.6‰)	3.8566 (0.29)	0.5082 (1.6%)	0.0506 (1.2%)
Disturbance item $\eta = 0.001$	0.2499 (0.4‰)	3.9347 (0.31)	0.5113 (2.2%)	0.0510 (2.0%)

Note: All the simulation results are based on the statistical average results of extreme point values in the time range [0 s, 300 s].

are better in terms of all the indicators. From the overall obtained results, we can see that the proposed method has good performance against the time-series noise.

3.3 *Simulation results in the presence of system disturbance*

Here, we shall consider the influence of system disturbance on the estimation performance of the proposed scheme in chaotic systems. Suppose that we have

$$\dot{x}_i = F_i(x, \theta_i) + \mu_i, \quad i \in \{1, 2, \ldots, n\}, \tag{39}$$

where μ_i represents the unknown disturbance item of the system. In our simulations, we assume that $\mu_i = k\sin(10t)$ and we shall show the different impacts of k.

For Example 1, we keep all the conditions invariant and add the disturbance into the state variables x_i $(i = 1, 2, 3)$ of the chaotic finance system. We still use identifiers (11), (14) and (17) to evaluate the unknown parameters. Figure 7 shows the corresponding estimation results when $k = 0.1$, 0.01 and 0.001. Table 7 lists the corresponding results with respect to figure 7.

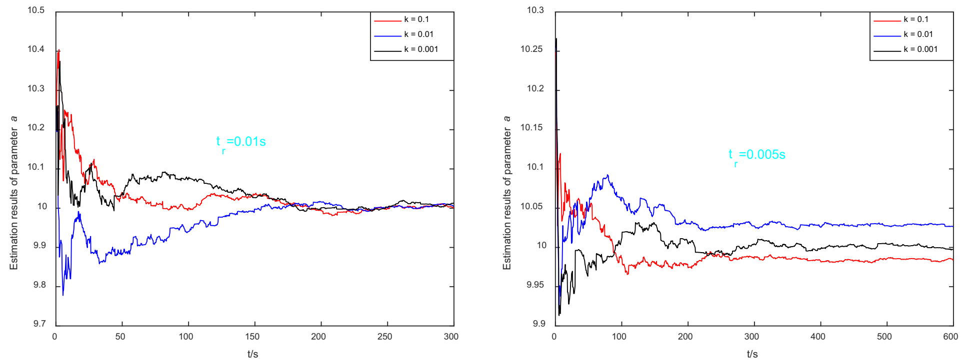
For Example 2, we keep all the conditions invariant and add the disturbance into the state variables x_i $(i = 1, 2, 3)$ of the hyperchaotic Rossler system. We still use identifiers (20), (22), (27) and (28) to evaluate the unknown parameters. Figure 8 shows the corresponding estimation results when $k = 0.1$, 0.01 and 0.001. Table 8 lists the corresponding results with respect to figure 8.

For Example 3, we keep all the conditions invariant and add the disturbance into the state variables x_i $(i = 1, 2, 3)$ of the classical Lorenz system. We still use identifiers (32), (33), (35) and (37) to evaluate the unknown parameters. Figure 9 shows the corresponding estimation results when $k = 0.1$, 0.01 and 0.001. Table 9 lists the corresponding results with respect to figure 9.

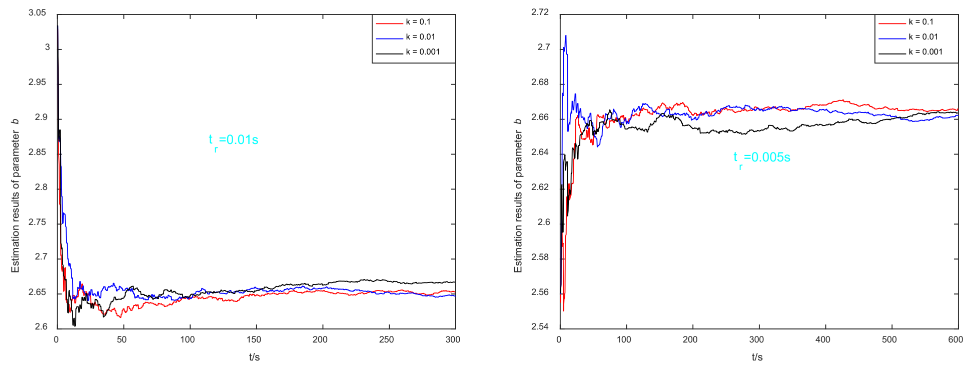
4. Discussion

Chaotic systems have a remarkable property that complicated behaviours always emerge from a set of irregular orbits. This leads to the phenomenon that most of the movement complications can be explained by a simple statistical mathematical analysis in a record time series, which motivated us to design a statistic and systematic scheme based on least squares estimation to identify the multiple parameters in chaotic systems, thus avoiding the pitfalls of traditional local analysis method.

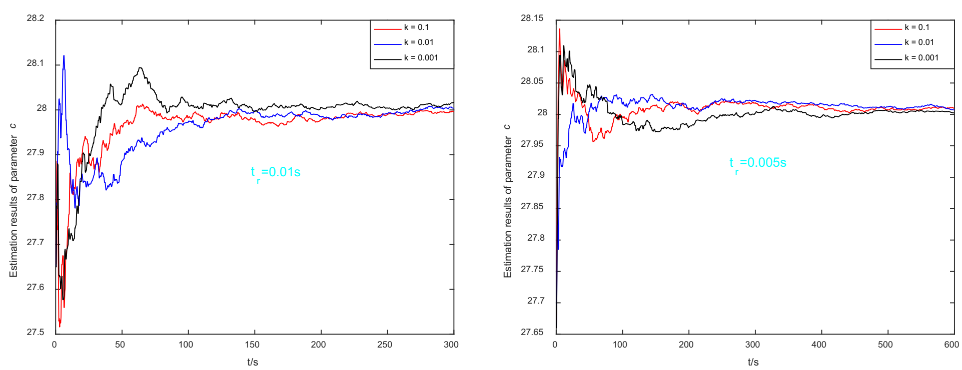
The critical point of this parameter estimation method as shown in eqs (5)–(7) is to find enough extreme value points and make sure that every extreme value point is independent. For a large class of chaotic systems, the above two items are always satisfied, which give us a sufficient condition and new insight to apply our proposed estimation method into chaotic systems. In comparison with [19–21] based on statistical method, the proposed estimation methods use only the measured time-series data without the requirement for a complicated theoretical analysis, which significantly decreased the difficulties in practical implementation. In addition, the absence of the adjustment parameters in observers, which are proposed in [22–25], will enhance the simplicity of the parameter estimation design and decrease the computational cost.



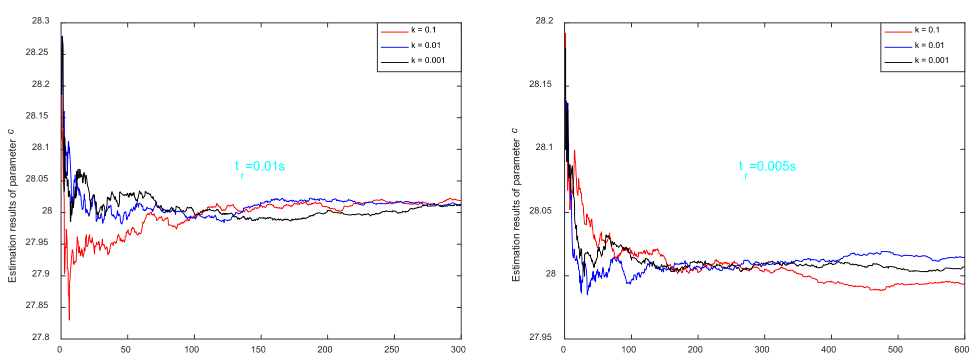
(a) The observation value of $\hat{a}(t)$ by using the identifier (32) in the presence of system disturbance



(b) The observation value of $\hat{b}(t)$ by using the identifier (35) in the presence of system disturbance



(c) The observation value of $\hat{c}(t)$ by using the identifier (33) in the presence of system disturbance



(d) The observation value of $\hat{c}(t)$ by using the identifier (37) in the presence of sytem disturbance

Figure 9. Parameter estimation results in disturbance-based classical Lorenz system.

Table 9. The results of parameter estimation in disturbance-based classical Lorenz system.

Parameters	a	b	c_1	c_2
		Sample time interval $t_r = 0.01$ s		
Real values	10	8/3	28	28
Disturbance item $\eta = 0.1$	10.027 (2.7‰)	2.6475 (7.2‰)	27.963 (1.3‰)	27.997 (0.1‰)
Disturbance item $\eta = 0.01$	9.9696 (3.0‰)	2.6567 (3.9‰)	27.960 (1.4‰)	28.013 (0.5‰)
Disturbance item $\eta = 0.001$	10.037 (3.7‰)	2.6586 (3.0‰)	27.990 (0.4‰)	28.008 (0.3‰)
		Sample time interval $t_r = 0.005$ s		
Real values	10	8/3	28	28
Disturbance item $\eta = 0.1$	9.9930 (0.7‰)	2.6625 (1.5‰)	28.009 (0.3‰)	28.008 (0.3‰)
Disturbance item $\eta = 0.01$	10.036 (3.6‰)	2.6631 (1.3‰)	28.011 (0.4‰)	28.011 (0.4‰)
Disturbance item $\eta = 0.001$	10.001 (0.1‰)	2.6560 (4.0‰)	28.002 (0.1‰)	28.012 (0.4‰)

Note: All the simulation results are based on the statistical average results of extreme point values in the time range [0 s, 300 s]

5. Conclusion

On the basis of least squares estimation, a novel off-line parameter estimation approach for different chaotic systems is proposed in this paper. This approach gives a systematic procedure for estimating parameters from the measured time series on the basis of central limit theorem, where all unknown system parameters can be estimated dynamically. Three numerical chaotic system simulations have been conducted to verify the validity of the proposed method, and the results show that the proposed method has good estimation results for different types of chaotic systems. Furthermore, the numerical results also verify the fact that the proposed estimation method is robust to the time-series noise and system disturbance in various chaotic systems.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Nos 41571514, U1765201) and the Fundamental Research Funds for the Central Universities (No. 2017KFYXJJ204).

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