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CONSTRUCTION OF STABLE RANK 2 BUNDLES ON $\mathbb{P}^3$ VIA SYMPLECTIC BUNDLES

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Abstract: In this article we study the Gieseker–Maruyama moduli spaces $\mathscr{B}(e,n)$ of stable rank 2 algebraic vector bundles with Chern classes $c_1 = e \in \{-1, 0\}$ and $c_2 = n \geq 1$ on the projective space $\mathbb{P}^3$. We construct the two new infinite series Σ_0 and Σ_1 of irreducible components of the spaces $\mathscr{B}(e,n)$ for $e = 0$ and $e = -1$, respectively. General bundles of these components are obtained as cohomology sheaves of monads whose middle term is a rank 4 symplectic instanton bundle in case $e = 0$, respectively, twisted symplectic bundle in case $e = -1$. We show that the series Σ_0 contains components for all big enough values of n (more precisely, at least for $n \geq 146$). Σ_0 yields the next example, after the series of instanton components, of an infinite series of components of $\mathscr{B}(0,n)$ satisfying this property.

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Introduction

Given $e \in \{-1, 0\}$ and $n \in \mathbb{Z}_+$ let $\mathscr{B}(e,n)$ be the Gieseker–Maruyama moduli space of stable rank 2 algebraic vector bundles with Chern classes $c_1 = e$ and $c_2 = n$ on the projective space $\mathbb{P}^3$. Hartshorne [1] showed that $\mathscr{B}(e,n)$ is a quasiprojective scheme, nonempty for arbitrary $n \geq 1$ in case $e = 0$ and, respectively, for even $n \geq 2$ in case $e = -1$, and the deformation theory predicts that each irreducible component of $\mathscr{B}(e,n)$ has dimension at least $8n - 3 + 2e$.

In case $e = 0$ it is known by now (see [1–8]) that the scheme $\mathscr{B}(0,n)$ contains an irreducible component I_n of expected dimension $8n - 3$, and this component is the closure of the smooth open subset of I_n constituted by the so-called mathematical instanton vector bundles. Historically, $\{I_n\}_{n \geq 1}$ was the first known infinite series of the irreducible components of $\mathscr{B}(0,n)$ having the expected dimension $\dim I_n = 8n - 3$. In [1, Example 4.3.2] Hartshorne constructed a first infinite series $\{\mathscr{B}_0(-1, 2m)\}_{m \geq 1}$ of the irreducible components $\mathscr{B}_0(-1, 2m)$ of $\mathscr{B}(-1, 2m)$ having the expected dimension $\dim \mathscr{B}_0(-1, 2m) = 16m - 5$.

The other infinite series of families of vector bundles of dimension $3k^2 + 10k + 8$ from $\mathscr{B}(0, 2k+1)$ was constructed in 1978 by Barth and Hulek, and Ellingsrud and Strømme in [2, (4.6), (4.7)] showed that these families are open subsets of irreducible components distinct from the instanton components I_{2k+1}. Later in 1985 and 1987 Vedernikov [9, 10] constructed three infinite series of families of bundles from $\mathscr{B}(0,n)$, and one infinite family of bundles from $\mathscr{B}(-1, 2m)$. A more general series of rank 2 bundles depending on triples of integers a, b, c, appeared in 1984 in the paper of Prabhakar Rao [11]. Soon after that, in 1988, Ein [12] independently studied these bundles and proved that they constitute open parts of irreducible components of $\mathscr{B}(e,n)$ for both $e = 0$ and $e = -1$.

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A new progress in the description of the spaces $\mathscr{B}(0,n)$ was achieved in 2017 by Almeida, Jardim, A. Tikhomirov, and S. Tikhomirov in [13], where they constructed a new infinite series of irreducible components Y_a of the spaces $\mathscr{B}(0,1+a^2)$ for $a \in \{2\} \cup \mathbb{Z}_{\geq 4}$. These components have dimensions $\dim Y_a = 4\binom{a+3}{3} - a - 1$ which for $a \geq 4$ is larger than expected. General bundles from these components are obtained as the cohomology bundles of rank 1 monads whose middle term is a rank 4 symplectic instanton with $c_2 = 1$, and the left-hand and right-hand terms are $\mathscr{O}_{\mathbb{P}^3}(-a)$ and $\mathscr{O}_{\mathbb{P}^3}(a)$, respectively.

The aim of the present article is to provide the two new infinite series of irreducible components $\mathscr{M}_n$ of $\mathscr{B}(e,n)$: one, for $e = 0$ and another, for $e = -1$ which in some sense generalizes the above construction from [13]. Namely, in case $e = 0$ we construct an infinite series Σ_0 of irreducible components $\mathscr{M}_n$ of $\mathscr{B}(0,n)$, such that a general bundle of $\mathscr{M}_n$ is a cohomology bundle of a monad of the type by analogy to the above, the middle term of which is a rank 4 symplectic instanton with arbitrary second Chern class. The first main result of the article, Theorem 1, states that the series Σ_0 contains components $\mathscr{M}_n$ for all big enough values of n (more precisely, at least for $n \geq 146$). The series Σ_0 is a first example, besides the instanton series $\{I_n\}_{n\geq 1}$, of the series with this property. (For all other series mentioned above the question whether they contain components with all big enough values of the second Chern class n is open.)

In case $e = -1$ we construct in a similar way an infinite series Σ_1 of irreducible components $\mathscr{M}_n$ of $\mathscr{B}(-1,n)$, such that a general bundle of $\mathscr{M}_n$ is a cohomology bundle of a monad of the type by analogy to the above, in which the left-hand and right-hand terms are $\mathscr{O}_{\mathbb{P}^3}(-a-1)$ and $\mathscr{O}_{\mathbb{P}^3}(a)$, respectively, and the middle term is a twisted rank 4 symplectic bundle with the first Chern class -2. The second main result of the article, Theorem 2, states that Σ_1 contains components $\mathscr{M}_n$ asymptotically for almost all big enough values of n. (A precise statement about the behavior of the set of values of n for which $\mathscr{M}_n$ is contained in Σ_1 is given in Remark 3.)

We will give a brief sketch of the contents of the article. In Section 1 we study some properties of pairs $([\mathscr{E}_1],[\mathscr{E}_2])$ of mathematical instanton bundles and prove the vanishing of certain cohomology groups of their twists by line bundles $\mathscr{O}_{\mathbb{P}^3}(a)$ and $\mathscr{O}_{\mathbb{P}^3}(-a)$ (see Proposition 1). The direct sum $\mathbb{E} = \mathscr{E}_1 \oplus \mathscr{E}_2$ is then used in Section 2 as a test rank 4 symplectic instanton bundle. This bundle and its deformations are used as middle terms of anti-self-dual monads of the form $0 \to \mathscr{O}_{\mathbb{P}^3}(-a) \to \mathbb{E} \to \mathscr{O}_{\mathbb{P}^3}(a) \to 0$, the cohomology bundles of which provide general bundles of the components $\mathscr{M}_n$ of the series Σ_0 (see Theorem 1). In Section 3 we study the direct sums $\mathbb{E} = \mathscr{E}_1 \oplus \mathscr{E}_2$ of vector bundles, $\mathscr{E}_i$ are the bundles from the Hartshorne series $\{\mathscr{B}_0(-1,2n)\}_{n\geq 1}$ mentioned above. We prove certain vanishing properties for the cohomology of twists of $\mathscr{E}_i$ (see Proposition 2). These properties are then used in Theorem 2 in the construction of general vector bundles of components $\mathscr{M}_n$ of Σ_1. In Section 4 we give the list of $\mathscr{M}_n \in \Sigma_0$ for $n \leq 20$ and $\mathscr{M}_n \in \Sigma_1$ for $n \leq 40$.

Conventions and notation. Everywhere in this paper we work over the base field of complex numbers $\mathbf{k} = \mathbb{C}$, and $\mathbb{P}^3$ is a projective 3-space over $\mathbf{k}$. For a stable rank 2 vector bundle E with $c_1(E) = e$ and $c_2(E) = n$ on $\mathbb{P}^3$, we denote by $[E]$ its isomorphism class in $\mathscr{B}(e,n)$.

§ 1. Some Properties of Mathematical Instantons

Let a and m be two positive integers, where $a \geq 2$, and let $\varepsilon \in \{0,1\}$. In this section we prove the following proposition about mathematical instanton vector bundles which will be used in the proof of Theorem 1.

Proposition 1. *A general pair*

$$([\mathscr{E}_1],[\mathscr{E}_2]) \in I_m \times I_{m+\varepsilon} \tag{1}$$

of instanton vector bundles satisfies the conditions:

$$[\mathscr{E}_1] \neq [\mathscr{E}_2]; \tag{2}$$

for $i=1,\ m\le a+1$, *respectively,* $i=2,\ m+\varepsilon\le a+1$,

$$h^1(\mathscr{E}_i(a))=0, \tag{3}$$

$$h^2(\mathscr{E}_i(-a))=0 \quad \textit{if } a\ge 12; \tag{4}$$

for $i=1,\ m\le a-4,\ a\ge 5$, *respectively,* $i=2,\ m+\varepsilon\le a-4,\ a\ge 5$,

$$h^2(\mathscr{E}_i(-a))=0; \tag{5}$$

for $j\ne 1$

$$h^j(\mathscr{E}_1\otimes\mathscr{E}_2)=0. \tag{6}$$

Proof. It is clearly enough to settle the case $i=1$, as the case $i=2$ is settled completely similarly. Consider two instanton vector bundles such that (2) can be evidently achieved. Show that (3) can also be satisfied for the general bundles $[\mathscr{E}_1]\in I_m$ and $[\mathscr{E}_2]\in I_{m+\varepsilon}$. To this end, consider a smooth quadric surface $S\in\mathbb{P}^3$, together with an isomorphism $S\simeq\mathbb{P}^1\times\mathbb{P}^1$, and let $Y=\bigsqcup_{i=1}^{m+1}l_i$ be a union of $m+1$ distinct projective lines l_i in $\mathbb{P}^3$ belonging to one of the two rulings on S. Considering Y as a reduced scheme, we have $\mathscr{I}_{Y,S}\simeq\mathscr{O}_{\mathbb{P}^1}(-m-1)\boxtimes\mathscr{O}_{\mathbb{P}^1}$. Thus the exact triple

$$0\to\mathscr{I}_{S,\mathbb{P}^3}\to\mathscr{I}_{Y,\mathbb{P}^3}\to\mathscr{I}_{Y,S}\to 0$$

can be rewritten as

$$0\to\mathscr{O}_{\mathbb{P}^3}(-2)\to\mathscr{I}_{Y,\mathbb{P}^3}\to\mathscr{O}_{\mathbb{P}^1}(-m-1)\boxtimes\mathscr{O}_{\mathbb{P}^1}\to 0. \tag{7}$$

Tensor multiplication of (7) by $\mathscr{O}_{\mathbb{P}^3}(a+1)$, respectively by $\mathscr{O}_{\mathbb{P}^3}(a-3)$, yields the exact triple

$$0\to\mathscr{O}_{\mathbb{P}^3}(a-1)\to\mathscr{I}_{Y,\mathbb{P}^3}(a+1)\to\mathscr{O}_{\mathbb{P}^1}(a-m)\boxtimes\mathscr{O}_{\mathbb{P}^1}(a+1)\to 0, \tag{8}$$

$$0\to\mathscr{O}_{\mathbb{P}^3}(a-5)\to\mathscr{I}_{Y,\mathbb{P}^3}(a-3)\to\mathscr{O}_{\mathbb{P}^1}(a-4-m)\boxtimes\mathscr{O}_{\mathbb{P}^1}(a-3)\to 0. \tag{9}$$

By the Künneth formula $h^1(\mathscr{O}_{\mathbb{P}^1}(a-m)\boxtimes\mathscr{O}_{\mathbb{P}^1}(a+1))=0$ for $a\ge 2$ and $m\le a+1$, and (8) implies that

$$h^1(\mathscr{I}_{Y,\mathbb{P}^3}(a+1))=0. \tag{10}$$

Now consider an extension of $\mathscr{O}_{\mathbb{P}^3}$-sheaves of the form

$$0\to\mathscr{O}_{\mathbb{P}^3}(-1)\to\mathscr{E}_1\to\mathscr{I}_{Y,\mathbb{P}^3}(1)\to 0. \tag{11}$$

Such extensions are classified by the vector space $V=\operatorname{Ext}^1(\mathscr{I}_{Y,\mathbb{P}^3}(1),\mathscr{O}_{\mathbb{P}^3}(-1))$, and it is known that for a general point $\xi\in V$ the extension sheaf $\mathscr{E}_1$ in (11) is a locally free instanton sheaf from I_m (see, e.g., [14]) called a *special Hooft instanton.* Now tensoring (11) with $\mathscr{O}_{\mathbb{P}^3}(a)$ and passing to cohomology, in view of (10) we obtain (3) for $i=1$.

To prove (5), assume that $m\le a-4$; then by analogy to (10) we obtain on using (9)

$$h^1(\mathscr{I}_{Y,\mathbb{P}^3}(a-3))=0,\quad m\le a-4. \tag{12}$$

Tensoring (11) with $\mathscr{O}_{\mathbb{P}^3}(a-4)$ we have the triple

$$0\to\mathscr{O}_{\mathbb{P}^3}(a-5)\to\mathscr{E}_1(a-4)\xrightarrow{e}\mathscr{I}_{Y,\mathbb{P}^3}(a-3)\to 0. \tag{13}$$

From (12) and (13) it follows that

$$h^1(\mathscr{E}_1(a-4))=0,\quad m\le a-4. \tag{14}$$

This together with Serre duality for $\mathscr{E}_1$ yields (5) for $i=1$.

To prove (4), consider a smooth quadric surface $S' \subset \mathbb{P}^3$, together with an isomorphism $S' \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and let $Z = \bigsqcup_{i=1}^{d} \tilde{l}_i$ be a union of d distinct projective lines $\tilde{l}_i$ in $\mathbb{P}^3$, belonging to one of the two rulings on S', where $1 \le d \le 5$. Considering Z as a reduced scheme, we have $\mathscr{I}_{Z,S'} \simeq \mathscr{O}_{\mathbb{P}^1}(-d) \boxtimes \mathscr{O}_{\mathbb{P}^1}$. Without loss of generality we may assume that $Z \cap Y = \varnothing$ and that Z intersects the quadric surface S treated above in $2d$ distinct points $x_1, \dots, x_{2d}$ such that the points $\mathrm{pr}_2(x_i),\ i = 1, \dots, 2d$, are also distinct, where $\mathrm{pr}_2 : S \simeq \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection onto the second factor. Tensoring the exact triple (7) with $\mathscr{O}_{\mathbb{P}^3}(a-3)$ and restricting it onto Z we obtain a commutative diagram of exact triples

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathscr{O}_Z(a-5) \arrow[r] \arrow[u] & \mathscr{O}_Z(a-3) \arrow[r] \arrow[u] & \bigoplus_1^{2d} \mathbf{k}_{x_j} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathscr{O}_{\mathbb{P}^3}(a-5) \arrow[r] \arrow[u, "f"] & \mathscr{I}_{Y,\mathbb{P}^3}(a-3) \arrow[r] \arrow[u, "g"'] & \mathscr{O}_{\mathbb{P}^1}(a-m-4) \boxtimes \mathscr{O}_{\mathbb{P}^1}(a-3) \arrow[r] \arrow[u, "h"] & 0,
\end{tikzcd} \tag{15}$$

where f, g, and h are the restriction mappings. The sheaf $\ker f = \mathscr{I}_{Z,\mathbb{P}^3}(a-5)$ by analogy to (8) satisfies the exact triple

$$0 \to \mathscr{O}_{\mathbb{P}^3}(a-5) \to \ker f \to \mathscr{O}_{\mathbb{P}^1}(a-d-5) \boxtimes \mathscr{O}_{\mathbb{P}^1}(a-5) \to 0.$$

Passing to the cohomology of this triple, we obtain in view of the conditions $1 \le d \le 5$ and $a \ge 12$ that $h^1(\ker f) = 0$, i.e.

$$h^0(f) : H^0(\mathscr{O}_{\mathbb{P}^3}(a-5)) \to H^0(\mathscr{O}_Z(a-5))$$

is an epimorphism. On the other hand, we have (i) $a - m - 4 \ge 0$, (ii) $a - 3 \ge 2d - 1$, since $a \ge 12$ and $d \le 5$, and (iii) the points $\mathrm{pr}_2(x_i),\ i = 1, \dots, 2d$, are distinct. Therefore,

$$h^0(h) : H^0(\mathscr{O}_{\mathbb{P}^1}(a-m-4) \boxtimes \mathscr{O}_{\mathbb{P}^1}(a-3)) \to H^0\Big(\bigoplus_1^{2d} \mathbf{k}_{x_j}\Big)$$

is also an epimorphism. Whence by (15) we obtain an epimorphism

$$h^0(g) : H^0(\mathscr{I}_{Y,\mathbb{P}^3}(a-3)) \twoheadrightarrow H^0(\mathscr{O}_Z(a-3)). \tag{16}$$

Consider the $g \circ e : \mathscr{E}(a-4) \twoheadrightarrow \mathscr{O}_Z(a-3)$, where e is the epimorphism in the triple (13) and put $E := \ker(g \circ e) \otimes \mathscr{O}_{\mathbb{P}^3}(4-a)$. Thus, since $\mathscr{O}_Z = \bigoplus^{d} \mathscr{O}_{\tilde{l}_i}$, we have the exact triple

$$0 \to E(a-4) \to \mathscr{E}(a-4) \xrightarrow{g \circ e} \bigoplus_1^{d} \mathscr{O}_{\tilde{l}_i}(a-3) \to 0. \tag{17}$$

From the triple (13) it follows that

$$h^0(e) : H^0(\mathscr{E}(a-4)) \to H^0(\mathscr{I}_{Y,\mathbb{P}^3}(a-3))$$

is an epimorphism; hence by (16)

$$h^0(g \circ e) : H^0(\mathscr{E}(a-4)) \to H^0\Big(\bigoplus_1^{d} \mathscr{O}_{\tilde{l}_i}(a-3)\Big)$$

is also an epimorphism. This together with (17) and (14) yields

$$h^1(E(a-4)) = 0. \tag{18}$$

Note that from (17) it follows also that

$$c_2(E) = c_2(\mathscr{E}) + d = m + d \le a + 1, \tag{19}$$

since $d \le 5$ and $m \le a - 4$.

Show that

$$[E] \in \bar{I}_{m+d}, \tag{20}$$

where $\bar{I}_{m+d}$ is the closure of I_{m+d} in the Gieseker–Maruyama moduli scheme $M(0, m+d, 0)$ of semistable rank 2 coherent sheaves with Chern classes $c_1 = c_3 = 0$ and $c_2 = m + d$. (Recall that $M(0, m+d, 0)$ is a projective scheme containing $\mathscr{B}(0, m+1)$ as an open subscheme; see, e.g., [1, 15].) It is enough to treat the case $d = 2$, since the argument for any $d \leq 5$ is completely similar. Consider (17) and denote by E_0' the kernel of the composition

$$\mathscr{E} \xrightarrow{g \circ e} \mathscr{O}_{\tilde{l}_1}(1) \oplus \mathscr{O}_{\tilde{l}_2}(1) \xrightarrow{\mathrm{pr}_1} \mathscr{O}_{\tilde{l}_1}(1).$$

We then obtain the exact triple

$$0 \to E \to E_0' \xrightarrow{e'} \mathscr{O}_{\tilde{l}_2}(1) \to 0. \tag{21}$$

Now we invoke one of the main results of the paper [16] according to which the sheaf E_0' lies in the closure $\bar{I}_{m+1}$ of I_{m+1} in the Gieseker–Maruyama moduli scheme $M(0, m+1, 0)$. This implies that there exists a punctured curve $(C, 0) \in \bar{I}_{m+1}$ and a flat over C coherent $\mathscr{O}_{\mathbb{P}^3 \times C}$-sheaf $\mathbb{E}'$ such that the sheaf $E_t' := \mathbb{E}'|_{\mathbb{P}^3 \times \{t\}}$ is an instanton bundle from I_{m+1} for $t \neq 0$ and coincides with E_0' for $t = 0$. Now, without loss of generality, after possible shrinking the curve C, we can extend the epimorphism ε' in (21) to an epimorphism $\mathbf{e} : \mathbb{E}' \twoheadrightarrow \mathscr{O}_{\tilde{l}_2}(1) \boxtimes \mathscr{O}_C$ such that $\mathbf{e} \otimes \mathbf{k}(0) = e'$. Put $\mathbb{E} = \ker \mathbf{e}$ and denote $E_t = \mathbb{E}|_{\mathbb{P}^3 \times \{t\}}$, $t \in C$. As for $t \neq 0$ the sheaf E_t' is an instanton bundle from I_{m+1}, and it fits the exact triple

$$0 \to E_t \to E_t' \to \mathscr{O}_{\tilde{l}_2}(1) \to 0,$$

the above-mentioned result from [16] yields $[E_t] \in \bar{I}_{m+2}$ for $t \neq 0$. Hence, since $[E_t] \in \bar{I}_{m+2}$ is projective, it follows that $E_0 \in \bar{I}_{m+2}$. Now by construction $E_0 \simeq E$. Thus, $[E] \in \bar{I}_{m+2}$, i.e. we obtain the desired result (20) for $d = 2$. Formula (4) now follows from (18) for a general $\mathscr{E}$ by semicontinuity and Serre duality.

To prove the vanishing of (6), consider the triple (11) twisted by $\mathscr{E}_2$:

$$0 \to \mathscr{E}_2(-1) \to \mathscr{E}_1 \otimes \mathscr{E}_2 \to \mathscr{E}_2 \otimes \mathscr{I}_{Y,\mathbb{P}^3}(1) \to 0, \tag{22}$$

and the exact triple

$$0 \to \mathscr{E}_2 \otimes \mathscr{I}_{Y,\mathbb{P}^3}(1) \to \mathscr{E}_2(1) \to \bigoplus_{i=1}^{m+1} (\mathscr{E}_2|_{l_i}) \to 0.$$

Since $\mathscr{E}_2$ is an instanton bundle, it follows that

$$h^2(\mathscr{E}_2(1)) = 0, \quad h^2(\mathscr{E}_2(-1)) = 0. \tag{23}$$

On the other hand, without loss of generality, by the Grauert–Mülich Theorem [17, Chapter 2] we may assume that $\mathscr{E}_2|_{l_i} \simeq 2\mathscr{O}_{\mathbb{P}^1}$. This together with the last exact triple and the first equality of (23) yields $h^2(\mathscr{E}_2 \otimes \mathscr{I}_{Y,\mathbb{P}^3}(1)) = 0$. Therefore, in view of (22) and the second equality of (23) we obtain (6) for $j = 1$. Finally, this equality for $j = 0, 3$ follows from (2) and the stability of $\mathscr{E}_1$ and $\mathscr{E}_2$. $\square$

Remark 1. Note that, under the conditions of Proposition 1, (6) together with the Riemann–Roch Theorem yield

$$h^1(\mathscr{E}_1 \otimes \mathscr{E}_2) = 8m + 4\varepsilon - 4. \tag{24}$$

§ 2. Construction of Stable Rank 2 Bundles with Even Determinant

We first recall the notion of symplectic instanton. By a *symplectic structure* on a vector bundle E on a scheme X we mean an anti-self-dual isomorphism $\theta : E \xrightarrow{\sim} E^\vee$, $\theta^\vee = -\theta$, considered modulo proportionality. Clearly, a symplectic vector bundle E has even rank:

$$\operatorname{rk} E = 2r, \quad r \geq 1,$$

and, if $X = \mathbb{P}^3$, vanishing odd Chern classes:

$$c_1(E) = c_3(E) = 0.$$

Following [13], we call a symplectic vector bundle E on $\mathbb{P}^3$ a *symplectic instanton* if

$$h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0, \tag{25}$$

$$c_2(E) = n, \quad n \geq 1.$$

Consider the instanton bundles $\mathscr{E}_1$ and $\mathscr{E}_2$ introduced in Section 1 (see Proposition 1). Since $\det \mathscr{E}_1 \simeq \det \mathscr{E}_2 \simeq \mathscr{O}_{\mathbb{P}^3}$, there are symplectic structures $\theta_i : \mathscr{E}_i \xrightarrow{\simeq} \mathscr{E}_i^\vee$, $i = 1, 2$, which yield a symplectic structure on the direct sum $\mathbb{E} = \mathscr{E}_1 \oplus \mathscr{E}_2$:

$$\theta = \theta_1 \oplus \theta_2 : \mathbb{E} = \mathscr{E}_1 \oplus \mathscr{E}_2 \xrightarrow{\simeq} \mathscr{E}_1^\vee \oplus \mathscr{E}_2^\vee = \mathbb{E}^\vee. \tag{26}$$

Clearly, $\mathbb{E}$ is a symplectic instanton. Since $\mathscr{E}_1$ and $\mathscr{E}_2$ are instanton bundles, for $a \geq 2$ we have

$$h^1(\mathscr{E}_i(-a)) = h^j(\mathscr{E}_i(a)) = 0, \quad i = 1, 2, \ j = 2, 3;$$

hence by (26)

$$h^1(\mathbb{E}(-a)) = 0, \quad a \geq 2, \tag{27}$$

$$h^j(\mathbb{E}(a)) = 0, \quad j = 2, 3, \ a \geq 2. \tag{28}$$

Similarly, in view of (3),

$$h^1(\mathbb{E}(a)) = 0, \quad m + \varepsilon \leq a + 1, \ a \geq 2. \tag{29}$$

This together with (28) and the Riemann–Roch Theorem yields

$$h^0(\mathbb{E}(a)) = \chi(\mathbb{E}(a)) = 4\binom{a+3}{3} - (2m + \varepsilon)(a + 2), \quad m + \varepsilon \leq a + 1, \ a \geq 2. \tag{30}$$

Respectively,

$$h^0(\mathscr{E}_1(a)) = \chi(\mathscr{E}_1(a)) = 2\binom{a+3}{3} - m(a + 2), \quad m \leq a + 1, \ a \geq 2, \tag{31}$$

and a similar formula holds for $h^0(\mathscr{E}_2(a))$.

Show now that, for a general pair $([\mathscr{E}_1], [\mathscr{E}_2]) \in I_m \times I_{m+\varepsilon}$ and the general sections $s_i : \mathscr{O}_{\mathbb{P}^3} \to \mathscr{E}_i(a)$ the conditions are satisfied:

$$\dim(s_i)_0 = 1, \ i = 1, 2, \quad (s_1)_0 \cap (s_2)_0 = \varnothing. \tag{32}$$

In view of (31) and the irreducibility of I_m it is enough to pick $\mathscr{E}_1$ and a section $s : \mathscr{O}_{\mathbb{P}^3} \to \mathscr{E}_1(a)$ such that

$$\dim(s)_0 = 1; \tag{33}$$

the rest of (32) is clear.

To this end, take for $\mathscr{E}_1$ a special Hooft bundle defined as an extension (11), where Y is a disjoint union of $m + 1$ lines lying on a smooth quadric surface S. As $Y \subset S$, it follows from the triple (11) twisted by $\mathscr{O}_{\mathbb{P}^3}(1)$ that $h^0(\mathscr{E}_1(1)) = 2$. Choose a basis, say, t_1 and t_2 of $H^0(\mathscr{E}_1(1))$. Next, as $a \geq 2$, we can choose two surfaces D_1 and D_2 of degree $a - 1$, intersecting along a smooth curve C of degree $(a - 1)^2$ not lying on a quadric S. Let $D_i = \{f_i = 0\}$ be the equations of surfaces D_i, where $f_i \in H^0(\mathscr{O}_{\mathbb{P}^3}(a - 1))$, $i = 1, 2$. Consider the multiplication map

$$\mu : H^0(\mathscr{E}_1(1)) \otimes H^0(\mathscr{O}_{\mathbb{P}^3}(a - 1)) \to H^0(\mathscr{E}_1(a))$$

and consider a 2-dimensional subspace V of $H^0(\mathscr{E}_1(a))$ spanned by the sections $s_i = \mu(t_i \otimes f_i)$, $i = 1, 2$. To prove (33), it is enough to show that $\dim(s)_0 = 1$ for a general section $s \in V$.

Suppose the contrary, i.e. $\dim(s)_0 = 2$ for any $s \in V$. Given $\lambda = \mathbf{k}s \in P(V)$, denote by D_λ the divisorial part of the scheme $(s)_0$. Since all sections from V vanish along the smooth irreducible curve $C = D_1 \cap D_2$, it follows that D_λ with equation, say, $\{f_\lambda = 0\}$ passes through C and so belongs to a pencil of surfaces of degree $a-1$ spanned by D_1 and D_2. Let $\mathbb{P}^1 = P(W)$ be the base of this pencil. We thus have a well-defined morphism $\varphi:\ P(V) \to P(W),\ \lambda \mapsto D_\lambda$ which is nonconstant since $D_1 \neq D_2$. Then by our assumption, for all $\lambda = (\lambda_1 : \lambda_2) \in P(V)$ we have, since μ is a $\mathbf{k}$-linear,

$$\lambda_1\mu(t_1 \otimes f_1) + \lambda_2\mu(t_2 \otimes f_2) = \mu(\lambda_1 t_1 \otimes f_1 + \lambda_2 t_2 \otimes f_2) = \mu(t_\lambda \otimes f_\lambda) \tag{34}$$

for some $t_\lambda \in H^0(\mathscr{E}_1(1))$. We therefore obtain a well-defined morphism

$$\psi : P(V) \to P(H^0(\mathscr{E}_1(1))), \quad \lambda \mapsto \mathbf{k}t_\lambda.$$

This morphism is also nonconstant, since t_1 and t_2 are linearly independent. The right-hand side of (34) is a polynomial of degree $\deg\psi + \deg\phi \geq 2$ on variables λ_1 and λ_2. On the other hand, the left-hand side of (34) is linear in λ_1 and λ_2; a contradiction.

Condition (32) implies that the section

$$s = (s_1, s_2) : \mathscr{O}_{\mathbb{P}^3}(-a) \to \mathbb{E} \tag{35}$$

is a subbundle morphism, hence its transpose

$${}^t s := s^\vee \circ \theta : \mathbb{E} \to \mathscr{O}_{\mathbb{P}^3}(a)$$

is a surjection. As θ in (26) is symplectic, the composition ${}^t s \circ s : \mathscr{O}_{\mathbb{P}^3}(-a) \to \mathscr{O}_{\mathbb{P}^3}(a)$ is also symplectic. Since $\mathscr{O}_{\mathbb{P}^3}(\pm a)$ are line bundles, it follows that ${}^t s \circ s = 0$. Therefore, the complex

$$K^\cdot : 0 \to \mathscr{O}_{\mathbb{P}^3}(-a) \xrightarrow{s} \mathbb{E} \xrightarrow{{}^t s} \mathscr{O}_{\mathbb{P}^3}(a) \to 0 \tag{36}$$

is a monad and its cohomology sheaf

$$E = \frac{\ker({}^t s)}{\mathrm{im}(s)} \tag{37}$$

is locally free. Note that, since the instanton bundles $\mathscr{E}_1$ and $\mathscr{E}_2$ are stable, they have zero spaces of global sections, hence also $h^0(\mathbb{E}) = 0$, and (36) and (37) yield $h^0(E) = 0$, i.e. E as a rank 2 vector bundle with $c_1 = 0$ is stable. Moreover, since $c_2(\mathbb{E}) = c_2(\mathscr{E}_1) + c_2(\mathscr{E}_2) = 2m + \varepsilon$, it follows from (36) that $c_2(E) = 2m + \varepsilon + a^2$. Thus, $[E] \in \mathscr{B}(0, 2m + \varepsilon + a^2)$ and the deformation theory yields

$$\dim \mathscr{M} \geq 1 - \chi(\mathscr{E}nd\ E) = 8(2m + \varepsilon + a^2) - 3$$

for any irreducible component $\mathscr{M}$ of $\mathscr{B}(0, 2m + \varepsilon + a^2)$.

Next,

$$\mathscr{E}nd\ \mathbb{E} \simeq \mathbb{E} \otimes \mathbb{E} \simeq S^2\mathbb{E} \oplus \wedge^2\mathbb{E}, \tag{38}$$

and it follows from (26) that

$$S^2\mathbb{E} \simeq S^2\mathscr{E}_1 \oplus (\mathscr{E}_1 \otimes \mathscr{E}_2) \oplus S^2\mathscr{E}_2, \quad \wedge^2\mathbb{E} \simeq \wedge^2\mathscr{E}_1 \oplus (\mathscr{E}_1 \otimes \mathscr{E}_2) \oplus \wedge^2\mathscr{E}_2. \tag{39}$$

Now, since

$$\mathscr{E}nd\ \mathscr{E}_i \simeq \mathscr{E}_i \otimes \mathscr{E}_i \simeq S^2\mathscr{E}_i \oplus \wedge^2\mathscr{E}_i,\ \wedge^2\mathscr{E}_i \simeq \mathscr{O}_{\mathbb{P}^3}, \quad i = 1, 2,$$

it follows from [5] that

$$h^1(\mathscr{E}nd\ \mathscr{E}_1) \simeq h^1(S^2\mathscr{E}_1) = 8m - 3, \quad h^1(\mathscr{E}nd\ \mathscr{E}_2) \simeq h^1(S^2\mathscr{E}_2) = 8m + 8\varepsilon - 3,$$

and

$$h^j(\mathscr{E}nd\ \mathscr{E}_i) = h^j(S^2\mathscr{E}_i) = 0, \quad i = 1, 2,\ j \geq 2.$$

This together with (38), (39), (3) and (24) implies that

$$h^1(\mathscr{E}nd\ \mathbb{E}) = 32m + 16\varepsilon - 14, \quad h^1(S^2\mathbb{E}) = 24m + 12\varepsilon - 10, \tag{40}$$

$$h^i(\mathscr{E}nd\ \mathbb{E}) = h^i(S^2\mathbb{E}) = 0, \quad i \geq 2. \tag{41}$$

Assume now that

$$\text{either } 5 \leq a \leq 11,\ 1+\varepsilon \leq m+\varepsilon \leq a-4, \text{or } a \geq 12,\ 1+\varepsilon \leq m+\varepsilon \leq a+1. \tag{42}$$

It follows from (4), (5), and (26) that

$$h^2(\mathbb{E}(-a)) = 0. \tag{43}$$

Consider the total complex $T^{\cdot}$ of the double complex $K^{\cdot} \otimes K^{\cdot}$, where $K^{\cdot}$ is the monad (36):

$$T^{\cdot}: 0 \to \mathscr{O}_{\mathbb{P}^3}(-2a) \overset{d_{-2}}{\to} 2\mathbb{E}(-a) \overset{d_{-1}}{\to} \mathbb{E}\otimes\mathbb{E} \oplus 2\mathscr{O}_{\mathbb{P}^3} \overset{d_0}{\to} 2\mathbb{E}(a) \overset{d_1}{\to} \mathscr{O}_{\mathbb{P}^3}(2a) \to 0,$$

$$E \otimes E = \frac{\ker(d_0)}{\operatorname{im}(d_{-1})}.$$

Following Le Potier [18], consider the symmetric part $ST^{\cdot}$ of $T^{\cdot}$:

$$ST^{\cdot}: 0 \to \mathbb{E}(-a) \overset{\alpha}{\to} S^2\mathbb{E} \oplus \mathscr{O}_{\mathbb{P}^3} \overset{{}^t\alpha}{\to} \mathbb{E}(a) \to 0, \quad S^2E = \frac{\ker({}^t\alpha)}{\operatorname{im}(\alpha)}, \tag{44}$$

where α is the induced subbundle map. The inclusion of complexes $ST^{\cdot} \hookrightarrow T^{\cdot}$ induces the commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{E}(-a) \arrow[r] \arrow[d, hook] & \ker({}^t\alpha) \arrow[r] \arrow[d, hook] & S^2E \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \operatorname{im}(d_{-1}) \arrow[r] & \ker(d_0) \arrow[r] & E\otimes E \arrow[r] & 0,
\end{tikzcd} \tag{45}$$

$$\begin{tikzcd}
0 \arrow[r] & \ker({}^t\alpha) \arrow[r] \arrow[d, hook] & S^2\mathbb{E}\oplus\mathscr{O}_{\mathbb{P}^3} \arrow[r] \arrow[d, hook] & \mathbb{E}(a) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \ker(d_0) \arrow[r] & \mathbb{E}\otimes\mathbb{E}\oplus 2\mathscr{O}_{\mathbb{P}^3} \arrow[r] & \operatorname{im}(d_0) \arrow[r] & 0
\end{tikzcd} \tag{46}$$

and the exact triple

$$0 \to \mathscr{O}_{\mathbb{P}^3}(-2a) \overset{d_{-2}}{\to} 2\mathbb{E}(-a) \to \operatorname{im}(d_{-1}) \to 0. \tag{47}$$

Passing to cohomology in (45)–(47) and using (27), (43), (29) and the equality

$$h^0(S^2\mathbb{E}) = 0, \tag{48}$$

we obtain the equality $h^0(\operatorname{coker}\alpha) = 1$ and the exact sequence

$$0 \to H^0(\mathbb{E}(a))/\mathbb{C} \to H^1(S^2E) \overset{\mu}{\to} H^1(S^2\mathbb{E}) \to 0, \tag{49}$$

which fits the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbb{E}(a))/\mathbb{C} \arrow[r] & H^1(S^2E) \arrow[r, "\mu"] \arrow[d, equal] & H^1(S^2\mathbb{E}) \arrow[r] \arrow[d, hook] & 0 \\
 & & H^1(E\otimes E) \arrow[r] & H^1(\mathbb{E}\otimes\mathbb{E}). &
\end{tikzcd} \tag{50}$$

From (30), (40), and (50) it follows that

$$h^1(S^2E) = h^0(\mathbb{E}(a)) + 24m + 12\varepsilon - 11 = 4\binom{a+3}{3} + (2m+\varepsilon)(10-a) - 11. \tag{51}$$

Note that, since E is a stable rank 2 bundle, $H^1(\mathscr{E}nd\ E) = H^1(S^2E)$ is isomorphic to the Zariski tangent space $T_{[E]}\mathscr{B}(0, 2m+\varepsilon+a^2)$:

$$\theta_E : T_{[E]}\mathscr{B}(0, 2m+\varepsilon+a^2) \xrightarrow{\sim} H^1(\mathscr{E}nd\ E) = H^1(S^2E). \tag{52}$$

(Here θ_E is the Kodaira–Spencer isomorphism.) Thus, we can rewrite (51) as

$$\dim T_{[E]}\mathscr{B}(0, 2m+\varepsilon+a^2) = 4\binom{a+3}{3} + (2m+\varepsilon)(10-a) - 11. \tag{53}$$

We will now prove the main result of this section.

Theorem 1. *Under condition* (42)*, there exists an irreducible family $\mathscr{M}_n(E) \subset \mathscr{B}(0,n)$, where $n = 2m+\varepsilon+a^2$, of dimension given by the right-hand side of* (53) *and containing the above constructed point $[E]$. Hence the closure $\mathscr{M}_n$ of $\mathscr{M}_n(E)$ in $\mathscr{B}(0,n)$ is an irreducible component of $\mathscr{B}(0,n)$. The set Σ_0 of these components $\mathscr{M}_n$ is an infinite series distinct from the series of instanton components $\{I_n\}_{n\geq 1}$ and from the series of components described in* [12] *and* [13]*. Furthermore, at least for each $n \geq 146$ there exists an irreducible component $\mathscr{M}_n$ of $\mathscr{B}(0,n)$ belonging to the series Σ_0.*

Proof. According to Bingener [8, Appendix], the equality $h^2(\mathscr{E}nd\,\mathbb{E}) = 0$ (see (41)) implies that there exists (over $\mathbf{k} = \mathbb{C}$) a versal deformation of the bundle $\mathbb{E}$, i.e. a smooth variety B of dimension $\dim B = h^1(\mathscr{E}nd\,\mathbb{E})$, with a marked point $0 \in B$, and a locally free sheaf $\boldsymbol{\varepsilon}$ on $\mathbb{P}^3 \times B$ such that $\boldsymbol{\varepsilon}|_{\mathbb{P}^3\times\{0\}} \simeq \mathbb{E}$ and the Koaira–Spencer map $\theta : T_{[\mathbb{E}]}B \to H^1(\mathscr{E}nd\,\mathbb{E})$ is an isomorphism. For $b \in B$ denote $E_b := \boldsymbol{\varepsilon}|_{\mathbb{P}^3\times\{b\}}$ and consider in B the closed subset

$$U = \{b \in B \mid E_b \text{ is a symplectic instanton}\}.$$

By definition, $U = \widetilde{U} \cap B^*$, where

$$\widetilde{U} = \{b \in B \mid E_b \text{ is a symplectic bundle}\}$$

is a closed subset of B and

$$B^* = \{b \in B \mid E_b \text{ satisfies (25) and the condition}$$
$$h^0(E_b) = h^i(E_b(-a)) = h^j(E_b(a)) = h^k(S^2E_b) = 0,\ i = 1,2,\ j \geq 1,\ k = 0,2,3\}$$

is an open subset of B by semicontinuity. (Here a is taken from (42)). Since $\mathbb{E}$ is symplectic, so that $\mathscr{E}nd\mathbb{E} \simeq S^2\mathbb{E} \oplus \wedge^2\mathbb{E}$, it follows from [19] that the Kodaira–Spencer map θ yields an isomorphism $\theta : T_{[\mathbb{E}]}U = T_{[\mathbb{E}]}\widetilde{U} \xrightarrow{\sim} H^1(S^2\mathbb{E})$. Thus, U is a smooth variety of dimension

$$\dim U = h^1(S^2\mathbb{E}) = 24m + 12\varepsilon - 10.$$

(We use the Riemann–Roch Theorem and the vanishing of $h^i(S^2\mathbb{E})$, $i \neq 1$, by (48) and (41).)

Let $p : \mathbb{P}^3 \times B \to B$ be the projection. By the change of the base and the vanishing conditions that define B^* and U, the sheaf $\mathscr{A} := p_*(\boldsymbol{\varepsilon} \otimes \mathscr{O}_{\mathbb{P}^3}(a) \boxtimes \mathscr{O}_U)$ is a locally free sheaf of rank $\chi(\mathbb{E}(a)) = h^0(\mathbb{E}(a))$ given by (30). Hence, $\pi : \widetilde{X} = \mathbf{Proj}(S^{\cdot}_{\mathscr{O}_{\mathbb{P}^3}}\mathscr{A}^\vee) \to U$ is the projective bundle with the Grothendieck sheaf $\mathscr{O}_{\widetilde{X}/U}(1)$ and the morphism

$$\mathbf{s} : \mathscr{O}_{\mathbb{P}^3} \boxtimes \mathscr{O}_{\widetilde{X}/U}(-1) \to \tilde{\pi}^*(\boldsymbol{\varepsilon} \otimes \mathscr{O}_{\mathbb{P}^3}(a) \boxtimes \mathscr{O}_U),$$

defined as the composition of the canonical evaluation morphisms

$$\mathscr{O}_{\mathbb{P}^3} \boxtimes \mathscr{O}_{\widetilde{X}/U}(-1) \to \tilde{p}^*\pi^*\mathscr{A} \to \tilde{\pi}^*(\boldsymbol{\varepsilon} \otimes \mathscr{O}_{\mathbb{P}^3}(a) \boxtimes \mathscr{O}_U),$$

where $\widetilde{X} \xleftarrow{\tilde{p}} \mathbb{P}^3 \times \widetilde{X} \xrightarrow{\tilde{\pi}} \mathbb{P}^3 \times U$ are the induced projections.

Put $X = \{x \in \widetilde{X} \mid \mathbf{s}^\vee|_{\mathbb{P}^3\times\{x\}}$ is surjective$\}$. This is an open dense subset of the smooth irreducible variety $\widetilde{X}$ since it contains the point $x_0 = (s : \mathscr{O}_{\mathbb{P}^3} \to \mathbb{E}(a))$ given in (35). Hence X is smooth and irreducible. Moreover, since $\boldsymbol{\varepsilon}$ is a versal family of bundles, X is an open subset of the Quot-scheme $\mathrm{Quot}_{\mathbb{P}^3\times B/B}(\boldsymbol{\varepsilon}, P(n))$, where $P(n) := \chi(\mathscr{O}_{\mathbb{P}^3}(a+n))$. Therefore, by [15, Proposition 2.2.7] in view of (29) there is an exact triple

$$0 \to H^0(\mathbb{E}(a))/\mathbb{C} \to T_{x_0}X \overset{d\pi}{\to} T_{[\mathbb{E}]}B \to 0 \tag{54}$$

obtained as the cohomology sequence

$$0 \to H^0(\mathbb{E}(a))/\mathbb{C} \to H^1(\mathscr{H}om(F,\mathbb{E})) \to H^1(\mathscr{E}nd\,\mathbb{E}) \to 0 \tag{55}$$

of the exact triple $0 \to \mathscr{H}om(F,\mathbb{E}) \to \mathscr{H}om(\mathbb{E},\mathbb{E}) \to \mathbb{E}(a) \to 0$ obtained by applying the functor $\mathscr{H}om(-,\mathbb{E})$ to the exact triple $0 \to \mathscr{O}_{\mathbb{P}^3}(-a) \overset{s}{\to} \mathbb{E} \to F \to 0$, where $F := \mathrm{coker}(s)$.

Next, since $\boldsymbol{\varepsilon}$ is a versal family of bundles, $\mathbf{E} = \boldsymbol{\varepsilon}|_{\mathbb{P}^3\times U}$ is a versal family of symplectic instantons. Hence, putting $Y = U \times_B X$, we extend the exact triple (54) to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbb{E}(a))/\mathbb{C} \arrow[r] & T_{x_0}X \arrow[r] & T_{[\mathbb{E}]}B \arrow[r] & 0 \\
0 \arrow[r] & H^0(\mathbb{E}(a))/\mathbb{C} \arrow[r] \arrow[u, equal] & T_{x_0}Y \arrow[r, "d\pi"] \arrow[u, hook, "i_Y"] & T_{[\mathbb{E}]}U \arrow[r] \arrow[u, hook, "i_U"] & 0,
\end{tikzcd} \tag{56}$$

where i_Y and i_U are natural inclusions. (Note that, under the Kodaira–Spencer isomorphisms θ : $T_{[\mathbb{E}]}U \overset{\simeq}{\to} H^1(S^2\mathbb{E})$ and $T_{[\mathbb{E}]}B \overset{\simeq}{\to} H^1(\mathbb{E}\otimes\mathbb{E}) \simeq H^1(\mathscr{E}nd\,\mathbb{E})$ the rightmost inclusions in diagrams (50) and (56) coincide.) Consider the modular morphism

$$\Phi : Y \to \mathscr{B} := \mathscr{B}(0, 2m+\varepsilon+a^2), \quad (b,s) \mapsto \left[\frac{\mathrm{Ker}({}^t s)}{\mathrm{Im}(s)}\right],$$

where, as before, $s : \mathscr{O}_{\mathbb{P}^3}(-a) \to E_b$ is a subbundle morphism. Its differential $d\Phi$ composed with the Kodaira–Spencer map θ_E from (52) is a linear map

$$\phi = \theta_E \circ d\Phi : T_{x_0}Y \to H^1(S^2E) = H^1(E\otimes E).$$

Now from functorial properties of the Kodaira–Spencer maps ϕ and θ it follows that (49) and the lower triple in (56) fit the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathbb{E}(a))/\mathbb{C} \arrow[r] & H^1(S^2E) \arrow[r, "\mu"] & H^1(S^2\mathbb{E}) \arrow[r] & 0 \\
0 \arrow[r] & H^0(\mathbb{E}(a))/\mathbb{C} \arrow[r] \arrow[u, equal] & T_{x_0}Y \arrow[r, "d\pi"] \arrow[u, "\phi"] & T_{[\mathbb{E}]}U \arrow[r] \arrow[u, "\theta", "\simeq"'] & 0.
\end{tikzcd}$$

This diagram implies that ϕ is an isomorphism, so that, since Y is smooth at x_0 and irreducible, $\mathscr{M}_n(E) = \Phi(Y)$ is an open subset of an irreducible component $\mathscr{M}_n$ of $\mathscr{B}(0,n)$, of dimension given by (53).

It is easy to check that the dimension $\dim \mathscr{M}_n$ given by (53), with m, ε, and a subjected to (42), satisfies the strict inequality $\dim \mathscr{M}_n > 8n-3 = \dim I_n$. This shows that the series Σ_0 is distinct from $\{I_n\}_{n\geq 1}$. To distinguish Σ_0 from from the series of components described in [12], it is enough to see that the spectra of the general bundles of these two series are different. (We leave to the reader a direct verification of this fact.)

Note that, for each $a \geq 12$ we have $1 \leq m \leq a+1$ and $0 \leq \varepsilon \leq 1$, so that $n = 2m+\varepsilon+a^2$ ranges through the whole interval of positive integers $(a^2+2, (a+1)^2+1) \subset \mathbb{Z}_+$. Hence, n takes at least all positive values $\geq 12^2+2 = 146$. This shows that for each $n \geq 146$ there exists an irreducible component $\mathscr{M}_n \in \Sigma_0$.

Note finally that, for the series of components $\mathscr{M}_n$ described in [13], n takes values $n = 1+k^2$, $k \in \{2\} \cup (4,\infty)$. Hence this series is distinct from Σ_0. Theorem 1 is proved. $\square$

§ 3. Construction of Stable Rank 2 Bundles with Odd Determinant

In this section we will construct an infinite series of stable vector bundles from $\mathscr{B}(-1, 2m)$, $m \in \mathbb{Z}_+$. It is known from [1, Example 4.3.2] that, for each $m \geq 1$, there exists an irreducible component $\mathscr{B}_0(-1, 2m)$ of $\mathscr{B}(-1, 2m)$, of the expected dimension

$$\dim \mathscr{B}_0(-1, 2m) = 16m - 5, \tag{57}$$

which contains bundles $\mathscr{E}$ obtained via the Serre constructions as the extensions of the form

$$0 \to \mathscr{O}_{\mathbb{P}^3}(-2) \to \mathscr{E} \to \mathscr{I}_{Y,\mathbb{P}^3}(1) \to 0, \tag{58}$$

where Y is a union of $m + 1$ disjoint conics in $\mathbb{P}^3$.

Below we will need the following analog of Proposition 1.

Proposition 2. *Let $a, m \in \mathbb{Z}_+$, $a \geq 2$, and let $\varepsilon \in \{0, 1\}$. A general pair*

$$([\mathscr{E}_1], [\mathscr{E}_2]) \in \mathscr{B}_0(-1, 2m) \times \mathscr{B}_0(-1, 2(m + \varepsilon)) \tag{59}$$

of vector bundles satisfies the conditions:

$$[\mathscr{E}_1] \neq [\mathscr{E}_2];$$

for $i = 1$, $a \geq 2m + 4$, respectively, for $i = 2$, $a \geq 2(m + \varepsilon) + 4$,

$$h^1(\mathscr{E}_i(a)) = 0, \tag{60}$$

$$h^2(\mathscr{E}_i(-a)) = 0, \tag{61}$$

$$h^1(\mathscr{E}_i(-a)) = 0, \tag{62}$$

$$h^j(\mathscr{E}_1(1) \otimes \mathscr{E}_2) = 0, \quad j \neq 1. \tag{63}$$

Proof. Let $Y = \sqcup_1^{m+1} C_i$ be a disjoint union of conics $C_i = l_i \cup l'_i$ decomposable into pairs of distinct lines l_i and l'_i, such that

(i) there exist two smooth quadrics $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $S' \simeq \mathbb{P}^1 \times \mathbb{P}^1$ with the property that $l_1, \dots, l_{m+1}$, respectively, $l'_1, \dots, l'_{m+1}$ are the lines of one ruling on S, respectively, on S'; for instance, denoting $Y_0 = l_1 \sqcup \dots \sqcup l_{m+1}$ and $Y' = l'_1 \sqcup \dots \sqcup l'_{m+1}$, we may assume that

$$\mathscr{O}_S(Y_0) \simeq \mathscr{O}_{\mathbb{P}^1}(m + 1) \boxtimes \mathscr{O}_{\mathbb{P}^1}, \quad \mathscr{O}_{S'}(Y') \simeq \mathscr{O}_{\mathbb{P}^1}(m + 1) \boxtimes \mathscr{O}_{\mathbb{P}^1};$$

(ii) the set of $m + 1$ distinct points $Z = (Y' \cap S) \setminus (Y_0 \cap Y')$ satisfies the condition that $\mathrm{pr}_1(Z)$ is a union of $m + 1$ distinct points, where $\mathrm{pr}_1 : S' \simeq \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection. We then have the diagram similar to (15):

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathscr{O}_{Y'}(a-4) \arrow[r] \arrow[u] & \mathscr{O}_{Y'}(a-3) \arrow[r] \arrow[u] & \mathscr{O}_Z(a-3) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathscr{O}_{\mathbb{P}^3}(a-4) \arrow[r] \arrow[u, "f"] & \mathscr{I}_{Y_0,\mathbb{P}^3}(a-2) \arrow[r] \arrow[u, "g"] & \mathscr{O}_{\mathbb{P}^1}(a-m-3) \boxtimes \mathscr{O}_{\mathbb{P}^1}(a-2) \arrow[r] \arrow[u, "h"] & 0.
\end{tikzcd} \tag{64}$$

Under the assumptions $a \geq 2m + 4$ and $m \geq 2$, the cohomology of the lower triple of this diagram yields

$$h^1(\mathscr{I}_{Y_0,\mathbb{P}^3}(a - 2)) = 0. \tag{65}$$

Next, by analogy to (8) we have the exact triple

$$0 \to \mathscr{O}_{\mathbb{P}^3}(a-6) \to \mathscr{I}_{Y',\mathbb{P}^3}(a-4) \to \mathscr{O}_{\mathbb{P}^1}(a-5-m) \boxtimes \mathscr{O}_{\mathbb{P}^1}(a-4) \to 0,$$

which implies that $h^1(\mathscr{I}_{Y',\mathbb{P}^3}(a-4)) = 0$ since $a-5-m \geq 0$ for $a \geq 2m+4$ and $m \geq 1$. Since $\mathscr{I}_{Y',\mathbb{P}^3}(a-4) = \ker f$, the homomorphism

$$h^0(f) : H^0(\mathscr{O}_{\mathbb{P}^3}(a-4)) \to H^0(\mathscr{O}_{Y'}(a-4)) \tag{66}$$

is surjective. On the other hand, since $a-3-m \geq m+1 = h^0(Z)$, from the above condition (ii) on Z it follows that

$$h^0(h) : H^0(\mathscr{O}_{\mathbb{P}^1}(a-m-3) \boxtimes \mathscr{O}_{\mathbb{P}^1}(a-2)) \to H^0(\mathscr{O}_Z(a-3))$$

is surjective. This together with (66) and diagram (64) implies that $h^0(g) : H^0(\mathscr{I}_{Y_0,\mathbb{P}^3}(a-2)) \to H^0(\mathscr{O}_{Y'}(a-3))$ is surjective. Since $\ker g \simeq \mathscr{I}_{Y,\mathbb{P}^3}(a-2)$, it follows by (65) that

$$h^1(\mathscr{I}_{Y,\mathbb{P}^3}(a-2)) = 0. \tag{67}$$

Twisting the triple (58) by $\mathscr{O}_{\mathbb{P}^3}(a-3)$ and using (67) we obtain $h^1(\mathscr{E}_1(a-3)) = 0$; hence, by the Serre duality $h^2(\mathscr{E}_1(-a)) = 0$. Moreover, the equality $h^1(\mathscr{E}_1(a-3)) = 0$ and the above argument with a substituted by $a+3$ imply $h^1(\mathscr{E}_1(a)) = 0$, since $a \geq 2m+4$. Now, by semicontinuity, this yields (60) and (61) for the general $[\mathscr{E}_1] \in \mathscr{B}_0(-1, 2m)$. The same equalities are clearly true for $i = 2$.

Next, since $a \geq 2$, $h^0(\mathscr{O}_{C_i}(1-a)) = 0$ for any conic $C_i \subset Y$. Hence the cohomology of the triple

$$0 \to \mathscr{I}_{Y,\mathbb{P}^3}(1-a) \to \mathscr{O}_{\mathbb{P}^3}(1-a) \to \bigoplus_{i=1}^{m+1} \mathscr{O}_{C_i}(1-a) \to 0$$

yields $h^1(\mathscr{I}_{Y,\mathbb{P}^3}(1-a)) = 0$; this together with (58) and semicontinuity yields (62) for $i = 1$ and similarly for $i = 2$.

At last, (63) are proved like (6). $\square$

Remark 2. Note that, under the conditions of Proposition 2, (63) together with the Riemann–Roch Theorem yield

$$h^1(\mathscr{E}_1(1) \otimes \mathscr{E}_2) = 16m + 8\varepsilon - 6. \tag{68}$$

Now, to construct the new series of components of $\mathscr{B}(-1, 4m+2\varepsilon)$, we proceed along the same lines as in Section 2. We first introduce the notion of twisted symplectic structure on a vector bundle. By a *twisted symplectic structure* on a vector bundle E on $\mathbb{P}^3$ we mean an isomorphism $\theta : E \xrightarrow{\simeq} E^\vee(-1)$ such that $\theta^\vee(1) = -\theta$, considered modulo proportionality. (Here by definition $\theta^\vee(1) := \theta^\vee \otimes \mathrm{id}_{\mathscr{O}_{\mathbb{P}^3}(1)}$.) Clearly, a vector bundle E with twisted symplectic structure has even rank: $\mathrm{rk}\, E = 2r$, $r \geq 1$.

Consider the vector bundles $\mathscr{E}_1$ and $\mathscr{E}_2$ introduced in Proposition 2. Since $\det \mathscr{E}_1 \simeq \det \mathscr{E}_2 \simeq \mathscr{O}_{\mathbb{P}^3}(-1)$, there are twisted symplectic structures $\theta_i : \mathscr{E}_i \xrightarrow{\simeq} \mathscr{E}_i^\vee(-1)$, $i = 1, 2$, which yield a twisted symplectic structure on the direct sum $\mathbb{E} = \mathscr{E}_1 \oplus \mathscr{E}_2$:

$$\theta = \theta_1 \oplus \theta_2 : \mathbb{E} = \mathscr{E}_1 \oplus \mathscr{E}_2 \xrightarrow{\simeq} \mathscr{E}_1^\vee(-1) \oplus \mathscr{E}_2^\vee(-1) = \mathbb{E}^\vee(-1). \tag{69}$$

Choose the bundles $\mathscr{E}_1$ and $\mathscr{E}_2$ in such a way that there exist sections

$$s_i : \mathscr{O}_{\mathbb{P}^3} \to \mathscr{E}_i(a+1) \quad \text{such that } \dim(s_i)_0 = 1,\ i = 1, 2, \quad (s_1)_0 \cap (s_2)_0 = \varnothing. \tag{70}$$

(Such $[\mathscr{E}_1] \in \mathscr{B}_0(-1, 2m)$ and $[\mathscr{E}_2] \in \mathscr{B}_0(-1, 2(m+\varepsilon))$ always exist, since already for $a = 1$, hence also for $a \geq 2$ the two general bundles of the form (58) satisfy the property (70). The argument here repeats that of the proof of (33) in Section 2, therefore we omit the details.) From (70) it follows that the section $s = (s_1, s_2) : \mathscr{O}_{\mathbb{P}^3}(-a-1) \to \mathbb{E}$ is a subbundle morphism, hence its transpose ${}^t s := s^\vee(-1) \circ \theta : \mathbb{E} \to \mathscr{O}_{\mathbb{P}^3}(a)$ is an epimorphism. As θ in (69) is twisted symplectic, the composition ${}^t s \circ s : \mathscr{O}_{\mathbb{P}^3}(-a-1) \to \mathscr{O}_{\mathbb{P}^3}(a)$ is also twisted symplectic. Therefore, since $\mathscr{O}_{\mathbb{P}^3}(a)$ and $\mathscr{O}_{\mathbb{P}^3}(-a-1)$ are line bundles, ${}^t s \circ s = 0$, i.e. the complex

$$K^{\cdot} : 0 \to \mathscr{O}_{\mathbb{P}^3}(-a-1) \xrightarrow{s} \mathbb{E} \xrightarrow{{}^t s} \mathscr{O}_{\mathbb{P}^3}(a) \to 0, \quad E = \frac{\ker({}^t s)}{\mathrm{im}(s)}, \tag{71}$$

is a monad and its cohomology sheaf E is locally free. Note that, since the bundles $\mathscr{E}_1$ and $\mathscr{E}_2$ are stable, they have zero spaces of global sections. Hence also $h^0(\mathbb{E}) = 0$, and (71) yields $h^0(E) = 0$, i.e. E as a rank 2 vector bundle with $c_1 = -1$ is stable. Moreover, since $c_2(\mathbb{E}) = c_2(\mathscr{E}_1) + c_2(\mathscr{E}_2) = 4m + 2\varepsilon$, it follows from (71) that $c_2(E) = 4m + 2\varepsilon + a(a+1)$. Thus, $[E] \in \mathscr{B}(-1, 4m + 2\varepsilon + a(a+1))$, and the deformation theory yields

$$\dim \mathscr{M} \geq 1 - \chi(\mathscr{E}nd\ E) = 8(4m + 2\varepsilon + a(a+1)) - 5$$

for any irreducible component $\mathscr{M}$ of $\mathscr{B}(-1, 4m + 2\varepsilon + a(a+1))$.

Now, as in (44), consider the symmetric part of the total complex of the double complex $K^{\cdot} \otimes (K^{\cdot})^{\vee}$, where $K^{\cdot}$ is the monad (71):

$$0 \to \mathbb{E}(-a) \xrightarrow{\alpha} S^2\mathbb{E}(1) \oplus \mathscr{O}_{\mathbb{P}^3} \xrightarrow{{}^t\alpha} \mathbb{E}(a+1) \to 0, \quad S^2E(1) = \frac{\ker({}^t\alpha)}{\mathrm{im}(\alpha)}. \tag{72}$$

Here α is the induced subbundle map and $S^2E(1)$ is its cohomology sheaf. The monad (72) can be rewritten as a diagram of exact triples by analogy to (45) and (46):

$$\begin{tikzcd}
 & & & 0 & \\
 & & & \mathbb{E}(a+1) \arrow[u] & \\
0 \arrow[r] & \mathbb{E}(-a) \arrow[r, "\alpha"] & S^2\mathbb{E}(1) \oplus \mathscr{O}_{\mathbb{P}^3} \arrow[r] & \operatorname{coker} \alpha \arrow[r] \arrow[u] & 0. \\
 & & & S^2E(1) \arrow[u] & \\
 & & & 0 \arrow[u] &
\end{tikzcd} \tag{73}$$

Note that, by (62) and (69) we have

$$h^1(\mathbb{E}(-a)) = 0, \quad a \geq 2, \tag{74}$$

$$h^j(\mathbb{E}(a+1)) = 0, \quad j = 2, 3,\ a \geq 2(m+\varepsilon) + 3. \tag{75}$$

Similarly, in view of (60),

$$h^1(\mathbb{E}(a+1)) = 0, \quad a \geq 2(m+\varepsilon) + 3. \tag{76}$$

This, together with (75) and the Riemann–Roch Theorem, yields

$$h^0(\mathbb{E}(a+1)) = \chi(\mathbb{E}(a+1)) = 4\binom{a+3}{3} + 2\binom{a+3}{2} - (2m+\varepsilon)(2a+5). \tag{77}$$

Next,

$$\mathscr{E}nd\ \mathbb{E} \simeq \mathbb{E}(1) \otimes \mathbb{E} \simeq S^2\mathbb{E}(1) \oplus \wedge^2\mathbb{E}(1), \tag{78}$$

and it follows from (69) that

$$\begin{aligned} S^2\mathbb{E}(1) &\simeq S^2\mathscr{E}_1(1) \oplus (\mathscr{E}_1(1) \otimes \mathscr{E}_2) \oplus S^2\mathscr{E}_2(1), \\ \wedge^2\mathbb{E}(1) &\simeq \wedge^2\mathscr{E}_1(1) \oplus (\mathscr{E}_1(1) \otimes \mathscr{E}_2) \oplus \wedge^2\mathscr{E}_2(1). \end{aligned} \tag{79}$$

Now, since

$$\mathscr{E}nd\ \mathscr{E}_i \simeq \mathscr{E}_i(1) \otimes \mathscr{E}_i \simeq S^2\mathscr{E}_i(1) \oplus \wedge^2\mathscr{E}_i(1),\ \wedge^2\mathscr{E}_i \simeq \mathscr{O}_{\mathbb{P}^3}, \quad i = 1, 2,$$

it follows from [5] that

$$\begin{aligned} h^1(\mathscr{E}nd\ \mathscr{E}_1) &\simeq h^1(S^2\mathscr{E}_1(1)) = 16m - 5, \\ h^1(\mathscr{E}nd\ \mathscr{E}_2) &\simeq h^1(S^2\mathscr{E}_2(1)) = 16(m + \varepsilon) - 5, \end{aligned}$$

and

$$h^j(\mathscr{E}nd\ \mathscr{E}_i) = h^j(S^2\mathscr{E}_i(1)) = 0, \quad i = 1, 2,\ j \geq 2.$$

This, together with (78), (79), (60), and (68), implies that

$$\begin{aligned} h^1(\mathscr{E}nd\ \mathbb{E}) = 64m + 32\varepsilon - 22, \quad h^1(S^2\mathbb{E}(1)) = 48m + 24\varepsilon - 16, \\ h^i(\mathscr{E}nd\ \mathbb{E}) = h^i(S^2\mathbb{E}(1)) = 0, \quad i \geq 2. \end{aligned} \tag{80}$$

It follows from (61) and (69) that

$$h^2(\mathbb{E}(-a)) = 0. \tag{81}$$

Note that (74), (81), and (76), together with (73), yield the equality $h^0(\operatorname{coker}\alpha) = 1$ and the exact sequence

$$0 \to H^0(\mathbb{E}(a+1))/\mathbb{C} \to H^1(S^2E(1)) \xrightarrow{\mu} H^1(S^2\mathbb{E}(1)) \to 0.$$

Hence by (77) and (80) we have

$$\begin{aligned} h^1(S^2E(1)) &= h^0(\mathbb{E}(a+1)) + 48m + 24\varepsilon - 17 \\ &= 4\binom{a+3}{3} + 2\binom{a+3}{2} - (2m+\varepsilon)(2a-19) - 17. \end{aligned} \tag{82}$$

Note that, since E is a stable rank-2 bundle, $H^1(\mathscr{E}nd\ E) = H^1(S^2E(1))$ is isomorphic to the Zariski tangent space $T_{[E]}\mathscr{B}(-1, 4m + 2\varepsilon + a(a+1))$:

$$\theta_E : T_{[E]}\mathscr{B}(-1, 4m + 2\varepsilon + a(a+1)) \xrightarrow{\sim} H^1(\mathscr{E}nd\ E) = H^1(S^2E(1)) \tag{83}$$

(here θ_E is the Kodaira–Spencer isomorphism). Thus, we can rewrite (51) as

$$\dim T_{[E]}\mathscr{B}(-1, 4m + 2\varepsilon + a(a+1)) = 4\binom{a+3}{3} + 2\binom{a+3}{2} - (2m+\varepsilon)(2a-19) - 17. \tag{84}$$

Theorem 2. *For $n = 4m + 2\varepsilon + a(a+1)$, where $m \geq 1$, $\varepsilon \in \{0, 1\}$ and $a \geq 2(m+\varepsilon) + 3$, there exists an irreducible family of stable vector bundles $\mathscr{M}_n(E) \subset \mathscr{B}(-1, n)$ containing the vector bundle $[E]$ constructed in* (71). *The dimension of the family $\mathscr{M}_n(E)$ is given by the right-hand side of* (84). *The closure $\mathscr{M}_n$ of $\mathscr{M}_n(E)$ in the scheme $\mathscr{B}(-1, n)$ is an irreducible component of $\mathscr{B}(-1, n)$. The set Σ_1 of these components $\mathscr{M}_n$ is an infinite series distinct from the series $\{\mathscr{B}_0(-1, n)\}_{n\geq 1}$ and from the series of Ein components described in* [12].

The proof of Theorem 2 is completely analogous to that of Theorem 1 with obvious modifications due to the change from $c_1(E) = 0$ to $c_1(E) = -1$.

It is easy to check that the dimension $\dim \mathscr{M}_n$ given by (84), with m, ε and a as in Theorem 2, satisfies the strict inequality $\dim \mathscr{M}_n > 8n - 5 = \dim \mathscr{B}_0(-1, n)$ (cf. (57)). This shows that Σ_1 is distinct from $\{\mathscr{B}_0(-1,n)\}_{n\ge 1}$. To distinguish Σ_1 from the series of Ein components, it is enough to see that the spectra of the general bundles of these two series are different. (A direct verification of this fact is left to the reader.)

REMARK 3. Let $\mathscr{N}$ be the set of all values of n for which $\mathscr{M}_n \in \Sigma_1$, i.e.

$$\mathscr{N} = \{n \in 2\mathbb{Z}_+ \mid n = 4m + 2\varepsilon + a(a+1),\ \text{where } m \in \mathbb{Z}_+,\ \varepsilon \in \{0,1\},\ a \ge 2(m+\varepsilon)+3\}.$$

Then we easily see that

$$\lim_{r\to\infty} \frac{|\mathscr{N} \cap \{2,4,\dots,2r\}|}{r} = 1.$$

§ 4. Examples of Moduli Components of Stable Vector Bundles with Small Values of c_2

The conditions imposed on the triples of integers (m,ε,a) in Theorem 1 and, respectively, in Theorem 2 may fail to be satisfied for small values of these integers. However, (3), (4), and (5), respectively, (60), (61), and (62) are still true for some of small values of (m,ε,a). Hence our construction of irreducible components $\mathscr{M}_n \in \Sigma_0$, where $n = 2m+\varepsilon+a^2$, respectively, $\mathscr{M}_n \in \Sigma_1$, where $n = 4m+2\varepsilon+a(a+1)$, given in Sections 2 and 3 is still true for these values of (m,ε,a). A precise computation of these values is performed via using the Serre construction (11), respectively, (58) for the pairs $([\mathscr{E}_1],[\mathscr{E}_2])$ from (1) and (59), respectively. We thus provide below the list of irreducible components $\mathscr{M}_n$ of the series Σ_0 for $n \le 20$ and, respectively, of irreducible components $\mathscr{M}_n$ of the series Σ_1 for $n \le 40$.

4.1. Components $\mathscr{M}_n \in \Sigma_0$ for $n \le 20$. By Spec(E) we denote the spectrum of a general bundle E from $\mathscr{M}_n$. Below we use a standard notation Spec$(E) = (a^p, b^q, \dots)$ for the spectrum $(\underbrace{a\dots\dots a}_{p}, \underbrace{b\dots\dots b}_{q}, \dots)$.

(1) $n = 6$, $(m,\varepsilon,a) = (1,0,2)$. $\mathscr{M}_6$ is a component of the expected (by deformation theory) dimension $\dim \mathscr{M}_6 = 45$, and Spec$(E) = (-1, 0^4, 1)$. This corresponds to case 6(2) of Table 5.3 of Hartshorne and Rao [20].

(2) $n = 7$, $(m,\varepsilon,a) = (1,1,2)$. $\mathscr{M}_7$ is a component of the expected dimension $\dim \mathscr{M}_7 = 53$, and Spec$(E) = (-1, 0^5, 1)$ (cf. [20, Table 5.3, 7(2)]).

(3) $n = 8$, $(m,\varepsilon,a) = (2,0,2)$. $\mathscr{M}_8$ is a component of the expected dimension $\dim \mathscr{M}_8 = 61$, and Spec$(E) = (-1, 0^6, 1)$ (cf. [20, Table 5.3, 8(2)]).

(4) $n = 9$, $(m,\varepsilon,a) = (2,1,2)$. $\mathscr{M}_9$ is a component of the expected dimension $\dim \mathscr{M}_9 = 69$, and Spec$(E) = (-1, 0^7, 1)$.

(5) $n = 10$, $(m,\varepsilon,a) = (3,0,2)$. $\mathscr{M}_{10}$ is a component of the expected dimension $\dim \mathscr{M}_{10} = 77$, and Spec$(E) = (-1, 0^8, 1)$.

(6) $n = 11$, $(m,\varepsilon,a) = (3,1,2)$. $\mathscr{M}_{11}$ is a component of the expected dimension $\dim \mathscr{M}_{11} = 85$, and Spec$(E) = (-1, 0^9, 1)$.

(7) $n = 12$, $(m,\varepsilon,a) = (4,0,2)$. $\mathscr{M}_{12}$ is a component of the expected dimension $\dim \mathscr{M}_{12} = 93$, and Spec$(E) = (-1, 0^{10}, 1)$.

(8) $n = 18$, $(m,\varepsilon,a) = (1,0,4)$. $\mathscr{M}_{18}$ is a component of the expected dimension $\dim \mathscr{M}_{12} = 141$, and Spec$(E) = (-3, -2^2, -1^3, 0^6, 1^3, 2^2, 3)$.

4.2. Components $\mathscr{M}_n \in \Sigma_1$ for $n \le 40$.

(1) $n = 24$, $(m,\varepsilon,a) = (1,0,4)$. $\mathscr{M}_{24}$ is a component of the expected dimension $\dim \mathscr{M}_{24} = 187$, and Spec$(E) = (-4, -3^2, -2^3, -1^6, 0^6, 1^3, 2^2, 3)$.

(2) $n = 34$, $(m,\varepsilon,a) = (1,0,5)$. $\mathscr{M}_{34}$ is a component of dimension $\dim \mathscr{M}_{34} = 281$ larger than expected, and Spec$(E) = (-5, -4^2, -3^3, -2^4, -1^7, 0^7, 1^4, 2^3, 3^2, 4)$.

(3) $n = 36$, $(m,\varepsilon,a) = (1,1,5)$. $\mathscr{M}_{36}$ is a component of dimension $\dim \mathscr{M}_{36} = 290$ larger than expected, and Spec$(E) = (-5, -4^2, -3^3, -2^4, -1^8, 0^8, 1^4, 2^3, 3^2, 4)$.

(4) $n = 38$, $(m,\varepsilon,a) = (2,0,5)$. $\mathscr{M}_{38}$ is a component of the expected dimension $\dim \mathscr{M}_{38} = 299$, and Spec$(E) = (-5, -4^2, -3^3, -2^4, -1^9, 0^9, 1^4, 2^3, 3^2, 4)$.

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