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MULTIAGENT TEMPORAL LOGICS WITH MULTIVALUATIONS

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Abstract: We study multiagent logics and use temporal relational models with multivaluations. The key distinction from the standard relational models is the introduction of a particular valuation for each agent and the computation of the global valuation using all agents' valuations. We discuss this approach, illustrate it with examples, and demonstrate that this is not a mechanical combination of standard models, but a much more subtle and sophisticated modeling of the computation of truth values in multiagent environments. To express the properties of these models we define a logical language with temporal formulas and introduce the logics based at classes of such models. The main mathematical problem under study is the satisfiability problem. We solve it and find deciding algorithms. Also we discuss some interesting open problems and trends of possible further investigations.

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1. Introduction

Logical foundations of Information Sciences and Computer Sciences have been widely studied for reasoning about correctness, consistency, and reliability of information. In particular, the multiagent logics (e.g., with modalities interpreted as agents' operations, or oriented to model checking) were used for study interaction and autonomy, effects of cooperation (cf., e.g., [1–9]). For example, representation of agents' interaction (as a dual of common knowledge or information) was suggested in [9]. The concept of common knowledge for an agent was formalized and profoundly analyzed in [10] using agents' knowledge (S5-like) modalities as a base. Knowledge, as a concept itself, came from multiagency, since the individual knowledge may be obtained only from interaction of agents and learning.

The conception of knowledge was in a focus of AI and Logic in Computer Science for a long time. As a general field, knowledge representation is a part of AI which is devoted to designing computer representations for capture of information about the world which can be used to solve complex problems. The approach to model knowledge in terms of symbolic logic, probably, may be dated to the end of the 1950s. At 1962 Hintikka wrote the book: *Knowledge and Belief*, the first book-length work to suggest the use of modalities for capturing the semantics of knowledge. This book laid much of the groundwork for the subject, but a great deal of research has been conducted since then.

Some important feature of a multiagent environment is the observation that the acquiring of knowledge and interaction of agents occur during some intervals of time, and the length of such an interval might be very important. To capture this observation CS often use a symbolic (mathematical) temporal logic. Historically, the investigations of a temporal logic in the framework of mathematical and philosophical logics on using modal systems was originated by Prior in the late 1950s.

Since then temporal logic has been a very active area in mathematical logic, information sciences, AI and CS overall (see [11–13]). One of the important cases of these logics is the linear temporal logic $\mathcal{LTL}$, which was used for analyzing the protocols of computations and verification of consistency. The automaton technique for deciding satisfiability in this logic was developed in [14, 15]). Temporal ontology and temporal discourse were investigated and discussed in [16]. Further, to evolve the mathematical tools of $\mathcal{LTL}$, the solution of the admissibility problem for $\mathcal{LTL}$ was found in [17], while the basis for admissible

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rules of $\mathcal{LTL}$ was obtained in [18]. Modeling multiagency under assumption of nontransitive time was studied recently in [19, 20].

This paper studies a new approach to multiagency on using temporal relational models. These models have many valuations, a particular one for each agent, and the global one to be computed from an individual valuation by special rules. The main distinction from the standard approach is the new rules for computation of the truth values of formulas (which will use switches of valuations). We will illustrate by examples why this is not merely a mechanical combination of the usual rules. Using these models we define logics for classes of models and study the properties of the logics. The main mathematical problem we are dealing with is the satisfiability problem. We solve it and find deciding algorithms. In the final part of this paper we discuss some interesting open problems as well as possible trends of further investigations.

2. Motivation, Definitions, and Notation

Before defining the language of the logical systems that describe a multiagent environment, we preliminarily motivate the background for the introduction of the language. By way of recalling, we firstly outline the notion of relational model informally. These models are often used for the analysis and representation of information (cf. e.g. relational databases). A relational model usually may be viewed as a tuple $\langle W, \{R_i \mid i \in I\}, V\rangle$ that has the base set W—the set of worlds (or states) of these models, the set of binary relations $\{R_i \mid i \in I\}$ on these worlds (i.e., each R_i is a subset of $W \times W$), the valuation V of the set $Prop$ of propositional variables (letters) in these models; i.e., $V(p) \subseteq W$ for all $p \in Prop$. If $w \in W$ and $w \in V(p)$, then we say that p is true in the world w.

The relations R_i are usually referred to as particular accessibility relations between the worlds (or, alternatively, states). Then usually some logical language is introduced that is typically based on the Boolean logic and use the special logical operations which model the properties of these relations . The formulas of such a language are terms that are constructed from letters by means of logical operations; the formulas describe the properties of the models. The special rules are introduced for computation of the truth values of the formulas, and the logic is usually defined as the set of all formulas true in every world of the so-specified models.

Inspecting the general framework, we first discuss the way of embedding the multiagent approach. And the first idea is to consider many valuations $V_1, \dots, V_k$ in such models instead of the unique, single fixed one. Then each V_i represents the view of agent i about the truth of the atomic statements—propositional letters, and $w \in V_i(p)$ means that agent i consider p to be true in the world w.

Now, we introduce the relational models with which we will work. Let $Prop$ be the set of propositional letters.

Definition 1. A temporal linear k-model with agents' multivaluations is the structure

$$\mathscr{M} := \langle \mathscr{N}, \leq, \mathrm{Next}, V_1, \dots, V_k, V_0\rangle,$$

where $\mathscr{N}$ is the set of all naturals, $\leq$ is the standard order on $\mathscr{N}$, while Next is the binary relation as follows: $a \,\mathrm{Next}\, b$ if $b = a + 1$, and each V_j is a valuation of $Prop$ (i.e., $V_j(p) \subseteq \mathscr{N}$ for all $p \in Prop$).

We will use the convenient notation $\mathrm{Next}(n) = m$ to represent n Next m (i.e. to consider Next as a relation and a function).

These models have the wide range of applications: They can represent

(i) computational runs (in particular, threads, as often for the usual linear temporal logic),

(ii) surfing via networks, Internet, and databases collections (in this event $\mathscr{N}$ will represent a sequence of steps in the search),

(iii) sequences of queries for relational databases,

(iv) evolutions of social objects in time, etc. Each $a \in \mathscr{N}$ is called a state (or alternatively, as in Kripke semantics—a world), $V_j(p)$ represents the set of all states at which the atomic statement (proposition) p is considered true by agent j. Given $a \in \mathscr{N}$ and $p \in Prop$, we put

$$(\mathscr{M}, a) \Vdash_{V_j} p \;\Leftrightarrow\; a \in V_j(p)$$

and say that p is true at a with respect to V_j. But V_0 here is a special (global) valuation chosen by these models to fix the objective truth relation; this valuation, in a sense, summarizes the opinions of all agents. The ways to constructing V_0 from all V_j can differ. For instance, we may consider that

$$(\mathscr{M}, a) \Vdash_{V_0} p \Leftrightarrow \|\{j \mid (\mathscr{M}, a) \Vdash_{V_j} p, j \neq 0\}\| > \|\{j \mid (\mathscr{M}, a) \nVdash_{V_j} p, j \neq 0\}\|. \tag{I}$$

This means that the majority of agents believe in p being true.

$$(\mathscr{M}, a) \Vdash_{V_0} p \;\Leftrightarrow\; \|\{j \mid (\mathscr{M}, a) \Vdash_{V_j} p, j \neq 0\}\| \geq \|\{j \mid (\mathscr{M}, a) \nVdash_{V_j} p, j \neq 0\}\|. \tag{II}$$

This means that p is plausible.

$$(\mathscr{M}, a) \Vdash_{V_0} p \;\Leftrightarrow\; (\|\{j \mid (\mathscr{M}, a) \Vdash_{V_j} p, j \neq 0\}\|)/(\|\{j \mid (\mathscr{M}, a) \nVdash_{V_j} p, j \neq 0\}\|) > 3 \tag{III}$$

(if $\|\{j \mid (\mathscr{M}, a) \nVdash_{V_j} p,\ j \neq 0\}\| \neq 0$). This means that p is true from the viewpoint of an overwhelming majority of agents.

There are very many ways to express the meaning of the global valuation and the overwhelming majority of agents. Maybe, an agent's opinion may be considered with some prescribed appropriate weights or depends on different states, the rules for computing the global valuation can differ, etc. In the most extreme case we may assume V_0 to be arbitrary, which does not depend on any V_j—it is the opinion of the total dominant—the only truth that V_0 considers as true.

Now we discuss how to express the truth values of the statements describing the properties of models. To this end, we fix some logical language that uses the formulas that are built up from the (potentially infinite) set *Prop* of atomic propositions (synonymously-propositional letters, variables).

Definition 2. The set Form of all formulas for our multiagent logic contains *Prop* and is closed under the Boolean logical operations $\wedge, \vee \neg, \rightarrow$, the unary operations $\mathscr{N}_i$ (next) ($i \in [0, k]$), and the binary operations U_i, $i \in [0, k]$ (until, each one for agent i).

The formula $\mathscr{N}_i \varphi$ means that φ holds in the next time point (state) for agent i; further $\varphi U_i \psi$ can be read as follows: φ holds until ψ is true in the opinion of agent i.

Thus, we defined our semantics—models, and defined formulas—logical language. Now we need the rules for computing the truth values in our models for compound and long formulas. Assume given a temporal linear k-model with agents' multivaluations

$$\mathscr{M} := \langle \mathscr{N}, \leq, \text{ Next}, V_1, \ldots, V_k, V_0 \rangle$$

be given. That is, $V_i(p) \subseteq \mathscr{N}$ for all $p \in Prop$. If $a \in \mathscr{N}$ and $a \in V_i(p)$ then we write $(\mathscr{M}, a) \Vdash_{V_i} p$ and say that p is true at a with respect to V_i. The truth values may be expanded from letters to all formulas as follows:

Definition 3.

$$\forall p \in Prop\ (\mathscr{M}, a) \Vdash_{V_j} p \;\Leftrightarrow\; a \in \mathscr{N} \wedge\ a \in V_j(p);$$
$$(\mathscr{M}, a) \Vdash_{V_j} (\varphi \wedge \psi) \;\Leftrightarrow\; (\mathscr{M}, a) \Vdash_{V_j} \varphi \wedge (\mathscr{M}, a) \Vdash_{V_j} \psi;$$
$$(\mathscr{M}, a) \Vdash_{V_j} \neg\varphi \;\Leftrightarrow\; \neg[(\mathscr{M}, a) \Vdash_{V_j} \varphi];$$
$$(\mathscr{M}, a) \Vdash_{V_j} \mathscr{N}_i \varphi \;\Leftrightarrow\; \forall b[(a \text{ Next } b) \Rightarrow (\mathscr{M}, b) \Vdash_{V_i} \varphi]; \tag{$*$}$$

$$\begin{aligned}(\mathscr{M}, a) \Vdash_{V_j} (\varphi U_i \psi) \;\Leftrightarrow\; &\exists b[(a \leq b) \wedge ((\mathscr{M}, b) \Vdash_{V_i} \psi) \wedge \forall c[(a \leq c < b) \\ &\Rightarrow\; (\mathscr{M}, c) \Vdash_{V_i} \varphi]].\end{aligned} \tag{$**$}$$

We may define other logical operations on using the postulated ones. The modal operations $\Box_i$ (necessary for agent i) and $\Diamond_i$ (possible for agent i) might be defined via the temporal operations as follows: $\Diamond_i p := \top U_i p$ and $\Box_i p := \neg \Diamond_i \neg p$. It might be easily verified that

$$(\mathscr{M}, a) \Vdash_{V_j} \Diamond_i \varphi \;\Leftrightarrow\; \exists b \in \mathscr{N}[(a \leq b) \wedge (\mathscr{M}, b) \Vdash_{V_i} \varphi];$$
$$(\mathscr{M}, a) \Vdash_{V_j} \Box_i \varphi \;\Leftrightarrow\; \forall b \in \mathscr{N}[(a \leq b) \;\Rightarrow\; (\mathscr{M}, b) \Vdash_{V_i} \varphi].$$

Now, we will illustrate by examples that the chosen language is flexible to describe correctness of information in a multiagent environment.

Example 1. Agents (1) and (2) are in opposition for tomorrow:

$$(\mathscr{M}, a) \Vdash_{V_j} [\mathscr{N}_1 p \to \mathscr{N}_2 \neg p] \wedge [\mathscr{N}_2 p \to \mathscr{N}_1 \neg p].$$

This formula says that if one agent thinks that p will be true tomorrow, whereas the another one thinks the opposite.

Example 2. Agents (1) and (2) are in opposition about the truth of incontestable facts:

$$(\mathscr{M}, a) \Vdash_{V_j} [\Box_1 p \to \Box_2 \neg p] \wedge [\Box_2 p \to \Box_1 \neg p].$$

This formula says that now and always in future the agents have opposite opinions; one agent thinks that some fact is always true, whereas the others think it must be false always.

Example 3. Agent (1) eventually outargues agent (2)

$$(\mathscr{M}, a) \Vdash_{V_1} p \wedge \mathscr{N}_1 (p \wedge (p U_1 \Box_2 \neg p)).$$

The formula says that p is true now in the opinion of agent (1) and will be true during some time interval in future, but then p will be false in the opinion of (2).

Example 4. The fact is always possible in the opinion of at least one agent whereas it is never possible to be true in the opinion of all other agents

$$(\mathscr{M}, a) \Vdash_{V_0} \Box_0 \Big[\bigvee_{i \in [1,k]} \Diamond_i p \wedge \Box_0 \Big(p \to \neg \Big(\bigwedge_{i \in [1,k]} \Diamond_i p \Big) \Big) \Big].$$

Example 5. A fact p is always possible but suspicious:

$$(\mathscr{M}, a) \Vdash_{V_0} \Box_0 \Big[\bigvee_{i \in [1,k]} \Diamond_i p \Big] \wedge \neg \Box_0 \Diamond_0 p.$$

This says that the fact p is always (from viewpoint of the global agent 0) possible in the opinion of at least one agent. But p is not always possible in the opinion of the global agent (0).

Now, we pause briefly to discuss why the approach we offer is indeed innovative; why we cannot look at it as simply a mechanical combination of k examples of the standard linear temporal logic; why it is really new and interesting, and why the standard technique cannot work here directly.

The above is a consequence of the fact that in the definition of rules for computing the truth values of formulas; namely, $(*)$ and $(**)$. So, we switch here the valuations for the temporal operations: If the valuation is some V_j and we compute the truth value of a temporal operation with index i we switch j to i and use the valuation V_i further. That seems correct and well justified: If a temporal statement refers to agent i, then the opinion about truth for the future is its own. We give below illustrating examples.

Here (2) and (3) are agents' indexes, $\mathscr{N}_2$ and $\mathscr{N}_3$ are logical operations over the rules for computation their truth values defined above.

$$(\mathscr{M}, a) \Vdash_{V_1} p \wedge \mathscr{N}_2 (\neg p \wedge \Box_3 p);$$

$$(\mathscr{M}, a) \Vdash_{V_1} p \wedge \mathscr{N}_2 (\neg p \wedge \Box_3 (\neg p \mathscr{N}_3 p \to (\mathscr{N}_2 (\mathscr{N}_2 (p U_2 q))))).$$

As you may see the computation of truth values in these formulas switches the valuations. Therefore the standard technique cannot be applied here directly. That is in particular because the standard rule for replacing equivalents does not work.

Indeed, if for a model $\mathscr{M}$ we have

$$\forall a, (\mathscr{M}, a) \Vdash_{V_0} \Box_0 ((p \to q) \wedge (q \to p)),$$

then this does not imply in general that

$$\forall a, (\mathscr{M}, a) \Vdash_{V_0} \Box_1((p \to q) \wedge (q \to p)).$$

Assume that the class $\mathscr{K}$ of described models is given. We may assume that the rules of the definition of the global valuation V_0 via agents' valuations V_i, $1 \leq i \leq k$, are fixed and are the same for all models and all states of these models. Though the agents' valuations themselves may be various (which seems to be the most general case) but the rules imposed on the agents' valuations are the same for all states. For example, rules for the agents' valuations may be with the limitation: for all states a,

$$[\|\{i \mid (\mathscr{M}, a) \Vdash_{V_i} p\}\| > k/2 + 1] \;\Rightarrow [\forall i (1 \leq i \leq k \;\Rightarrow\; (\mathscr{M}, a) \Vdash_{V_i} p)]. \tag{1}$$

This means a uniform opinion, if majority of agents believe that a fact is true then all of them think likewise.

Definition 4. A formula φ is satisfiable in $\mathscr{K}$ if there are a model $\mathscr{M} \in \mathscr{K}$ and a state $a \in \mathscr{M}$ such that $(\mathscr{M}, a) \Vdash_{V_j}$ for some j.

The satisfiability problem for $\mathscr{K}$ is to resolve for any given formula if it is satisfiable in some model from $\mathscr{K}$. Assuming that $\mathscr{K}$ is chosen we may define the logic $\mathscr{L}(\mathscr{K})$ of this class, e.g., as follows:

$$\mathscr{L}(\mathscr{K}) := \{\varphi \mid \varphi \in Form, \forall \mathscr{M} \in \mathscr{K},\ \forall a \in \mathscr{M}, \forall V_j [(\mathscr{M}, a) \Vdash_{V_j} p]\}.$$

Assuming that all V_j are equal and V_0 is the same as each V_j and all of them are arbitrary, we see that $\mathscr{L}(\mathscr{K})$ is just the standard linear temporal logic $\mathcal{LTL}$. Moreover, each j-fragment of every logic $\mathscr{L}(\mathscr{K})$ for the valuation V_j will be $\mathcal{LTL}$. But, if the combinations of different temporal and modal operations for distinct agents are allowed, then the possibility of describing the properties of multiagent reasoning are much wider. For example, if (1) holds we have

$$\Big[\bigvee_{X, X \subseteq \{1,\ldots,k\}, \|X\| > k/2+1} \Big[\bigwedge_{i \in X} \mathscr{N}_i p\Big] \;\Rightarrow\; \bigwedge_{i \in \{1,\ldots,k\}} \mathscr{N}_i p\Big] \;\in\; \mathscr{L}(\mathscr{K}). \tag{2}$$

The satisfiability problem for the logic $\mathscr{L}(\mathscr{K})$ generated by some $\mathscr{K}$, is the satisfiability problem for the class $\mathscr{K}$ itself. For brevity we will write $\mathscr{L}$ instead $\mathscr{L}(\mathscr{K})$ on assuming $\mathscr{K}$ fixed. By a model $\mathscr{M}$ (if not specified otherwise) we understand a model from $\mathscr{K}$.

3. Satisfiability Problem

We will need the following special modification of the k-models; the models $\mathscr{M}_{+Circle}$. Recall that if $n, m \in \mathscr{N}$ with $n < m$, then we let $[n, m]$ denote the closed interval of all numbers between n and m and these numbers n and m themselves.

Definition 5. Every $\mathscr{M}_{+Circle}$ model has the following structure. If $n, c(m), m \in \mathscr{N}$, where $0 < n < c(m) \leq m$, then $\mathscr{M}_{+Circle} = \langle [n, m], \leq, \mathrm{Next}, V_1, \ldots, V_k, V_0\rangle)$ where $\mathrm{Next}(m) := c(m)$.

The rules for computing the truth values of formulas in such models with respect to any V_j are defined exactly as earlier in the models; simply for the states bigger than $c(m)$ the order $\leq$ is replaced by the possible runs via sequences by Next. More precisely, we define $(\mathscr{M}_{+Circle}, a) \Vdash_{V_j} (\varphi U_i \psi)$ as follows: If $a \in [0, c(m)]$ then the definition is as earlier; if $a > c(m)$ then

$$\begin{gathered}(\mathscr{M}_{+Circle}, a) \Vdash_{V_j} (\varphi U_i \psi) \;\Leftrightarrow \exists b[(a \leq b \leq m) \wedge ((\mathscr{M}_{+Circle}, b) \Vdash_{V_i} \psi) \\ \wedge \forall c[(a \leq c < b) \Rightarrow (\mathscr{M}_{+Circle}, c) \Vdash_{V_i} \varphi]] \vee \exists d[(d > c(m)) \wedge ((\mathscr{M}_{+Circle}, d) \Vdash_{V_i} \psi) \\ \wedge \forall c[(a \leq c \leq m) \Rightarrow (\mathscr{M}_{+Circle}, c) \Vdash_{V_i} \varphi] \wedge \forall c[(c(m) \leq c < d) \\ \Rightarrow (\mathscr{M}_{+Circle}, c) \Vdash_{V_i} \varphi]].\end{gathered}$$

So, these rules act in accordance with the intuition of what is circled bypath by Next. Given a formula φ, we let $Sub(\varphi)$ stand for the set of all its subformulas.

Let $Tm(\varphi)$ be the temporal degree of φ. Recall that the temporal degree of formulas is defined inductively:

(i) the temporal degree of letters is 0;

(ii) the temporal degree of each formula with a temporal operation as the main one is the maximal temporal degree of the components plus 1;

(iii) the temporal degree of each formula with a Boolean logic operation as the main one is the maximal temporal degree of the components.

Recall that k is the number of agents in our models. Put $f(\varphi) := 2 \times 2^{\|Sub(\varphi)\|} \times k + 3$. By the *size of a model* we mean the number of states in this model.

Theorem 6. *If a formula φ is satisfiable in a model $\mathscr{M}$ at a state by a valuation V_j, then there exists a finite model of kind $\mathscr{M}_{+Circle}$ with size at most $f(\varphi)$ satisfying φ at the world 0 by its own V_j.*

Proof. Let $\mathscr{M} := \langle \mathscr{N}, \leq, \mathrm{Next}, V_1, \ldots, V_k, V_0 \rangle$ be given and $(\mathscr{M}, a) \Vdash_{V_j} \varphi$. Evidently, we may assume that $a = 0$. Given $b \in \mathscr{M}$, for all $j \in [0, k]$ put

$$Sub_j(b) := \{\alpha \in Sub(\varphi) \mid (\mathscr{M}, b) \Vdash_{V_j} \alpha\};$$

$$Desc(b) := \{Sub_j(b) \mid j \in [0, k]\};$$

$$Ftr(b) := \{Desc(c) \mid c \geq b\}.$$

A simple observation is that there is some $c_m \in \mathscr{M}, c_m > 3$ such that $Ftr(d) = Ftr(g)$ for all $d, g \geq c_m$. This is the case because the sets $Ftr(d)$ may only decrease with increasing d. Take such minimal c_m.

For all $x \geq 1$, the track of realizers from x is the minimal interval $[x, y]$ (denoted in the sequel by $[x, Rls(x)]$) starting from x such that

$$\begin{aligned} &\forall(\varphi_1 U_j \varphi_2) \in Sub(\varphi)[\exists i(\mathscr{M}, x) \Vdash_{V_i} (\varphi_1 U_j \varphi_2) \wedge (\mathscr{M}, x) \Vdash_{V_i} \neg\varphi_2 \\ &\Rightarrow \exists y \in Rls(x)((\mathscr{M}, y) \Vdash_{V_i} \varphi_2) \wedge \forall z(x \leq z < y)(\mathscr{M}, z) \Vdash_{V_i} \varphi_1)] \\ &\wedge\forall(\mathscr{N}_j \varphi_1) \in Sub(\varphi)[\exists i(\mathscr{M}, x) \Vdash_{V_i} \mathscr{N}_j \varphi_1 \Rightarrow [(x+1) \in Rls(x)]. \end{aligned}$$

That minimal interval might be large but nonetheless it exists; we denote it by $[x, Rls(x)]$. Now we consider c_m and $[c_m, Rls(c_m)]$.

By the definition of c_m there is $d_m > Rls(c_m) + 2$ such that $Desc(d_m) = Desc(c_m)$. Take such smallest d_m and define now that $Next(d_m) := c_m + 1$ and delete all states from $\mathscr{M}$ that are strictly bigger than d_m. Denote the so-obtained model by $\mathscr{M}_{+Circle}$. As we noted before the formulation of our theorem, the rules for the truth values of formulas in such model with respect to any V_i are defined exactly as earlier in the original k-models, simply for the states bigger than c_m the order $\leq$ to be replaced by all possible sequences of states by Next.

Lemma 7. *Given $\psi \in Sub(\varphi)$, $a \in \mathscr{M}_{+Circle}$, with $a \geq c_m$, and V_j, we have*

$$(\mathscr{M}, a) \Vdash_{V_i} \psi \;\Leftrightarrow\; (\mathscr{M}_{+Circle}, a) \Vdash_{V_i} \psi.$$

Proof. We will proceed it by induction on the length of ψ. For letters, the claim is evidently true. The inductive steps for Boolean logical operations are evident as well. Let $\psi = \mathscr{N}_j \varphi_1$. If $a \geq c_m$ and $a < d_m$ then the conclusion

$$(\mathscr{M}, a) \Vdash_{V_i} \mathscr{N}_j \varphi_1 \;\Leftrightarrow\; (\mathscr{M}_{+Circle}, a) \Vdash_{V_i} \mathscr{N}_j \varphi_1$$

is immediate from the inductive assumption.

If $a = d_m$ then $\mathrm{Next}(d_m) := c_m + 1$ and by the indicative hypothesis

$$(\mathscr{M}, c_m + 1) \Vdash_{V_i} \varphi_1 \Leftrightarrow (\mathscr{M}_{+Circle}, c_m + 1) \Vdash_{V_i} \varphi_1.$$

Therefore, from $Desc(d_m) = Desc(c_m)$ we obtain

$$(\mathscr{M}, d_m) \Vdash_{V_i} \mathscr{N}_j\varphi_1 \Leftrightarrow (\mathscr{M}_{+Circle}, d_m) \Vdash_{V_i} \mathscr{N}_j\varphi_1.$$

Thus, the inductive proof for $\mathscr{N}_i$ is complete.

Consider now that case that $\psi = \varphi_1 U_j \varphi_2 \in Sub(\varphi)$. Assume first that $(\mathscr{M}, a) \Vdash_{V_i} \varphi_1 U_j \varphi_2$. Then there exists a smallest $b \in \mathscr{M}$ such that $b \geq a$ and $(\mathscr{M}, b) \Vdash_{V_i} \varphi_2$, and else either

(i): for all c, where $a \leq c < b$, $(\mathscr{M}, b) \Vdash_{V_j} \varphi_1$, or otherwise

(ii): $b = a$.

Assume first that $b \leq d_m$; then from the inductive assumption we obtain $(\mathscr{M}_{+Circle}, b) \Vdash_{V_i} \varphi_2$.

If (ii) holds (that is $b = a$) then $(\mathscr{M}_{+Circle}, a) \Vdash_{V_i} \varphi_2$ and so we immediately obtain $(\mathscr{M}_{+Circle}, a) \Vdash_{V_i} \varphi_1 U_j \varphi_2$. If (ii) is not a case but (i) is, then we have that $(\mathscr{M}_{+Circle}, b) \Vdash_{V_i} \varphi_1$ for all c, where $a \leq c < b$, by the inductive hypothesis, and consequently $(\mathscr{M}_{+Circle}, a) \Vdash_{V_i} \varphi_1 U_i \varphi_2$.

Assume now that $b > d_m$. Then $(\mathscr{M}, d_m) \Vdash_{V_i} \varphi_1 U_j \varphi_2$. Applying $Desc(d_m) = Desc(c_m)$, we see that $(\mathscr{M}, c_m) \Vdash_{V_i} \varphi_1 U_i \varphi_2$. Using $d_m > Rls(c_m)$, $a \leq d_m$ and the inductive assumption, from the fact that $(\mathscr{M}, a) \Vdash_{V_i} \varphi_1 U_j \varphi_2$ we have that there is a path in $\mathscr{M}_{+Circle}$ by Next leading from a to a state in $[c_m, Rls(c_m)]$ where φ_2 is true with respect to V_i in $\mathscr{M}_{+Circle}$ and that along this path always φ_1 is true with respect to V_i in $\mathscr{M}_{+Circle}$. So we obtain $(\mathscr{M}_{+Circle}, a) \Vdash_{V_i} \varphi_1 U_j \varphi_2$, which is what we need.

In the opposite direction, assume now that $(\mathscr{M}_{+Circle}, a) \Vdash_{V_i} \varphi_1 U_j \varphi_2$. By the definition of $\mathscr{M}_{+Circle}$ we have that $a \leq d_m$ and there is a path by Next in the $\mathscr{M}_{+Circle}$ from a into some closest $b \leq d_m$ satisfying φ_2 with respect to V_i where φ_1 is always true along this path with respect to V_i in the model $\mathscr{M}_{+Circle}$. If

$$\text{the path does not go via } c_m + 1, \tag{3}$$

then for all c where $a \leq c < b \leq d_m$, $(\mathscr{M}_{+Circle}, c) \Vdash_{V_i} \varphi_1$. Then using the inductive assumption we conclude that $(\mathscr{M}, c) \Vdash_{V_j} \varphi_1$ for all such c, and so $(\mathscr{M}, a) \Vdash_{V_i} \varphi_1 U_j \varphi_2$.

Assume now that

$$\text{the path goes via } c_m + 1. \tag{4}$$

Then $(\mathscr{M}_{+Circle}, c_m + 1) \Vdash_{V_i} \varphi_1 U_j \varphi_2$ and using (3) we obtain $(\mathscr{M}, c_m + 1) \Vdash_{V_i} \varphi_1 U_j \varphi_2$. Applying the inductive assumption we conclude $(\mathscr{M}, a) \Vdash_{V_i} \varphi_1 U_j \varphi_2$. $\square$

Lemma 8. *For all $\psi \in Sub(\varphi)$, $a \in \mathscr{M}_{+Circle}$, and V_j, we have*

$$(\mathscr{M}, a) \Vdash_{V_j} \psi \Leftrightarrow (\mathscr{M}_{+Circle}, a) \Vdash_{V_j} \psi.$$

PROOF. The proof is immediate from Lemma 7 and the fact that the initial part of the model $\mathscr{M}$ before c_m while its transformation into $\mathscr{M}_{+Circle}$ stays intact. So the verification is a routine standard computation by induction on the length of the formulas. $\square$

Thus, by Lemma 8 we have now that the model $\mathscr{M}_{+Circle}$ also satisfies the formula φ and this model is finite. We only need now to reduce the size of this model to a bound computable from the size of φ.

Lemma 9. *There is a model $\mathscr{M}_{+Circle}$ satisfying φ and having size at most $f(\varphi)$.*

PROOF. We will use the previous notation and the above-proved facts. Thus, $\mathscr{M}_{+Circle}$ satisfies φ. Take the smallest state 0 from $\mathscr{M}_{+Circle}$. First, recall that $2 \leq c_m - 1$. Choose the biggest $b \in [1, c_m - 1]$ such that $Desc(1) = Desc(b)$, if one exists. In particular, it may happen that $b = 1$, then we do nothing at this stage. Otherwise we delete all states from $[1, b)$ in $\mathscr{M}_{+Circle}$ and denote the resulting model by $\mathscr{M}_{+Circle}(1, b)$. We will show that

$$(\mathscr{M}_{+Circle}, s) \Vdash_{V_i} \psi \Leftrightarrow (\mathscr{M}_{+Circle}(1, b), s) \Vdash_{V_i} \psi \tag{5}$$

for all $s \in \mathscr{M}_{+Circle}(1,b)$, $\psi \in Sub(\varphi)$, and i. For $s \geq b$ this statement is evident. It remains only to consider the case that $s = 0$. For ψ to be a letter it is evident, and the inductive steps of the proof by the length of ψ for Boolean operations are again evident. Assume now that for ψ (5) is proven and $\mathscr{N}_j\psi \in Sub(\varphi)$. We claim that

$$(\mathscr{M}_{+Circle}, 0) \Vdash_{V_i} \mathscr{N}_j\psi \;\Leftrightarrow\; (\mathscr{M}_{+Circle}(1,b), 0) \Vdash_{V_i} \mathscr{N}_j\psi. \tag{6}$$

This follows from (5) and our choice of b above with $Desc(1) = Desc(b)$.

Let for ψ_1 and ψ_2 the statement (5) be proven and $\psi_1 U_j \psi_2 \in Sub(\varphi)$.

If $(\mathscr{M}_{+Circle}, 0) \Vdash_{V_i} \psi_1 U_j \psi_2$, then we have either $(\mathscr{M}_{+Circle}, 0) \Vdash_{V_i} \psi_2$ and by inductive assumption we receive $(\mathscr{M}_{+Circle}(1,b), 0) \Vdash_{V_i} \psi_2$ and we obtain $(\mathscr{M}_{+Circle}(1,b), 0) \Vdash_{V_i} \psi_1 U_j \psi_2$.

Or otherwise $(\mathscr{M}_{+Circle}, 0) \Vdash_{V_i} \neg\psi_2$ and so $(\mathscr{M}_{+Circle}, 0) \Vdash_{V_i} \psi_1$, and, moreover, $(\mathscr{M}_{+Circle}, 1) \Vdash_{V_i} \psi_1 U_j \psi_2$. Then by the choice of b above with $Desc(1) = Desc(b)$ we obtain $(\mathscr{M}_{+Circle}, b) \Vdash_{V_i} \psi_1 U_j \psi_2$. Therefore,

$$(\mathscr{M}_{+Circle}(1,b), b) \Vdash_{V_i} \psi_1 U_j \psi_2.$$

By the inductive assumption, $(\mathscr{M}_{+Circle}, 0) \Vdash_{V_i} \psi_1$ implies $(\mathscr{M}_{+Circle}(1,b), 0) \Vdash_{V_i} \psi_1$. This overall gives that $(\mathscr{M}_{+Circle}(1,b), 0) \Vdash_{V_i} \psi_1 U_j \psi_2$.

In the opposite direction, assume that

$$(\mathscr{M}_{+Circle})(1,b), 0) \Vdash_{V_i} \psi_1 U_j \psi_2.$$

If we assume that $(\mathscr{M}_{+Circle}(1,b), 0) \Vdash_{V_i} \psi_2$, then by the inductive assumption this yields the statement $(\mathscr{M}_{+Circle}, 0) \Vdash_{V_i} \psi_2$ and so $(\mathscr{M}_{+Circle}, 0) \Vdash_{V_i} \psi_1 U_j \psi_2$.

Assume now that $(\mathscr{M}_{+Circle}(1,b), 0) \Vdash_{V_i} \neg\psi_2$. Then $(\mathscr{M}_{+Circle}(1,b), 0) \Vdash_{V_i} \psi_1$ and

$$(\mathscr{M}_{+Circle}(1,b), b) \Vdash_{V_i} \psi_1 U_j \psi_2.$$

Therefore,

$$(\mathscr{M}_{+Circle}, b) \Vdash_{V_j} \psi_1 U_j \psi_2$$

since $b \geq b$ and (5). Also, $(\mathscr{M}_{+Circle}(1,b), 0) \Vdash_{V_i} \psi_1$ yields $(\mathscr{M}_{+Circle}, 0) \Vdash_{V_i} \psi_1$. So, $(\mathscr{M}_{+Circle}), 0) \Vdash_{V_i} \psi_1 U_j \psi_2$. Thus, we proved that

$$(\mathscr{M}_{+Circle}, 0) \Vdash_{V_j} \psi_1 U_j \psi_2 \Leftrightarrow (\mathscr{M}_{+Circle}(1,b), 0) \Vdash_{V_j} \psi_1 U_j \psi_2, \tag{7}$$

This statement concludes the proof of (5).

Now we continue the proof of Lemma 9. Considering b as we initially did above with 0 in $\mathscr{M}_{+Circle}$, and subsequently making similar reformations that move to c_m, we do as much steps as much various $Descr(s)$ may happen—so some finite, effectively bounded amount of steps. So, we then receive a model similar to $\mathscr{M}_{+Circle}$, but which has at most $2^{\|Sub(\varphi)\|} \times k + 3$ states before c_m. Since this stage, we make similar rarefication in the loop path in $\mathscr{M}_{+Circle}$ from $c_m + 1$ to itself c_{m+1}. This concludes the proof of Theorem 6.

Theorem 10. *If φ is satisfiable in a finite model $\mathscr{M}_{+Circle}$ then φ is satisfiable in some k-model $\mathscr{M}$.*

PROOF. Let

$$\mathscr{M}_{+Circle} = \langle [n, c(m)] \cup [c(m), m], \leq, \mathrm{Next}, V_1, \ldots, V_k, V_0 \rangle,$$

where $\mathrm{Next}(m) := c(m)$, and $(\mathscr{M}_{+Circle}) \Vdash_{V_i} \varphi$ for some i. Consider the infinite k-model $\mathscr{M}$ with the following structure: The base set $\mathscr{N}$ of this model is the sequence of all states $[n, c(m)] \cup [c(m)+1, m]$ and the infinite amount of the states combined from the intervals of the states situated in $[c(m)+1, m]$ repeated one by one, where $\mathrm{Next}(m) = c(m)+1$. The valuations V_j on this model are just transferred from the model $\mathscr{M}_{+Circle}$. This is immediate to show (using simple by induction on the length of the formulas) that, for every (absolutely every) formula ψ constructed out of letters from φ, we have

$$\forall a \in \mathscr{M}_{+Circle}, \forall V_i \; [(\mathscr{M}_{+Circle}, a) \Vdash_{V_i} \psi \;\Leftrightarrow\; (\mathscr{M}, a) \Vdash_{V_i} \psi].$$

So, $\mathscr{M}$ satisfies φ. $\square$

Recall that a logic $\mathscr{L}(\mathscr{K})$ is decidable if for every formula φ we may compute whether $\varphi \in \mathscr{L}(\mathscr{K})$. Observe that $\varphi \in \mathscr{L}(\mathscr{K})$ iff $\neg\varphi$ is not satisfiable in $\mathscr{L}(\mathscr{K})$. From Theorems 6 and 10 we immediately obtain

Theorem 11. *The satisfiability problem for $\mathscr{L}(\mathscr{K})$is decidable (so the logic $\mathscr{L}(\mathscr{K})$ is decidable). For a formula φ to be satisfiable it is sufficient to check the satisfiability of φ in the models $\mathscr{M}_{+Circle}$ of size at most $f(\varphi)$.*

4. Multiagent Temporal Interval Linear Logic

In this section we will consider the case when the models are not linear and even nontransitive, but are compound from some fragments of our temporal models from the previous section. We think that the case is indeed interesting and useful for applications. The matter is that the assumption that all computational runs are linear and potentially infinite is too strong. In fact, all resources are always limited, while they may be sufficiently big but with some assumed upper bound. We aim to represent such limitation as follows:

Let we chop the set of all naturals N into the infinite sequence of closed intervals: $[s_i, s_i + k_i], i \in N$; i.e., we assume that $N = \bigcup_{i\in N}[s_i, s_i + k_i]$, $s_i < s_{i+1}$, and that $s_i + k_i \geq s_{i+1}$, $s_i + k_i < s_{i+1} + k_{i+1}$.

So, we admit that the intervals may have possible nonempty not one state overlap; i.e., it could be that $[s_i, s_i + k_i] \cap (s_{i+1}, s_{i+1} + k_{i+1}] \neq \varnothing$, and so these intervals may have a nonempty and not one state common part.

The temporal interval linear k-model with agents' multivaluations is the structure

$$\mathscr{M} := \Big\langle \bigcup_{i\in N}[s_i, s_i + k_i], \leq, \mathrm{Next}, V_0, V_1, \ldots, V_k \Big\rangle,$$

where $\leq$ is the standard linear order on N, while Next is the standard next binary relation, and each V_i is a valuation. But for V_0 we assume earlier that V_0 is a global valuation computed via valuations of the agents by some common rules.

The models are intended to describe real computation in bounded time. In the case that $s_{i+1} = s_i + 1$ it is just somewhat like immediate transfer of information. In case that $[s_i, s_i + k_i] \cap (s_{i+1}, s_{i_1} + k_{i+1}] \neq \varnothing$, this models the situation when it might be that the next computational run started before the previous one was completed and they work sharing the resources and the information.

For such models we may define the truth values of formulas in exactly the same way as in the previous section—for pure linear time, with only a distinction in the definition of the truth values of the formulas containing operations U_m—until ones. The definition is as follows:

$$(\mathscr{M}, a) \Vdash_{V_j} (\varphi U_m \psi) \;\Leftrightarrow\; \exists b \in \mathscr{M}[(a \leq b \leq s_{i+1} + k_{i+1})$$
$$\wedge((\mathscr{M}, b) \Vdash_{V_m} \psi) \wedge \forall c[(a \leq c < b) \Rightarrow ((\mathscr{M}, c) \Vdash_{V_m} \varphi)]].$$

Thus, every U_m works as usual but is bounded by the upper boundary of the local run—by $s_i + k_i$. This agrees very well with the usual intuition concerning the computational procedures and runs—the solution (a state satisfying the formula) should (if it exists) be reached before the end of computation for the current local computational process. We think that the structures of such models and their properties are clear. But nonetheless we will provide some illustrative examples.

Examples.

(I) $\mathscr{M} := \langle \bigcup_{i\in N}[s_i, s_i + k_i], \leq, \mathrm{Next}, V_0, V_1, \ldots, V_k \rangle$, where $s_i := i^2, k_i := (i+1)^2 - i^2$ so bounds are squares of numbers. Here the intersection of the time intervals are only bounds—the numbers i^2.

(II) $\mathscr{M} := \langle \bigcup_{i\in N}[s_i, s_i + k_i], \leq, \mathrm{Next}, V_0, V_1, \ldots, V_k \rangle$, with $s_i := (10\times i), k_i := (10\times(i+1)) - (10\times i) + 5$. Now the intersection of the time intervals are not only bounds; e.g. $[s_0, s_0 + k_0] \cup (s_1, s_1 + k_1] = [11, 15]$.

As in Section 2, we denote an arbitrary class of all such models by $\mathscr{K}$ and denote the logic generated by this class by $\mathscr{L}(\mathscr{K})$; the satisfiability of formulas and decidability of the logic defined as earlier.

Given a model $\mathscr{M}$ described above, the model $\mathscr{M}^{(-)}$ is the one obtained from $\mathscr{M}$ by deleting all states of all intervals $[s_i, s_{i+1}]$ situated strictly far than a certain fixed number n in $(s_m, s_m + k_m)$, and by defining $Next(n) = s_m$.

Lemma 12. *If a formula φ is satisfiable in a model $\mathscr{M}$ at 0 by a valuation V_j, then there exists a finite model of kind $\mathscr{M}^{(-)}$ satisfying φ in the world 0 by its own V_j where the size of $\mathscr{M}^{(-)}$ is at most $2^{\|Sub(\varphi)\|} \times k + 3$ and the number of the sates s_i in this model is at most the temporal degree of φ plus 2.*

Proof. Let the model

$$\mathscr{M} := \Big\langle \bigcup_{i \in N} [s_i, s_i + k_i], \leq, \mathrm{Next}, V_0, V_1, \ldots, V_k \Big\rangle$$

satisfies a formula φ: $(\mathscr{M}, 0) \Vdash_{V_j} \varphi$. Let the temporal degree of φ be m. Using the standard argument on the temporal degree of a formula, we may assume that the model is shortened now by deleting all states strictly bigger than $s_{m+2} + k_{m+2}$ and putting $\mathrm{Next}(s_{m+2} + k_{m+2}) = s_{m+2} + k_{m+2}$, where m is the temporal degree of φ. And this model will satisfy φ at 0 as well.

So, we assume now that $\mathscr{M}$ has this structure. The number of the intervals $[s_i, s_i + k_i]$ in this model is at most $m+2$. We will rarefy this model starting from the bottom interval $[s_0, s_0 + k_0]$. For $[s_0, s_0 + k_0]$ we carry out the proof exactly as in Lemma 9 starting from considering the interval $[s_0, s_0 + k_0]$ as $[1, c_m - 1]$ in Lemma 9 making rarefication as it is shown there. This transformation will not change the truth values of subformulas of φ, and, in particular, s_0 and $s_0 + k_0$ will remain intact. Since this point we continue this rarefication procedure for resulting $[s_1, s_1 + k_1]$ and so forth. In at most $m+2$ steps this procedure will be completed. And the resulting model will satisfy φ at 0. $\square$

Lemma 13. *If a formula φ is satisfiable in a finite model $\mathscr{M}^{(-)}$ described in Lemma 12 at the state 0 by a valuation V_j, then φ may be satisfied in an infinite model $\mathscr{M}$ of our class.*

Proof. This is a standard argument using the temporal degree of formulas. $\square$

From Lemmas 12 and 13 we immediately infer

Theorem 14. *The satisfiability problem for the logic $\mathscr{L}(\mathscr{K})$ is decidable. For verification that a formula φ is satisfiable it is sufficient to check its satisfiability in the models $\mathscr{M}^{(-)}$ of size at most $2^{\|Sub(\varphi)\|} \times k + 3$.*

5. Conclusion

We think that the research of this paper may be essentially extended, since many interesting problems remain open. The case when the global valuation would be computed via the valuations of agents at states not uniformly, but by the rules specific for any state, is not considered yet. Another venue not explored yet is the computation of truth at states when we consider many valued values (e.g., from some intervals of possible truth values, as e.g. in Łukasiewicz logics or Fuzzy Logic). The yet open question is to consider models with lacunaes in computational runs. That is the one when the agents can fail to see the whole future but have some lacunaes of invisible intervals that are unseen, and when rules for computation truth values for temporal and modal operations are accordingly enrolled. The extension of our results to the branching time logic is a very interesting task. Of much interest is the investigation of admissibility for rules and validity of rules in these logics. The admissibility of rules was an area of most attraction for the author for a long time (cf. [17, 21, 22]) and also many strong results about admissibility were obtained by other researchers (cf. e.g. [23–26]). That area is very close to unification problematics (cf. [27, 28, 23, 29]) and it is very interesting to extend the unification theory to logics within the framework of this paper.

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