

Existence of positive solutions to semilinear elliptic problems with nonlinear boundary condition

CHAN-GYUN KIM and EUN KYOUNG LEE*

Department of Mathematics Education, Pusan National University, Busan 609-735, Korea

*Corresponding author. E-mail: eklee@pusan.ac.kr

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Abstract. In this paper, a semilinear elliptic equation with a nonlinear boundary condition and a perturbation in the reaction term is studied. The existence of a positive solution and another non-zero solution to the problem is proved when $|\lambda|$ is small enough without any specific assumptions on the perturbation term. Moreover, it is shown that the non-zero solution becomes a positive one for small $\lambda > 0$ under suitable assumptions on the perturbation term.

Keywords. Semilinear elliptic problem; nonlinear boundary condition; perturbed problem; positive solution; multiplicity.

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1. Introduction

In mathematical modeling, elliptic partial differential equations are used together with boundary conditions specifying the solution on the boundary of the domain. Commonly, problems with boundary conditions that are linear functions of the values or normal derivatives of the solutions on the boundary have been studied extensively (see [1, 8, 9, 14, 17, 21, 23, 32] and references therein). However models that require nonlinear boundary conditions have been increasing and consequently, problems with nonlinear boundary conditions have attracted great interest in recent studies. For example, Mavinga and Nkashama [19] studied the solvability of nonlinear second-order elliptic partial differential equations with nonlinear boundary conditions where nonlinearities are interacted with high order eigenvalues. More recently, Kim *et al.* [15] studied the Laplace equation with a superlinear nonlinear boundary condition on a bounded domain. Among other results, using variational method, the existence of a non-constant positive solution for any sufficiently large parameter value was shown. For more interesting results, we refer the reader to [3, 5–7, 10–13, 18, 20, 22, 24–30] and references therein.

In this paper, we are concerned with the following semilinear elliptic equation with a nonlinear boundary condition

$$\begin{cases} -\Delta u + u = \lambda g(x, u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} = f(x, u) & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a smooth bounded domain in \mathbb{R}^N with $N \geq 2$, η is the outer normal to $\partial\Omega$, $\lambda \in \mathbb{R}$ is a parameter, and the nonlinearities $f : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$, $g : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.

Under suitable assumptions on the boundary term $f(x, s)$ (a sign-changing nonlinearity), we study the existence of a positive mountain pass solution and another non-zero solution to the problem for $|\lambda| > 0$ small enough without any specific assumptions on $g(x, s)$. Moreover, it is shown that the non-zero solution is a positive one for small $\lambda > 0$ under suitable assumptions on $g(x, s)$ (see F_6 below).

To state the main result of this paper explicitly, let us introduce the following assumptions on the functions f and g :

- (F₁) there exist $\alpha_1 \in (2, 2_*)$ and $C > 0$ such that $|f(x, s)| \leq C(s^{\alpha_1-1} + 1)$ for $(x, s) \in \partial\Omega \times [0, \infty)$, where $2_* = \infty$ if $N = 2$ and $2_* = \frac{2(N-1)}{N-2}$ if $N \geq 3$;
- (F₂) there exist $\alpha > 2$, $\theta \in [1, 2)$ and $C > 0$ such that $\alpha F(x, s) - sf(x, s) \leq Cs^\theta + C$ for $s \geq 0$ and $x \in \partial\Omega$, where $F(x, s) = \int_0^s f(x, \tau) d\tau$;
- (F₃) there exist $x_0 \in \partial\Omega$ and $\delta_0 > 0$ such that $\lim_{s \rightarrow \infty} (\min_{x \in B(x_0, \delta_0) \cap \partial\Omega} \frac{F(x, s)}{s^2}) = \infty$, where $B(x_0, \delta_0)$ denotes an open ball centered at x_0 with radius δ_0 in \mathbb{R}^N ;
- (F₄) $f(x, s) = 0$ for all $(x, s) \in \partial\Omega \times (-\infty, 0]$ and $\limsup_{s \rightarrow 0^+} (\max_{x \in \partial\Omega} \frac{f(x, s)}{s}) < \mu_1$, where $\mu_1 > 0$ is the principal eigenvalue of $-\Delta u + u = 0$ in Ω , $\frac{\partial u}{\partial \eta} = \mu u$ on $\partial\Omega$;
- (F₅) f is locally Lipschitz continuous on $\partial\Omega \times [0, \infty)$, i.e., for any $M > 0$, there exist $\alpha_2 = \alpha_2(M) \in (0, 1]$ and $L = L(M) > 0$ such that $|f(x_1, s_1) - f(x_2, s_2)| < L(|x_1 - x_2|^{\alpha_2} + |s_1 - s_2|^{\alpha_2})$ for all $(x_1, s_1), (x_2, s_2) \in \partial\Omega \times [0, M]$;
- (F₆) $g(x, 0) \geq 0$ for all $x \in \Omega$ and $\liminf_{s \rightarrow 0} (\max_{x \in \Omega} \frac{g(x, s) - g(x, 0)}{s}) > -\infty$.

We give the examples of $f(x, s)$ satisfying all the hypotheses (F₁)–(F₅).

Example 1.1. Let $f(x, s) = a(x)s^{p-1} + b(x)s^{q-1}$ for $(x, s) \in \partial\Omega \times [0, \infty)$ and $f(x, s) = 0$ for $(x, s) \in \partial\Omega \times (-\infty, 0)$. Here $2 < q < p < 2_*$ and a, b are locally Lipschitz continuous on $\partial\Omega$ satisfying either (i) $a(x_0) > 0$ for some $x_0 \in \partial\Omega$ and $b(x) \leq 0$ for all $x \in \partial\Omega$ or (ii) $a(x) \geq 0, \neq 0$ in $\partial\Omega$ and $b(x)$ is any function. Then it can be easily verified that (F₁), (F₃), (F₄) and (F₅) hold. If we take $\alpha = p$ for the case (i) and $\alpha = q$ for the case (ii), then $\alpha F(x, s) - sf(x, s) \leq 0$ for all $(x, s) \in \partial\Omega \times [0, \infty)$. Thus (F₂) holds.

Let X denote the Sobolev space $W^{1,2}(\Omega)$ with norm $\|u\|_X = (\int_\Omega (|\nabla u|^2 + u^2) dx)^{\frac{1}{2}}$. Now we give the main result in this paper.

Theorem 1.2. Assume that (F₁)–(F₅) hold. Then the following assertions hold:

- (i) There exists $\lambda^* > 0$ such that (1.1) has a positive solution u_λ when $|\lambda| \leq \lambda^*$. Furthermore, for any sequence $(\lambda_n) \subseteq \mathbb{R}$ satisfying $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$, there exist a subsequence (λ_{n_k}) of (λ_n) and a sequence (u_{n_k}) of positive solutions to problem (1.1) with $\lambda = \lambda_{n_k}$ satisfying $u_{n_k} \rightarrow u_0$ in $C^1(\bar{\Omega})$ as $n_k \rightarrow \infty$, where u_0 is a mountain pass solution of (1.1) with $\lambda = 0$.
- (ii) If $g(x, 0) \not\equiv 0$ in Ω , then there exists $\lambda_* > 0$ such that (1.1) has another non-zero solution v_λ for $\lambda \in (-\lambda_*, 0) \cup (0, \lambda_*)$ such that $v_\lambda \rightarrow 0$ in X as $\lambda \rightarrow 0$. Moreover, if we assume (F₆) also holds, $v_\lambda > 0$ in $\bar{\Omega}$ for $\lambda \in (0, \lambda_*)$.

2. Main result

Throughout this paper, we assume that the assumptions (F_1) and (F_4) hold. For (1.1) with $\lambda = 0$, we define the Lagrangian functional $I_0 : X \rightarrow \mathbb{R}$ by $I_0(u) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + u^2) dx - \int_{\partial\Omega} F(x, u) d\sigma$ for $u \in X$, where σ is the surface measure on the boundary. Note that the trace map $X \hookrightarrow L^q(\partial\Omega)$ is compact for $q \in [1, 2_*)$ (see [4, Theorem 1.1]). Then, by (F_1) , I_0 is well-defined and $I_0 \in C^1(X, \mathbb{R})$ with $I'_0(u)\phi = \int_{\Omega} (\nabla u \cdot \nabla \phi + u\phi) dx - \int_{\partial\Omega} f(x, u)\phi d\sigma$ for $u, \phi \in X$ (see [19, Lemma 4.2]).

Lemma 2.1. *Assume that (F_1) – (F_4) hold. Then the following assertions hold:*

(1) *There exist $r_0 > 0$ and $\rho_0 > 0$ such that*

$$I_0(u) \geq \rho_0 \quad \text{when } \|u\|_X = r_0. \quad (2.1)$$

(2) *There exists $v_1 \in C^1(\bar{\Omega})$ such that $v_1 \geq 0$, $v_1(x_0) > 0$, $\|v_1\|_X > r_0$ and $I_0(v_1) < 0$, where $x_0 \in \partial\Omega$ is in the assumption (F_3) .*

(3) *I_0 satisfies the Palais–Smale condition, i.e., let $\{u_n\}$ be any sequence in X such that $|I_0(u_n)|$ is uniformly bounded and $I'_0(u_n) \rightarrow 0$ as $n \rightarrow \infty$, then $\{u_n\}$ has a convergent subsequence in X .*

Proof.

(1) From (F_4) , there exist $s_0 > 0$ and $\mu \in (0, \mu_1)$ such that $\frac{f(x,s)}{s} < \mu$ for $|s| < s_0$. Then $F(x, s) \leq \frac{\mu}{2}s^2$ for $|s| < s_0$. Combining this with (F_1) , there exist $\alpha_1 \in (2, 2_*)$ and $C > 0$ such that $F(x, s) \leq \frac{\mu}{2}s^2 + C|s|^{\alpha_1}$ for $s \in \mathbb{R}$. Since μ_1 is the principal eigenvalue of $-\Delta u + u = 0$ in Ω , $\frac{\partial u}{\partial \eta} = \mu u$ on $\partial\Omega$, $\mu_1 \int_{\partial\Omega} u^2 d\sigma \leq \|u\|_X^2$ for all $u \in X$. Then, for all $u \in X$,

$$I_0(u) \geq \frac{1}{2} \left(1 - \frac{\mu}{\mu_1}\right) \|u\|_X^2 - \tilde{C} \|u\|_X^{\alpha_1} = \|u\|_X^2 \left[\frac{1}{2} \left(1 - \frac{\mu}{\mu_1}\right) - \tilde{C} \|u\|_X^{\alpha_1-2} \right], \quad (2.2)$$

where \tilde{C} is the positive constant which comes from the continuous embedding $X \rightarrow L^{\alpha_1}(\partial\Omega)$. Thus, there exist $r_0 > 0$ and $\rho_0 > 0$ satisfying (2.1).

(2) Let $x_0 \in \partial\Omega$ and $\delta_0 > 0$ be in (F_3) . Choose a function $\psi \in C^1(\bar{\Omega})$ such that $\psi \geq 0$, $\psi(x_0) > 0$ and the support of ψ is in $B(x_0, \delta_0)$. Put $A = \{x \in \partial\Omega \cap B(x_0, \delta_0) : \psi(x) > \frac{\psi(x_0)}{\sqrt{2}}\}$ and $B = \{x \in \partial\Omega \cap B(x_0, \delta_0) : \psi(x) \leq \frac{\psi(x_0)}{\sqrt{2}}\}$. Then $A \cup B = \partial\Omega \cap B(x_0, \delta_0)$ and the surface measure of A is positive. By (F_3) , $\min \left\{ \frac{F(x,s)}{s^2} : x \in A \cup B \right\} \rightarrow \infty$ as $s \rightarrow \infty$. Consequently, $\min \left\{ \frac{F(x, t\psi(x))}{(t\psi(x))^2} : x \in A \right\} \rightarrow \infty$ as $t \rightarrow \infty$ and $\int_B F(x, t\psi(x)) d\sigma \geq 0$ for sufficiently large t . Then

$$\begin{aligned} I_0(t\psi) &= \frac{t^2}{2} \|\psi\|_X^2 - \int_{\partial\Omega \cap B(x_0, \delta_0)} F(x, t\psi) d\sigma = \frac{t^2}{2} \|\psi\|_X^2 \\ &\quad - t^2 \int_A \frac{F(x, t\psi)}{t^2 \psi^2} \psi^2 d\sigma - \int_B F(x, t\psi) d\sigma \\ &\leq \frac{t^2}{2} \left(\|\psi\|_X^2 - [\psi(x_0)]^2 \int_A \frac{F(x, t\psi)}{t^2 \psi^2} d\sigma \right) \rightarrow -\infty \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Thus, there exists $t_0 > 0$ such that $v_1 := t_0 \psi \in C^1(\bar{\Omega})$ satisfies $v_1 \geq 0$, $v_1(x_0) > 0$, $\|v_1\|_X > r_0$ and $I_0(v_1) < 0$.

(3) Let (u_n) be any sequence in X such that $I_0(u_n)$ is bounded and $I'_0(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Then $\alpha I_0(u_n) - I'_0(u_n)u_n = \frac{\alpha-2}{2}\|u_n\|_X^2 - \int_{\partial\Omega} (\alpha F(x, u_n) - u_n f(x, u_n)) d\sigma$. Since $|I_0(u_n)|$ and $\|I'_0(u_n)\|$ are bounded, by (F₂) and (F₄), for some constants M , $M' > 0$,

$$\begin{aligned} \frac{\alpha-2}{2}\|u_n\|_X^2 &= \alpha I_0(u_n) - I'_0(u_n)u_n + \int_{\partial\Omega} (\alpha F(x, u_n) - u_n f(x, u_n)) d\sigma \\ &\leq M + M\|u_n\|_X + M\|u_n\|_{L^\theta(\partial\Omega)}^\theta \\ &\leq M + M\|u_n\|_X + M'\|u_n\|_X^\theta. \end{aligned}$$

From the facts $\alpha > 2$ and $\theta \in [1, 2)$, it follows that $\|u_n\|_X$ is bounded. By standard arguments, I_0 satisfies the Palais–Smale condition (see Proposition 4.3 of [19]). \square

Put $\Gamma := \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = v_1\}$, where v_1 is the function in Lemma 2.1(2). Then $c_0 := \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I_0(\gamma(t)) > 0$ is a critical value of I_0 in view of the mountain-pass theorem of Ambrosetti and Rabinowitz [2]. Thus I_0 has a critical point $u_0 \in X \setminus \{0\}$ with $I_0(u_0) = c_0 > 0$. We call u_0 a *mountain pass solution* associated with I_0 . In general, a mountain pass solution is not necessarily unique, but an *a priori* estimate for all mountain pass solutions is given as follows.

Lemma 2.2. *Assume that (F₁)–(F₄) hold. Then there exists $M_1 > 0$ such that $\|u\|_{L^\infty(\Omega)} \leq M_1$ for any mountain pass solution u associated with I_0 . If, in addition, we assume (F₅) also holds, then there exist constants $\beta \in (0, 1)$ and $M_2 > 0$ such that $\|u\|_{C^{1,\beta}(\bar{\Omega})} \leq M_2$ for any mountain pass solution u associated with I_0 .*

Proof. Let u be any mountain pass solution associated with I_0 . Since $I'_0(u) = 0$ and $I_0(u) = c_0$,

$$\begin{aligned} \frac{\alpha-2}{2}\|u\|_X^2 &= \alpha I_0(u) - I'_0(u)u - \int_{\partial\Omega} (\alpha F(x, u) - u f(x, u)) d\sigma \\ &\leq \alpha c_0 + M\|u\|_X^\theta + M \end{aligned}$$

for some $M > 0$, and thus $\|u\|_X \leq M_0$ for some $M_0 > 0$. By Moser's iteration technique [29, 31], $\|u\|_{L^\infty(\Omega)} \leq M_1$ for some $M_1 > 0$. Assume that (F₅) also holds. Then, by Lieberman's regularity result (Theorem 2 of [16]), there exist constants $\beta \in (0, 1)$ and $M_2 > 0$ such that $\|u\|_{C^{1,\beta}(\bar{\Omega})} \leq M_2$ for any mountain pass solution u associated with I_0 . \square

Note that 0 is the unique constant solution to problem (1.1) with $\lambda = 0$. Next, we show that any non-constant solutions to problem (1.1) with $\lambda = 0$ are indeed positive solutions.

Lemma 2.3. *Assume that (F₁)–(F₅) hold. Let u be any non-constant solution to problem (1.1) with $\lambda = 0$. Then $u > 0$ in $\bar{\Omega}$.*

Proof. Let u be any non-constant solution to problem (1.1) with $\lambda = 0$. Assume, on the contrary, that there exists $x_1 \in \bar{\Omega}$ such that $u(x_1) \leq 0$. Since u is not a constant solution, by strong maximum principle, u cannot attain its non-positive minimum in Ω . Then there exists $x_2 \in \partial\Omega$ such that $u(x_2) < u(x)$ for all $x \in \Omega$ and $u(x_2) \leq 0$. By Hopf's lemma, $\frac{\partial u}{\partial \eta}(x_2) < 0$, which contradicts the fact that $\frac{\partial u}{\partial \eta}(x_2) = f(x, u(x_2)) = 0$. Then $u > 0$ in $\bar{\Omega}$. \square

Recall that M_1 is a constant in Lemma 2.2. Let \hat{g} be a continuous function on $\bar{\Omega} \times \mathbb{R}$ such that it coincides with g on $\bar{\Omega} \times [-2M_1, 2M_1]$ and has its support in $\bar{\Omega} \times [-3M_1, 3M_1]$. For $\lambda \in \mathbb{R}$, define $I_\lambda : X \rightarrow \mathbb{R}$ by, for $u \in X$,

$$\begin{aligned} I_\lambda(u) &:= \int_{\Omega} \left(\frac{1}{2} (|\nabla u|^2 + u^2) - \lambda \hat{G}(x, u) \right) dx - \int_{\partial\Omega} F(x, u) d\sigma \\ &= I_0(u) - \lambda \int_{\Omega} \hat{G}(x, u) dx, \end{aligned}$$

where $\hat{G} : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by $\hat{G}(x, s) := \int_0^s \hat{g}(x, \tau) d\tau$ for $(x, s) \in \bar{\Omega} \times \mathbb{R}$. Then I_λ is well defined and $I_\lambda \in C^1(X, \mathbb{R})$ with $I'_\lambda(u)\phi = \int_{\Omega} (\nabla u \cdot \nabla \phi + u\phi) dx - \int_{\partial\Omega} f(x, u)\phi d\sigma - \lambda \int_{\Omega} \hat{g}(x, u)\phi dx$ for $u, \phi \in X$. A critical point of I_λ is a (weak) solution of $-\Delta u + u = \lambda \hat{g}(x, u)$ in Ω and $\frac{\partial u}{\partial \eta} = f(x, u)$ on $\partial\Omega$.

Lemma 2.4. Assume that (F₁)–(F₄) hold. Then the following assertions hold:

(1) There exists $\lambda^0 > 0$ such that for all λ with $|\lambda| \leq \lambda^0$,

$$I_\lambda(u) \geq \frac{\rho_0}{2} \quad \text{when } \|u\|_X = r_0 \quad (2.3)$$

and $I_\lambda(v_1) < 0$, where r_0 , ρ_0 and v_1 are as in Lemma 2.1.

(2) For each $\lambda \in \mathbb{R}$, I_λ satisfies the Palais–Smale condition.

Proof.

(1) Since $\hat{G}(x, s)$ is bounded on $\bar{\Omega} \times \mathbb{R}$, we have

$$I_0(u) - |\lambda|C \leq I_\lambda(u) \leq I_0(u) + |\lambda|C \quad \text{for } u \in X, \quad (2.4)$$

where $C > 0$ is independent of λ and u . Then, by (2.2) and (2.4), there exists $\lambda^0 > 0$ such that for all λ with $|\lambda| \leq \lambda^0$, $I_\lambda(v_1) \leq I_0(v_1) + |\lambda|C < 0$ and $I_\lambda(u) \geq \rho - |\lambda|C \geq \frac{\rho}{2}$ for $\|u\|_X = r_0$.

(2) Since $\hat{g}(x, s)$ and $\hat{G}(x, s)$ are bounded by the same argument as in Lemma 2.1(3), one can show that for each $\lambda \in \mathbb{R}$, I_λ satisfies the Palais–Smale condition. \square

In view of the mountain pass theorem of Ambrosetti and Rabinowitz [2], by Lemma 2.4, we define a mountain pass value c_λ of I_λ by $c_\lambda := \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I_\lambda(\gamma(t)) > 0$. Then, by (2.4), $c_\lambda \rightarrow c_0$ as $\lambda \rightarrow 0$.

Lemma 2.5. Assume that (F₁)–(F₅) hold. Let (λ_n) be a sequence in \mathbb{R} such that $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$ and let (u_n) be a sequence in X such that u_n is a mountain pass solution associated with I_{λ_n} . Then there exists a subsequence (u_{n_k}) of (u_n) satisfying $u_{n_k} \rightarrow u_0$ in $C^1(\bar{\Omega})$ as $n_k \rightarrow \infty$, where u_0 is a mountain pass solution associated with I_0 .

Proof. Since $I_{\lambda_n}(u_n) = c_{\lambda_n}$ and $I'_{\lambda_n}(u_n) = 0$, using the same argument as in Lemma 2.2 with the boundedness of c_{λ_n} , the $C^{1,\beta}(\bar{\Omega})$ norm of u_n is uniformly bounded. By the compact embedding $C^{1,\beta}(\bar{\Omega}) \hookrightarrow C^1(\bar{\Omega})$, a subsequence of u_n converges to a limit u_0 in $C^1(\bar{\Omega})$. Then u_0 satisfies that $I_0(u_0) = c_0$ and $I'_0(u_0) = 0$. Consequently, u_0 is a mountain pass solution associated with I_0 . \square

In view of Lemmas 2.2, 2.3 and 2.5, we have the following lemma.

Lemma 2.6. *Assume that (F₁)–(F₅) hold. There exists $\lambda^* \in (0, \lambda^0)$ such that every mountain pass solution u associated with I_λ with $|\lambda| \leq \lambda^*$ satisfies $0 < u(x) \leq 2M_1$ in $\bar{\Omega}$, where λ^0 and M_1 are the constants in Lemma 2.4 and Lemma 2.2, respectively.*

Proof. First, we prove that every mountain pass solution u associated with I_λ for $|\lambda| > 0$ small enough satisfies $|u(x)| \leq 2M_1$ for all $x \in \bar{\Omega}$. Suppose, on the contrary, there exist sequences $(\lambda_n) \subseteq \mathbb{R}$ and $(u_n) \subseteq X$ such that $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$ and u_n is a mountain pass solution associated with I_{λ_n} satisfying $\|u_n\|_{L^\infty(\Omega)} > 2M_1$. By Lemma 2.5, a subsequence of (u_n) converges to a mountain pass solution u_0 associated with I_0 in $C^1(\bar{\Omega})$. Since $\|u_0\|_{L^\infty(\Omega)} \leq M_1$ by Lemma 2.2, it follows that $\|u_n\|_{L^\infty(\Omega)} \leq 2M_1$ for n large enough, which is a contradiction. Thus there exists $\lambda^1 > 0$ such that $\|u\|_{L^\infty(\Omega)} \leq 2M_1$ for any mountain pass solution u associated with I_λ with $|\lambda| < \lambda^1$. Similarly, one can prove that $u(x) > 0$ for all $x \in \bar{\Omega}$ in view of Lemmas 2.3 and 2.5. \square

Now, we give the proof of Theorem 1.2.

Proof of Theorem 1.2.

(i) Let u_λ be a mountain pass solution associated with I_λ with $|\lambda| \leq \lambda^*$, where λ^* is the constant in Lemma 2.6. Then $0 < u_\lambda(x) \leq 2M_1$ by Lemma 2.6. Thus $\hat{g}(x, u_\lambda) = g(x, u_\lambda)$, and u_λ is a positive solution to problem (1.1). Let (λ_n) be any sequence in \mathbb{R} such that $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Then, by Lemma 2.5, there exist a subsequence (λ_{n_k}) of (λ_n) and a sequence (u_{n_k}) of positive solutions to problem (1.1) with $\lambda = \lambda_{n_k}$ satisfying $u_{n_k} \rightarrow u_0$ in $C^1(\bar{\Omega})$ as $n_k \rightarrow \infty$, where u_0 is a mountain pass solution u_0 associated with I_0 .

(ii) We now assume that $g(x, 0) \not\equiv 0$ in Ω . Let \bar{B}_{r_0} be the closed ball centered at the origin with radius r_0 in X , where $r_0 > 0$ is the constant in Lemma 2.4. Then the minimum of I_λ in \bar{B}_{r_0} is achieved at an interior point v_λ of \bar{B}_{r_0} . Indeed, choose a sequence (v_n) in \bar{B}_{r_0} such that $I_\lambda(v_n)$ converges to the infimum of I_λ in \bar{B}_{r_0} and $v_n \rightharpoonup v_\lambda$ weakly in X for some $v_\lambda \in \bar{B}_{r_0}$. By the weakly lower semicontinuity of I_λ , we have $I_\lambda(v_\lambda) \leq \liminf_{n \rightarrow \infty} I_\lambda(v_n)$, which implies that v_λ is a minimum point in \bar{B}_{r_0} . Since $I_\lambda(0) = 0$, it follows that $I_\lambda(v_\lambda) \leq 0 < I_\lambda(u_\lambda)$, where u_λ is a mountain pass solution associated with I_λ . By (2.3), $\inf_{\|u\|_X=r_1} I_\lambda(u) \geq \frac{\rho}{2} > 0 = I_\lambda(0)$. Thus $\|v_\lambda\|_X < r_0$ and $v_\lambda \neq u_\lambda$. By (2.2) and (2.4), we may choose $r_0 > 0$ in Lemmas 2.1 and 2.4 so that $I_\lambda(u) \geq \frac{1}{4} \left(1 - \frac{\mu}{\mu_1}\right) \|u\|_X^2 - |\lambda|C$ for all u with $\|u\|_X \leq r_0$. Note that $C > 0$ is independent of λ and u . Consequently, $\|v_\lambda\|_X \rightarrow 0$ as $\lambda \rightarrow 0$, by the fact that $I_\lambda(v_\lambda) \leq 0$. By Moser's iteration technique [29, 31] as in Lemma 2.2, there exists $\lambda_* > 0$ such that $\|v_\lambda\|_{L^\infty(\Omega)} \leq 2M_1$ for $|\lambda| < \lambda_*$. Hence $\hat{g}(x, v_\lambda) = g(x, v_\lambda)$, and thus v_λ is a nonzero solution of (1.1) for $\lambda \in (-\lambda_*, 0) \cup (0, \lambda_*)$, since $g(x, 0) \not\equiv 0$.

Assume that (F₆) also holds. Then there exists $C > 0$ such that $g(x, s) - g(x, 0) \geq -Cs$ for $-2M_1 \leq s \leq 2M_1$ and $x \in \Omega$. Let $\lambda \in (0, \lambda_*)$ be fixed. Then $-\Delta v_\lambda + (1 + \lambda C)v_\lambda$

$= \lambda(g(x, v_\lambda) - g(x, 0) + C v_\lambda) + \lambda g(x, 0) \geq 0$. Since $v_\lambda \not\equiv 0$, by the same argument as in Lemma 2.3, it is concluded that $v_\lambda > 0$ in $\bar{\Omega}$ for $\lambda \in (0, \lambda_*)$. \square

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