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Connectivity of the Julia sets of singularly perturbed rational maps

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Abstract. We consider a family of rational functions which is given by

$$f_\lambda(z) = \frac{z^n(z^{2n} - \lambda^{n+1})}{z^{2n} - \lambda^{3n-1}},$$

where $n \geq 2$ and $\lambda \in \mathbb{C}^* - \{\lambda : \lambda^{2n-2} = 1\}$. When $\lambda \neq 0$ is small, f_λ can be seen as a perturbation of the unicritical polynomial $z \mapsto z^n$. It was known that in this case the Julia set $J(f_\lambda)$ of f_λ is a Cantor set of circles on which the dynamics of f_λ is not topologically conjugate to that of any McMullen maps. In this paper, we prove that this is the unique case such that $J(f_\lambda)$ is disconnected.

Keywords. Julia sets; connectivity; Herman ring; Cantor circles.

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1. Introduction

The connectivity of the Julia sets of the polynomials has been studied thoroughly. It was known that the Julia set of a polynomial is connected if and only if all the critical points have bounded orbits (see [2]). The completely invariance of the immediate super-attracting basin centered at infinity plays a key role since the Julia set is equal to the boundary of it. For rational maps, there exists no general criterion to determine when the Julia sets are connected or not. Hence it is natural to consider this problem for some special families of rational maps. For example, the family of rational maps which has exactly one free critical orbit.

In this paper, we are interested in the following family which was studied in [4]:

$$f_\lambda(z) = \frac{z^n(z^{2n} - \lambda^{n+1})}{z^{2n} - \lambda^{3n-1}}, \tag{1.1}$$

where $n \geq 2$ and $\lambda \in \Lambda := \mathbb{C}^* - \{\lambda : \lambda^{2n-2} = 1\}$. We would like to mention that if $n = 1$, $\lambda = 0$ or $\lambda^{2n-2} = 1$, then f_λ degenerates to the polynomial $z \mapsto z^n$. The motivation of the study of this family is to construct a one-dimensional family of rational maps, such that the Julia sets are Cantor set of circles (if $\lambda \neq 0$ is small enough) and the dynamics on the Julia sets are not topologically conjugate to that of any McMullen maps.

It was known that the Julia set $J(f_\lambda)$ of f_λ is either a quasicircle, a Cantor set of circles, a Sierpiński carpet or a degenerated Sierpiński carpet, provided one of the free critical points of f_λ escapes to the origin or to infinity (see Lemma 2.2). The aim of this paper is to study the connectivity of the Julia sets of this family, not only considering the escaping case but also including the non-escaping situation. Our main result is the following.

Theorem 1.1. *For any $n \geq 2$ and $\lambda \in \Lambda$, the Julia set of f_λ is connected if and only if it is not a Cantor set of circles.*

Shishikura studied the connectivity of the Julia sets of the rational maps by considering the weak repelling fixed points [10]. His result can be applied to the rational functions arising from Newton's method for polynomials. In [14], Yin studied the connectivity of the Julia set of the quadratic rational maps. On the other hand, the connectivity of the Julia sets of one-dimensional family of rational maps has been studied extensively. Qiao and Li studied the family of renormalization transformation functions and obtained the connectivity of the Julia sets for all real parameter [7] and this result has been generalized to all complex parameter by Yang and Zeng in [15]. The connectivity of the Julia set of the McMullen maps has been studied in [12] and [3].

By Lemma 2.2(b), the Julia set of f_λ is a Cantor set of circles if and only if there is a critical value which lies in the immediate super-attracting basin of the origin or the infinity but the corresponding critical point does not. Comparing with the conclusions in Lemma 2.2, one can obtain a comparison between the connectivity of the Julia sets of f_λ and the classical quadratic polynomial $P_c(z) = z^2 + c$ with $c \in \mathbb{C}$. On the one hand, when the free critical orbit is not attracted to the fixed super-attracting basin, the Julia sets of f_λ and P_c are both connected. On the other hand, if it is attracted, the differences of the connectivity of the Julia sets of f_λ and P_c are presented by Lemma 2.2.

2. Preliminaries

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map with degree $d \geq 2$. We use $f^{\circ n}$ to denote the n-th iterate of f, where $n \in \mathbb{N}$. The *Fatou set* of f is defined as

$$F(f) := \{z \in \hat{\mathbb{C}} : \{f^{\circ n}\}_{n \in \mathbb{N}} \text{ forms a normal family in some neighborhood of } z\}.$$

Equivalently, $F(f)$ is also the maximal open subset of $\hat{\mathbb{C}}$ on which $\{f^{\circ n}\}_{n \in \mathbb{N}}$ is equicontinuous under the chordal metric. The complement of $F(f)$ is called *Julia set* $J(f)$. Each connected component of Fatou set is called a *Fatou component*. A Fatou component U is called *p-periodic* if there is an integer $p \geq 1$ such that $f^{\circ p}(U) = U$. It was known that there were only five types of periodic Fatou components: attracting basin, super-attracting basin, parabolic basin, Siegel disk and Herman ring. Moreover, all the Fatou components were iterated to one of these five types of periodic Fatou components eventually. For more details on the dynamics of holomorphic functions, the reader can refer to [1,2,6].

Now we review the property of dynamical symmetry of f_λ which has been shown in [4]. In the rest of this paper, we fix the integer $n \geq 2$ in the formula of f_λ in (1.1). Denote $\omega_0 = \mathrm{e}^{\frac{\mathrm{i}\pi}{n}}$ and let ω be a complex number satisfying $\omega^{2n} = 1$. For $\lambda \in \Lambda$, define

$$\tau(z) := \lambda^2/z.$$

For any $A \subset \hat{\mathbb{C}}$ and $a \in \mathbb{C}$, we denote $aA := \{az : z \in A\}$.

Lemma 2.1. *Let A be a Fatou* (*or Julia*) *component of* f_λ. *Then*

(*a*) $f_\lambda^{\circ k}(\omega z) = \omega^{n^k} f_\lambda^{\circ k}(z)$ *for all* $k \geq 1$ *and all* $z \in \hat{\mathbb{C}}$. *In particular*, $f_\lambda(\omega_0^{2k} z) = f_\lambda(z)$ *and* $f_\lambda(\omega_0^{2k-1} z) = -f_\lambda(z)$ *for all* $1 \leq k \leq n$ *and all* $z \in \hat{\mathbb{C}}$;
(*b*) $\tau \circ f_\lambda(z) = f_\lambda \circ \tau(z)$ *for all* $z \in \hat{\mathbb{C}}$;
(*c*) *For* $i \in \mathbb{Z}$, *both* $\omega_0^i A$ *and* $\tau(A)$ *are Fatou* (*or Julia*) *components of* f_λ;
(*d*) *Either the Fatou* (*or Julia*) *component* A *of* f_λ *surrounds the origin and satisfies* $\omega_0^i A = A$ *for all* $i \in \mathbb{Z}$, *or* A *does not surround the origin and there are* $2n$ *such Fatou* (*or Julia*) *components* $\omega_0^i A$ $(1 \leq i \leq 2n)$ *such that* $\omega_0^i A \cap \omega_0^j A = \emptyset$ *for all* $i \neq j \pmod{2n}$.

The above symmetry is very useful in the proof of the connectivity of the Julia set of f_λ. First of all, a direct observation shows that the points 0 and ∞ are two super-attracting fixed points of f_λ. We denote by B_0 and B_∞ the immediate attracting basins of 0 and ∞ respectively. Then the above symmetry shows that $\omega_0^i B_0 = B_0$, $\omega_0^i B_\infty = B_\infty$ for all $i \in \mathbb{Z}$, $\tau(B_0) = B_\infty$ and $\tau(B_\infty) = B_0$.

Now let us locate the positions of the critical orbits of f_λ which determine the dynamics of f_λ essentially. Since the degree of f_λ is $3n$, f_λ has $6n - 2$ critical points (counted with multiplicity). Since the local degrees of 0 and ∞ are both n, f_λ leaves $4n$ free critical points. The forward orbits of 0 and ∞ are trivial since they are fixed by f_λ. The dynamical symmetry in Lemma 2.1 implies that the remaining $4n$ critical points have the following form:

$$\mathrm{Crit}(f_\lambda) = \{\omega_0^j c_\lambda, \omega_0^j \lambda^2/c_\lambda : 0 \leq j \leq 2n-1\}, \tag{2.1}$$

where c_λ is a free critical point of f_λ. Therefore, the points in $\mathrm{Crit}(f_\lambda)$ behave symmetrically by the iterates of f_λ. This means that f_λ has exactly one free critical orbit essentially.

The following lemma gives a dynamical classification in the case that the free critical orbits are attracted by 0 and ∞.

Lemma 2.2 (*The escape quartation* [4]). *Suppose that the orbit of one free critical point* c_λ *of* f_λ *is attracted by* ∞ (*resp.* 0). *Then*

(*a*) *If* $c_\lambda \in B_\infty$ (*resp.* B_0), *then* $J(f_\lambda)$ *is a quasicircle*;
(*b*) *If* $f_\lambda(c_\lambda) \in B_\infty$ (*resp.* B_0) *but* $c_\lambda \notin B_\infty$ (*resp.* B_0), *then* $J(f_\lambda)$ *is a Cantor set of circles*;
(*c*) *If* $f_\lambda^{\circ k}(c_\lambda) \in B_\infty$ (*resp.* B_0) *for* $k \geq 2$ *and* $f_\lambda^j(c_\lambda) \notin B_\infty$ (*resp.* B_0) *for* $0 \leq j < k$, *and further*,

($c1$) *If* $\partial B_0 \cap \partial B_\infty = \emptyset$, *then* $J(f_\lambda)$ *is a Sierpiński carpet*;
($c2$) *If* $\partial B_0 \cap \partial B_\infty \neq \emptyset$, *then* $J(f_\lambda)$ *is a degenerated Sierpiński carpet.*

A set is called a *Cantor set of circles* (or *Cantor circles* in short) if it is homeomorphic to $\mathcal{C} \times \mathbb{S}^1$, where $\mathcal{C}$ is the Cantor middle third set and $\mathbb{S}^1$ is the unit circle. The *Sierpiński carpet* is defined to be a connected, locally connected, nowhere dense compact set which has the property that any two complementary domains are bounded by disjoint Jordan curves [11]. In [4], a compact set in $\mathbb{C}$ is called a *degenerated Sierpiński carpet* if it satisfies all the conditions of the Sierpiński carpet except it allows that the intersection of the boundaries of complementary domains can be non-empty.

According to Lemma 2.2, if the free critical orbits are attracted by 0 and ∞, then case (b) is the unique case such that $J(f_\lambda)$ is disconnected. In this case, the Julia set is a Cantor set of circles and there is a critical value which lies in B_∞ or B_0 but the corresponding critical point does not. The Julia sets which are Cantor set of circles was first found in the McMullen [5]. For a comprehensive study of the rational maps whose Julia sets are Cantor circles, see [8] and [9].

3. Proof of Theorem 1.1

If one wants to prove that the Julia set is connected, it is equivalent to proving that all the Fatou components of f_λ are simply connected. According to [1, §7.5], there are only two types of periodic Fatou components which are not simply connected. One is the Herman ring, and the other is the infinitely connected attracting or parabolic basin.

Note that in order to prove Theorem 1.1, the case that the free critical orbits are attracted by 0 and ∞ has been discussed. In Lemma 2.2, the Julia set $J(f_\lambda)$ of f_λ is always connected, if we exclude the case that there is a critical point c_λ, which satisfies $f_\lambda(c_\lambda) \in B_\infty$ (resp. B_0) but $c_\lambda \notin B_\infty$ (resp. B_0). Hence, we only need to prove that if the free critical orbits are not attracted by 0 and ∞, then $J(f_\lambda)$ is always connected.

3.1 *No infinitely connected Fatou component*

In this subsection, we will prove the following result.

PROPOSITION 3.1

The rational map f_λ *has no infinitely connected attracting or parabolic basin.*

Proof. Suppose f_λ has a cycle of periodic attracting or parabolic basins other than B_0 and B_∞ whose period is $p \geq 1$. According to [6, §8-10], there is a component U of this cycle which contains a free critical point c_λ. Let $k \geq 1$ denote the maximal number of critical points contained in any component of this cycle of periodic attracting or parabolic basins. The arguments will be divided into three cases:

Case I. Suppose $k = 1$. By using the Riemann–Hurwitz formula, a standard argument shows that U is simply connected. In fact, according to the local dynamics of the attracting and parabolic periodic points, there exists a simply connected neighborhood $V_0 \subset U$ of the periodic point z_0 (for the parabolic case, $V_0 \subset U$ is chosen such that $\partial V_0 \cap \partial U = \{z_0\}$)

such that $f_\lambda^{\circ p}(V_0) \subset V_0$. Then we consider the preimage V_1 of V_0 under f_λ which is contained in the p-periodic attracting or parabolic cycle. Since $k = 1$, the set V_1 contains at most one critical point of f_λ. If V_1 does not contain any critical point of f_λ, then it is easy to see that by the Riemann–Hurwitz formula, the degree of the branched covering from V_1 to V_0 is equal to 1. Hence V_1 is simply connected. If V_1 contains one critical point, again by the Riemann–Hurwitz formula, the degree of the branched covering from V_1 to V_0 is equal to 2. Hence V_1 is also simply connected. Repeating this process, one can obtain that $U = \cup_{k>0} f_\lambda^{-pk}(V_0)$ and U is also simply connected.

Case II. Suppose $k = 2$. This means that one component U of the p-periodic attracting or parabolic basin contains two critical points c_1 and c_2 of f_λ. We first claim that $c_2 = \omega_0^{i_0}\tau(c_1)$ for some $i_0 \in \mathbb{Z}$. Otherwise, by (2.1), we must have $c_2 = \omega_0^{j_0} c_1$ for some $j_0 \in \mathbb{Z}$. According to Lemma 2.1, the $2n$ critical points $\{\omega_0^j c_\lambda : 0 \le j \le 2n-1\}$ are also contained in U, which is a contradiction.

We then claim that the degree of the restriction of f_λ on the component U is 3. On the one hand, if the degree is greater than 3, then Lemma 2.1 guarantees that for any $z \in f_\lambda(U)$, the preimage of $f_\lambda(z)$ will be greater that $3n$ (counted with multiplicity). This contradicts with the degree of f_λ. On the other hand, let z_0 be the periodic point in U. We can find three preimages of $f_\lambda(z_0)$ in U as follows (for the parabolic case, one can find three preimages of $f_\lambda(z_0) \in f_\lambda(\partial U)$ in ∂U in a similar way). Note that $\cup_{j=0}^{2n-1}\omega_0^j U$ contains all the $4n$ free critical points. Also, Lemma 2.1 implies that $\omega_0^j\tau(U)$ $(0 \le j \le 2n-1)$ is also a Fatou component which contains two free critical points, $\omega_0^j\tau(c_1)$ and $\omega_0^j\tau(c_2) = \omega_0^{j-i_0}c_1$. This means that $U = \omega_0^{i_0}\tau(U)$. We also have $z_0 = \omega_0^{i_0}\tau(z_0)$ since points in the attracting or parabolic basin can be only attracted by one attracting or parabolic point. Let $z_1 \in U$ be a preimage of $f_\lambda(z_0)$ and $z_2 = \omega_0^{i_0}\tau(z_1)$. We claim that $f_\lambda(z_2) = f_\lambda(z_0)$. In fact,

$$
\begin{aligned}
f_\lambda(z_2) &= \omega_0^{ni_0}\tau(f_\lambda(z_1)) = \omega_0^{ni_0}\tau(f_\lambda(z_0)) \\
&= \omega_0^{ni_0}\tau(f_\lambda(\omega_0^{i_0}\tau(z_0))) = \omega_0^{ni_0}\tau(\omega_0^{ni_0}\tau(f_\lambda(z_0))) \\
&= \omega_0^{ni_0}\frac{\lambda^2}{\omega_0^{ni_0}\tau(f_\lambda(z_0))} = \tau^2(f_\lambda(z_0)) = f_\lambda(z_0).
\end{aligned}
$$

Suppose $z_1 \ne z_0$. We claim that $z_1 \ne z_2$. For otherwise, if $z_1 = z_2$, it is easy to see that $z_1 = \pm z_0$. Combining with the Lemma 2.1, we can exclude the case $z_1 = -z_0$, since this implies that U surrounds the origin and satisfies $\omega_0^i U = U$ for any $i \in \mathbb{Z}$, which is a contradiction. Hence, we get $z_1 \ne z_2$ and it is easy to see that the points z_0, z_1 and z_2 are all distinct. The fact that $U = \omega_0^{i_0}\tau(U)$ implies $z_2 \in U$. This means the degree of f_λ on U is 3. Now, suppose $z_1 = z_0$ and then $z_2 = z_0$. This implies that the free critical point equals to z_0 and its local degree is at least 3, thus equals to 3. In both cases, we get the degree of f_λ on U is 3.

Similar to Case I, by using the Riemann–Hurwitz formula, one can use a standard process to show that the periodic attracting or parabolic basin is simply connected. Note that in this process, one can take V_0 such that it satisfies $V_0 = \omega_0^{i_0}\tau(V_0)$. This guarantees that the two critical points c_1 and $c_2 = \omega_0^{j_0}c_1$ of f_λ are contained in V_1 at the same time.

Case III. Suppose $2 < k \le 4n$. Then Lemma 2.1 implies that there is a component of the p-periodic attracting or parabolic basin U which satisfies $\omega_0^i U = U$ for any $i \in \mathbb{Z}$ and contains exactly $2n$ or $4n$ critical points. These critical points have the form $\{\omega_0^j c_\lambda : 0 \leq j \leq 2n-1\}$ for some free critical point c_λ. Let $\gamma \subset U$ be a Jordan curve surrounding the origin, which satisfies $\omega_0^i \gamma = \gamma$ for any $i \in \mathbb{Z}$. Then Lemma 2.1 guarantees that $f_\lambda^{\circ np}(\gamma)$ always surrounds the origin for any positive integer n. Note that U is disjoint with the immediate attracting basins of 0 and ∞. This contradicts the fact that γ will converge uniformly to $z_0 (\neq 0, \infty)$ under the iteration of $f_\lambda^{\circ p}$.

The proof is finished if we notice that all the components of the p-periodic attracting or parabolic basin have the same connectivity. □

3.2 *No Herman rings*

Let $A \subset \mathbb{C}$ be an annulus. Recall that the *core curve* of A is defined as $\psi^{-1}(\sqrt{r})$, where $\psi : A \to \mathbb{A}_r := \{z \in \mathbb{C} : 0 < r < |z| < 1\}$ is a conformal isomorphism and denoted by $\partial_+ A$ and $\partial_- A$, the outer and inner boundaries of A, respectively. For any $z \in \hat{\mathbb{C}}$, denoted by $O_f(z) := \{f^{\circ n}(z) : z \in \mathbb{N}\}$, the forward orbit of z under f. We call the two forward orbits $O_f(z_1)$ and $O_f(z_2)$ as *disjoint* if the intersection of them are empty. We need the following lemma which has been proved in [13, Corollary 2.2].

Lemma 3.2. *Suppose that a rational map f has $p \geq 1$ fixed Herman rings $A_0, \ldots, A_{p-1}$. Denote by $\gamma_i \subset A_i$ the core curve whose union divides $\hat{\mathbb{C}}$ into $p+1$ connected components $V_0, V_1, \ldots, V_p$, where $0 \leq i \leq p-1$. Then f has at least $p+1$ disjoint infinite critical orbits $O_f(c_i)$ in $J(f)$ such that $O_f(c_i) \subseteq V_i \cap J(f)$, where c_i, $0 \leq i \leq p$ is the critical point of f.*

Now we prove as follows.

PROPOSITION 3.3

The rational map f_λ has no Herman rings.

Proof. Suppose that f_λ has a p'-periodic Herman ring, where $p' \geq 1$. Let U_0 be a component of the cycle of these Herman rings. Since f_λ has the symmetric properties in Lemma 2.1, we consider a new rational map obtained by semiconjugacy. Specifically, let $\varphi(z) = z^{2n}$ and define

$$g_\lambda(z) = z^n \left(\frac{z - \lambda^{n+1}}{z - \lambda^{3n-1}} \right)^{2n}.$$

Then φ is a semiconjugacy between f_λ and g_λ, i.e. $\varphi \circ f_\lambda = g_\lambda \circ \varphi$. Hence it is easy to verify that g_λ has also a cycle of Herman rings. Indeed, by Lemma 2.1(d) the component U_0 can not surround the origin since f_λ is injective in U_0. Again by Lemma 2.1(d), there are $2n$ Fatou components $\omega_0^i U_0$ $(1 \leq i \leq 2n)$ satisfying $\omega_0^i U \cap \omega_0^j U = \emptyset$ for any $i \neq j$ (mod $2n$). Hence the restriction of φ on U_0 is injective and then $\varphi(U_0)$ is a periodic Herman ring of g_λ. We assume that the period of $\varphi(U_0)$ is p (note that p is not necessarily equal to p' but p is a divisor of p'). We hope to obtain a contradiction by using Lemma 3.2.

Let $W_0 := \varphi(U_0)$ and $W_i = g_\lambda^{\circ i}(W_0)$, where $0 < i \leq p-1$. In particular, $g_\lambda(W_{p-1}) = W_0$. Let $\hat{\tau}(z) = \lambda^{4n}/z$. By Lemma 2.1, we have

$$\hat{\tau} \circ g_\lambda = g_\lambda \circ \hat{\tau}. \tag{3.1}$$

Since the degree of g_λ is $3n$, g_λ has $6n-2$ critical points (counted with multiplicity). Note that the local degrees of 0 and ∞ are both n. The local degrees of the zero at λ^{n+1} and the pole at λ^{3n-1} of g_λ are both $2n$. Hence, there leaves only two free critical points, say a_λ and $\hat{\tau}(a_\lambda)$ by (3.1), and the local degrees of them are both 2.

Note that for any $0 \le i \le p-1$, W_i is bounded and does not surround the origin. Now we consider the iteration $g_\lambda^{\circ p}$. Note that the free critical points of $g_\lambda^{\circ p}$ are

$$\left(\bigcup_{i=0}^{p-1} g_\lambda^{-i}(a_\lambda)\right) \cup \left(\bigcup_{i=0}^{p-1} g_\lambda^{-i}(\lambda^{4n}/a_\lambda)\right).$$

Hence there are at most $2p$ disjoint critical orbits of $g_\lambda^{\circ p}$, which have the following form:

$$\{O_{g_\lambda^{\circ p}}(c_0), O_{g_\lambda^{\circ p}}(\hat{\tau}(c_0)), \ldots, O_{g_\lambda^{\circ p}}(c_{p-1}), O_{g_\lambda^{\circ p}}(\hat{\tau}(c_{p-1}))\}.$$

Suppose the collection of core curves $\{\gamma_0, \gamma_1, \ldots, \gamma_{p-1}\}$ of the p-periodic Herman rings of g_λ divides $\hat{\mathbb{C}}$ into $p+1$ connected components $V_0, V_1, \ldots, V_p$. We divide our arguments into two cases:

Case I. Suppose that $\hat{\tau}(W_0) = W_0$. This means that $\hat{\tau}(W_i) = W_i$ for all $0 \le i \le p-1$ by (3.1). We first claim that $\hat{\tau}(\partial_+ W_i) = \partial_+ W_i$. In fact, since W_i is bounded and does not surround the origin, it follows that the points in $\overline{W_i}$ with the largest and smallest modulus are contained in $\partial_+ W_i$. By noticing that $\hat{\tau}$ maps the point of $\overline{W_i}$ with the largest modulus to the smallest one, we get $\hat{\tau}(\partial_+ W_i) \ne \partial_- W_i$ and hence $\hat{\tau}(\partial_+ W_i) = \partial_+ W_i$. This implies that $O_{g_\lambda^{\circ p}}(c_i)$ and $O_{g_\lambda^{\circ p}}(\hat{\tau}(c_i))$ always belong to the same component V_i. According to Lemma 3.2, $g_\lambda^{\circ p}$ has at least $p+1$ disjoint infinite critical orbits $O_{g_\lambda^{\circ p}}(c_i)$ in $J(g_\lambda^{\circ p})$ such that $O_{g_\lambda^{\circ p}}(c_i) \subseteq V_i \cap J(g_\lambda^{\circ p})$, while the $2p$ critical orbits of $g_\lambda^{\circ p}$ can only get into p of $p+1$ components of the collection $\{V_0, V_1, \cdots, V_p\}$, which is a contradiction.

Case II. Suppose that $\hat{\tau}(W_0) \ne W_0$. This means that $\hat{\tau}(W_i) \ne W_i$ for any $0 \le i \le p-1$ by (3.1). Then there are $2p$ disjoint fixed Herman rings of $g_\lambda^{\circ p}$. But there are only at most $2p$ disjoint critical orbits, which also contradicts with Lemma 3.2.
This completes the proof that g_λ and hence f_λ has no Herman rings. □

Proof of Theorem 1.1. If the free critical orbits of f_λ are attracted by 0 and ∞, then the Julia set of f_λ is connected if it is not a Cantor set of circles by Lemma 2.2. Suppose that the free critical orbits are not attracted by 0 and ∞. By Propositions 3.1 and 3.3, it follows that each of the periodic Fatou components of f_λ is simply connected. Since all the symmetric Fatou components have the same connectivity and all the other preimages of these periodic Fatou components do not contain any critical points, this means that they are all simply connected. This completes the proof of Theorem 1.1. □

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