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THE RAO–REITER CRITERION FOR THE AMENABILITY OF HOMOGENEOUS SPACES

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Abstract: We prove that a homogeneous space G/H, with G a locally compact group and H a closed subgroup of G, is amenable in the sense of Eymard–Greenleaf if and only if the quasiregular action π_Φ of G on the unit sphere of the Orlicz space $L^\Phi(G/H)$ for some N-function $\Phi \in \Delta_2$ satisfies the Rao–Reiter condition (P_Φ).

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1. Introduction

In this article, we assume that all topological groups are separated.

Let G be a locally compact group and let H be a closed subgroup of G. The homogeneous space G/H is called *amenable* (in the sense of Eymard–Greenleaf [1, 2]) if there exists a G-invariant mean on $L^\infty(G/H)$ or, equivalently, the pair (G, H) possesses the *fixed point property*: For every action of G by continuous affine transformations on a nonempty convex compact subset Q of a locally convex space W such that there is a fixed point for H in Q, there is also a fixed point for G in Q.

It follows from the definition (see [1]) that every homogeneous space of an amenable group is amenable and if H is an amenable subgroup of G then G/H is amenable if and only if G is amenable. This means that the most interesting case of the above definition is when G and $H \leq G$ are nonamenable but G/H is amenable. Examples of this situation are given by the homogeneous spaces $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ (a homogeneous space with positive invariant measure of finite volume) and $SL(2,\mathbb{R})/H$, where H is the second commutant of $SL(2,\mathbb{Z})$ (a homogeneous space of infinite volume).

Let V be a normed space of functions $f : G \to \mathbb{R}$ ($f : G \to \mathbb{C}$) such that if $f \in V$ then

$$\lambda_G(g)f(x) = f(g^{-1}x), \quad x \in G,$$

for all $g \in G$ lies in V and $\|\lambda_G(g)f\|_V = \|f\|_V$. Then $\lambda_G : G \to B(V)$ is called the *left regular representation* of G in V.

In [3] (see also [4, Theorem 8.3.2]), Stegeman proved that, for a locally compact group G, the following conditions (P_p) (called *Reiter's conditions*) are equivalent for all $p \geq 1$:

(P_p) for every compact set $F \subset G$ and every $\varepsilon > 0$, there exists $f \in L^p(G)$ with $f \geq 0$ and $\|f\|_{L^p(G)} = 1$ such that $\|\lambda_G(s)f - f\|_p < \varepsilon$ for all $s \in F$.

Here $\|\cdot\|_p$ stands for the L^p-norm and integration is carried out with respect to a left-invariant Haar measure on G (such a measure is defined up to a constant factor).

If G is a locally compact group, H is a closed subgroup of G, and $\pi : G \to G/H$ is the natural projection, then G acts on G/H continuously by the rule $g\pi(x) = \pi(gx)$, $g \in G$, $x \in G$. It is well known that the homogeneous space G/H admits the so-called "quasi-G-invariant measure" connected with chosen left-invariant Haar measures on G and H.

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In [1], Eymard defined a quasiregular representation π_p of a locally compact group G on $L^p(G/H)$ and established the equivalence of the amenability of a homogeneous space G/H, where G is a locally compact group and H is a closed subgroup of G, to all conditions of the form

(P_p) (respectively, (P_p^*)) for every compact (respectively, finite) set $F \subset G$ and every $\varepsilon > 0$, there exists $f \in L^p(G/H)$ with $f \geq 0$ and $\|f\|_p = 1$ such that $\|\pi_p(s)f - f\|_p < \varepsilon$ for all $s \in F$.

In [5, Proposition 2, pp. 387–389], Rao proved that a locally compact group G is amenable if and only if, for an N-function $\Phi \in \Delta_2$ and the corresponding Orlicz space $L^\Phi(G)$ with the gauge norm

$$\|f\|_{(\Phi)} = \inf\left\{k > 0 \, : \, \int_G \Phi\left(\frac{f(x)}{k}\right) d\mu_G(x) \leq 1\right\},$$

G satisfies the property

(P_Φ) for every compact set $F \subset G$ and every $\varepsilon > 0$, there exists $f \in L^\Phi(G)$ with $f \geq 0$ and $\|f\|_{(\Phi)} = 1$ such that $\|\lambda_G(s)f - f\|_{(\Phi)} < \varepsilon$ for all $s \in F$.

In [6], the author established the equivalence of the amenability of a closed subgroup H of a second-countable locally compact group G and the fulfillment of condition (P_Φ) for its left regular representation on $L^\Phi(G)$.

In this article, given a locally compact group G and a closed subgroup H of G, we introduce some "quasiregular" action π_Φ of G on the unit sphere $S_1(L^\Phi(G/H))$ of the Orlicz space $L^\Phi(G/H)$, where Φ is an N-function satisfying the Δ_2-condition, and prove the equivalence of the amenability of G/H and the Rao–Reiter conditions (P_Φ) and (P_Φ^*) for homogeneous spaces:

(P_Φ) $((P_\Phi^*))$ for every compact (finite) set $F \subset G$ and every $\varepsilon > 0$, there exists $f \in L^\Phi(G/H)$ with $f \geq 0$ and $\|f\|_{(\Phi)} = 1$ such that $\|\pi_\Phi(s)f - f\|_{(\Phi)} < \varepsilon$ for all $s \in F$.

The paper is organized as follows: In Section 2, we recall some basic notions about N-functions and Orlicz spaces. In Section 3, we provide some necessary information about integration on locally compact groups and homogeneous spaces and define the quasiregular action π_Φ. In Section 4, we prove our main result on the equivalence of amenability and conditions (P_Φ) and (P_Φ^*).

2. *N*-Functions and Orlicz Spaces

Definition. A nonnegative function $\Phi : \mathbb{R} \to \mathbb{R}$ is called an *N-function* if

(i) Φ is even and convex;

(ii) $\Phi(x) = 0 \Longleftrightarrow x = 0$;

(iii) $\lim_{x\to 0} \frac{\Phi(x)}{x} = 0$; $\quad \lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$.

An N-function Φ has left and right derivatives (which can differ only on an at most countable set; for instance, see [7, Theorem 1, p. 7]). The left derivative φ of Φ is left continuous, nondecreasing on $(0, \infty)$, and such that $0 < \varphi(t) < \infty$ for $t > 0$, $\varphi(0) = 0$, $\lim_{t\to\infty} \varphi(t) = \infty$. The function

$$\psi(s) = \inf\{t > 0 : \varphi(t) > s\}, \quad s > 0,$$

is called the *left inverse* of φ.

The functions Φ and Ψ defined by

$$\Phi(x) = \int_0^{|x|} \varphi(t)dt, \quad \Psi(x) = \int_0^{|x|} \psi(t)dt$$

are called *complementary N-functions*.

The N-function Ψ complementary to an N-function Φ can also be expressed as

$$\Psi(y) = \sup\{x|y| - \Phi(x) : x \geq 0\}, \quad y \in \mathbb{R}.$$

Definition. An N-function Φ is said to satisfy the Δ_2-*condition*, which is written as $\Phi \in \Delta_2$, if there exists a constant $K > 2$ such that $\Phi(2x) \le K\Phi(x)$ for all $x \ge 0$.

Henceforth, let Φ be an N-function and let (Ω, Σ, μ) be a measure space. Given a measurable function $f : \Omega \to \mathbb{R}$, put

$$\rho_\Phi(f) = \int\limits_\Omega \Phi(f) d\mu.$$

Definition. The vector space

$$L^\Phi(\Omega) = L^\Phi(\Omega, \Sigma, \mu) = \{f : \rho_\Phi(af) < \infty \text{ for } \textit{some } \ a > 0\}$$

is called an *Orlicz space* on (Ω, Σ, μ).

Let Ψ be the complementary N-function to Φ. As usual, we were identify two functions equal outside a negligible set. The functional $\|\cdot\|_\Phi$ (called the *Orlicz norm*), defined for $f \in L^\Phi(\Omega)$ as

$$\|f\|_\Phi = \sup\left\{\left|\int\limits_\Omega fg\, d\mu\right| : \rho_\Psi(g) \le 1\right\},$$

is a seminorm on $L^\Phi(\Omega)$. It becomes a norm if μ possesses the *finite subset property* (see [7, p. 59]): If $A \in \Sigma$ and $\mu(A) > 0$ then there is $B \in \Sigma$, $B \subset A$, such that $0 < \mu(B) < \infty$.

The *gauge norm* (or *Luxemburg norm*) of $f \in L^\Phi(\Omega)$ is defined by the formula

$$\|f\|_{(\Phi)} = \inf\left\{k > 0 : \rho_\Phi\left(\frac{f}{k}\right) \le 1\right\}.$$

This is a norm without any constraint on μ (see [7, Theorem 3, p. 54]).

Suppose that μ possesses the finite subset property. As was shown in [8, Chapter 10], a left-invariant Haar measure on a locally compact group has this property.

It is well known that the Orlicz and gauge norms are equivalent; namely (see, for example, [7, pp. 61–62]): $\|f\|_{(\Phi)} \le \|f\|_\Phi \le 2\|f\|_{(\Phi)}$.

We will need the following version of Hölder's inequality for Orlicz spaces [7, p. 62]:

Hölder's Inequality. *If Φ and Ψ are two complementary N-functions, $f \in L^\Phi$ and $g \in L^\Psi$, then $fg \in L^1$ and $\|fg\|_1 \le \|f\|_{(\Phi)}\|g\|_\Psi$ ($\|fg\|_1 \le \|f\|_\Phi\|g\|_{(\Psi)}$).*

3. The Quasiregular Action π_Φ

Recall some basic definitions and facts of the theory of integration on locally compact groups.

Let G be a locally compact group and let H be a closed subgroup of G. Denote by μ_G and μ_H left-invariant Haar measures on G and H respectively and by π, the projection $G \to G/H$.

The group G acts on the homogeneous space G/H by the rule:

$$s\bar{x} := \overline{sx}, \text{ where } s \in G, \quad \bar{x} = xH \in G/H.$$

Denote by Δ_G the modular function of a locally compact group G, i.e. the continuous function such that

$$\int\limits_G f(xs)\, dx = \frac{1}{\Delta_G(s)} \int\limits_G f(x)\, dx$$

for all $f \in L^1(G)$ and $s \in G$.

Given f and $u \in G/H$, take an arbitrary representative x of the coset u and consider the function $\alpha : y \mapsto f(xy)$ on H. If α is integrable over H then the left invariance of μ_H implies that $\int_H f(xy)\, d\mu_H(y)$ is independent of the choice of x with $\pi(x) = u$.

It is known that the homogeneous space G/H admits a *quasi-G-invariant* measure $\mu_{G/H}$ on H which is unique up to equivalence. Here the "quasi-G-invariance" means that all left translates of $\mu_{G/H}$ by the elements of G are equivalent to $\mu_{G/H}$. The measure $\mu_{G/H}$ can be described as follows (see [9, Chapter VII, 2.5] or [1]):

(a) There exists a positive continuous function ρ on G such that $\rho(xy) = \frac{\Delta_H(y)}{\Delta_G(y)}\rho(x)$ for all $x \in G$ and $y \in H$.

Put $\mu_{G/H} = (\rho\mu_G)/\mu_H$ (see [9, Chapter VII, 2.2, Definition 1]).

(b) If $f \in L^1(G, \rho\mu_G)$ then the set of $\overline{x} = \pi(x) \in G/H$ for which $y \mapsto f(xy)$ is not μ_H-integrable is $\mu_{G/H}$-negligible, the function $\overline{x} = \pi(x) \mapsto \int_H f(xy)\,d\mu_H(y)$ is $\mu_{G/H}$-integrable, and

$$\int\limits_G f(x)\rho(x)\,d\mu_G(x) = \int\limits_{G/H} d\mu_{G/H}(\bar{x}) \int\limits_H f(xy)\,d\mu_H(y).$$

(c) There exists a nonnegative continuous function h on G with $\int_H h(xy)\,dy = 1$ for all $x \in G$ such that a function w on G/H is $\mu_{G/H}$-measurable ($\mu_{G/H}$-integrable) if and only if $h(w\circ\pi)$ is $\rho\mu_G$-measurable ($\rho\mu_G$-integrable). If $w \in L^1(G/H) := L^1(G/H, \mu_{G/H})$ then

$$\int\limits_{G/H} w(\bar{x})\,d\mu_{G/H}(\bar{x}) = \int\limits_G h(x)w(\pi(x))\rho(x)\,d\mu_G(x).$$

(d) Let χ be the function on $G \times G/H$ well-defined by the formula

$$\chi(s, \bar{x}) = \frac{\rho(sx)}{\rho(x)} \quad s \in G,\ \bar{x} \in G/H,\ x \in \bar{x}.$$

For $f \in L^1(G/H)$, we then have

$$\int\limits_{G/H} f(s\bar{x})\,d\mu_G(\bar{x}) = \int\limits_{G/H} \chi(s^{-1}, x)f(\bar{x})\,d\mu_G(\bar{x}). \tag{1}$$

Considering the Orlicz space $L^\Phi(G/H)$, put

$$S_1(L^\Phi(G/H)) := \{f \in L^\Phi(G/H) \mid \|f\|_{(\Phi)} = 1\}.$$

Suppose that $\Phi \in \Delta_2$. By analogy with [1], for $f \in S_1(L^\Phi(G/H))$ and $\bar{x} \in G/H$, we put

$$[\pi_\Phi(s)f](\bar{x}) = \Phi^{-1}\{\chi(s^{-1}, \bar{x})\Phi(f(s^{-1}\bar{x}))\}.$$

Here Φ^{-1} stands for the preimage with the same sign as $f(s^{-1}\bar{x})$.

In the case of $\Phi(t) = \frac{|t|^p}{p}$, π_Φ extends to Eymard's quasiregular representation π_p on $L^p(G/H)$. Unlike $\pi_p(s)$, the mappings $\pi_\Phi(s)$ are nonlinear but π_Φ still defines an action of G on the unit sphere of $L^\Phi(G/H)$. Namely, we have

Lemma. *If $\Phi \in \Delta_2$ then π_Φ defines a continuous action of G on $S_1(L^\Phi(G/H))$.*

PROOF. Since $\Phi \in \Delta_2$, applying (1) and the remarks of [10, pp. 95–96] to $f \in S_1(L^\Phi(G/H))$, we infer that

$$1 = \|f\|_{(\Phi)} = \int\limits_{G/H} \Phi(f(\bar{x}))\,d\mu_{G/H}(\bar{x}) = \int\limits_{G/H} \chi(s^{-1}, x)\Phi(f(s^{-1}\bar{x}))\,d\mu_{G/H}(\bar{x})$$

$$= \int\limits_{G/H} \Phi[\Phi^{-1}\{\chi(s^{-1}, \bar{x})\Phi(f(s^{-1}\bar{x}))\}]\,d\mu_{G/H}(\bar{x}) = \|\Phi^{-1}\{\chi(s^{-1}, \cdot)\Phi(f(s^{-1}\cdot))\}\|_{(\Phi)} = \|\pi_\Phi(s)f\|_{(\Phi)}.$$

Thus, $\pi_\Phi(s)f \in S_1(L^\Phi(G/H))$. Further,

$$\pi_\Phi(s_1s_2)f(\bar{x}) = \Phi^{-1}\big\{\chi\big(s_2^{-1}s_1^{-1},\bar{x}\big)\Phi\big(f\big(s_2^{-1}s_1^{-1}\bar{x}\big)\big)\big\}$$
$$= \Phi^{-1}\big\{\chi\big(s_1^{-1},\bar{x}\big)\Phi\big[\Phi^{-1}\big(\chi\big(s_2^{-1},s_1^{-1}\bar{x}\big)\Phi\big(f\big(s_2^{-1}s_1^{-1}\bar{x}\big)\big)\big)\big]\big\} = [\pi_\Phi(s_1)\pi_\Phi(s_2)f](\bar{x}).$$

Here we used the equality

$$\chi\big(s_2^{-1}s_1^{-1},\bar{x}\big) = \chi\big(s_2^{-1},s_1^{-1}\bar{x}\big)\chi\big(s_1^{-1},\bar{x}\big),$$

which is immediate from the definition of χ.

Thus, $\pi_\Phi(s_1s_2) = \pi_\Phi(s_2)\circ\pi_\Phi(s_1)$, and so π_Φ indeed defines an action of G on $S_1(L^\Phi(G/H))$. The continuity of π_Φ is clear since Φ is continuous and strictly monotone implying that Φ^{-1} is continuous. $\square$

Refer to π_Φ as the *quasiregular action* of G on $S_1(L^\Phi(G/H))$.

4. Amenability of Homogeneous Spaces and the Rao–Reiter Conditions

The main result of the article is as follows:

Theorem. *Assume that Φ is an N-function satisfying the Δ_2-condition. Let G be a locally compact group and let H be a closed subgroup in G. Then the homogeneous space G/H is amenable if and only if G/H satisfies the Rao–Reiter condition (P_Φ) or (P_Φ^*):*

(P_Φ) $((P_\Phi^*))$ *for every compact (finite) set $F\subset G$ and every $\varepsilon>0$, there exists a function $f\in S_1(L^\Phi(G/H))$ with $f\geq 0$ such that $\|\pi_\Phi(s)f-f\|_{(\Phi)}<\varepsilon$ for all $s\in F$.*

Proof. It is well known (see [1]) that G/H is amenable if and only if G/H satisfies Reiter's condition (P_1). We will prove that

$$(P_\Phi) \Longleftrightarrow (P_1), \tag{2}$$

by modifying the arguments of Eymard in the proof of the equivalence $(P_p)\Longleftrightarrow(P_1)$ and the ideas that were involved by Rao in [5] in his proof of (2) for locally compact groups G. The equivalence $(P_\Phi^*)\Longleftrightarrow(P_1^*)$ is proved in absolutely the same manner; now, the familiar equivalence $(P_1)\Longleftrightarrow(P_1^*)$ (see [1]) gives $(P_\Phi)\Longleftrightarrow(P_\Phi^*)$.

Let φ be the left derivative of Φ and let Ψ be the complementary N-function to Φ.

$(P_1)\Longrightarrow(P_\Phi)$ Take arbitrary $\varepsilon>0$ and a compact set $F\subset G$. Since $\Phi\in\Delta_2$, by [11, Theorem 7, p. 16],

$$\lim_{\rho_\Phi(f)\to 0}\|f\|_{(\Phi)} = 0$$

and hence there is $\delta>0$ such that if $\rho_\Phi(f)<\delta$ then $\|f\|_{(\Phi)}<\varepsilon$.

Choose $g\in L^1(G/H)$ such that $g\geq 0$, $\|g\|_1=1$, $\|\pi_1(s)g-g\|_1<\delta$ for all $s\in F$. Put $f=\Phi^{-1}\circ g$. Then $f\geq 0$ and

$$\int\limits_{G/H}\Phi(f(\bar{x}))\,d\mu_{G/H}(\bar{x}) = \int\limits_{G/H}\Phi(\Phi^{-1}(g(\bar{x})))\,d\mu_{G/H}(\bar{x}) = \|g\|_1 = 1,$$

whence $\|f\|_{(\Phi)}=1$ (see [10, pp. 95–96]). Since

$$\Phi(a-b)\leq|\Phi(a)-\Phi(b)|,\quad a\geq 0,\ b\geq 0$$

(see, for example, [5, p. 388]), we infer

$$\int\limits_{G/H}\Phi\left\{[\pi_\Phi(s)f](\bar{x})-f(\bar{x})\right\}d\mu_{G/H}(\bar{x}) = \int\limits_{G/H}\Phi[\Phi^{-1}\{\chi(s^{-1},\bar{x})\Phi(f(s^{-1}\bar{x}))\}-f(\bar{x})]d\mu_{G/H}(\bar{x})$$
$$\leq \int\limits_{G/H}|\chi(s^{-1},\bar{x})g(s^{-1}\bar{x})-g(\bar{x})|\,d\mu_{G/H}(\bar{x}) = \|\pi_1(s)g-g\|_1<\delta.$$

Therefore, $\|\pi_\Phi(s)f-f\|_{(\Phi)}<\varepsilon$ and so G/H satisfies condition (P_Φ).

$(P_\Phi) \Longrightarrow (P_1)$ Since $\Phi \in \Delta_2$, by [7, p. 79, Proposition 8], we conclude that

$$S := \sup\{\rho_\Psi(\varphi \circ |v|) : v \in L^\Phi(G/H), \|v\|_{(\Phi)} \le 1\} < \infty. \tag{3}$$

Fix arbitrary $\varepsilon > 0$ and a compact set $F \subset G$. Then $g \in S_1(L^\Phi(G/H))$ such that $\|\pi_\Phi(s)g - g\|_{(\Phi)} < \frac{\varepsilon}{2(S+1)}$ for all $s \in F$, where S is as in (3). Put $f = \Phi \circ g$. Then, since $\Phi \in \Delta_2$, we have

$$\|f\|_1 = \int\limits_{G/H} \Phi(g(\bar{x}))\, d\mu_{G/H}(\bar{x}) = \|g\|_{(\Phi)} = 1.$$

Applying Hölder's inequality and the inequality

$$|\Phi(a) - \Phi(b)| \le |a - b|(\varphi(a) + \varphi(b)), \quad a, b \ge 0$$

(cf. [5, p. 388]), we get

$$\|\pi_1(s)f - f\|_1 = \int\limits_{G/H} |\chi(s^{-1}, \bar{x})\Phi(g(s^{-1}\bar{x})) - \Phi(g(\bar{x}))|\, d\mu_{G/H}(\bar{x})$$

$$\le \int\limits_{G/H} |\Phi^{-1}\{\chi(s^{-1}, \bar{x})\Phi(g(s^{-1}\bar{x}))\} - g(\bar{x})|\, |\varphi(\Phi^{-1}\{\chi(s^{-1}, \bar{x})\Phi(g(s^{-1}\bar{x})\}) + \varphi(g(\bar{x}))|\, d\mu_{G/H}(\bar{x})$$

$$\le \|\pi_\Phi(s)g - g\|_{(\Phi)} \|\varphi(\Phi^{-1}\{\chi(s^{-1}, \bar{x})\Phi(g(s^{-1}\bar{x}))\}) + \varphi(g(\bar{x}))\|_\Psi$$

$$\le \|\pi_\Phi(s)g - g\|_{(\Phi)} [\|\varphi \circ \pi_\Phi(s)g\|_\Psi + \|\varphi \circ g\|_\Psi]. \tag{4}$$

Using the Lemma, we see that $\|\pi_\Phi(s)g\|_{(\Phi)} = \|g\|_{(\Phi)} = 1$. Applying (4), (3), and the familiar inequality $\|v\|_\Psi \le \rho_\Psi(v) + 1$, we obtain

$$\|\pi_1(s)f - f\|_1 \le \|\pi_\Phi(s)g - g\|_{(\Phi)}\| \, [\|\varphi \circ \pi_\Phi(s)g\|_\Psi + \|\varphi \circ g\|_\Psi]$$

$$\le \|\pi_\Phi(s)g - g\|_{(\Phi)}\|[\rho_\Psi(\varphi \circ \pi_\Phi(s)g) + \rho_\Psi(\varphi \circ g) + 2] < \frac{\varepsilon}{2(S+1)} 2(S+1) = \varepsilon.$$

Thus, $\|\pi_1(s)f - f\|_1 \le \varepsilon$. $\square$

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