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STRUCTURE OF SOME UNITAL SIMPLE JORDAN SUPERALGEBRAS WITH ASSOCIATIVE EVEN PART

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Abstract: Studying the unital simple Jordan superalgebras with associative even part, we describe the unital simple Jordan superalgebras such that every pair of even elements induces the zero derivation and every pair of two odd elements induces the zero derivation of the even part. We show that such a superalgebra is either a superalgebra of nondegenerate bilinear form over a field or a four-dimensional simple Jordan superalgebra.

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The study of simple Jordan superalgebras with associative even part originates in [1], where the simple $(-1, 1)$-superalgebras were described. In [1] it was shown in particular that the even part of such a superalgebra is an associative commutative algebra, and the odd part is an associative commutative module over the even part. Following [2], we call a superalgebra with this property *abelian.*

The Kantor doubling process is one of the main constructions of the Jordan superalgebras from associative supercommutative superalgebras equipped with a Jordan bracket, for example, a vector type bracket or a Poisson bracket (see [3, 4]). If a Jordan bracket is given on an associative commutative algebra then the so-obtained Jordan superalgebra is abelian. Note that the Jordan superalgebras of vector type were constructed starting from a derivation given on an associative supercommutative superalgebra. Various properties of the superalgebras of vector type and the Jordan brackets were studied in [5–14].

Special unital simple infinite-dimensional Jordan superalgebras with associative even part were studied in [15, 16]. In particular, it was shown that for the even part A and the odd part M the associators (a, m, b) are zero, where $a, b \in A$ and $m \in M$; i.e., every pair of even elements induces the zero derivation. Moreover, in [15, 16] the special unital simple abelian Jordan superalgebras were described, which are nonisomorphic to a superalgebra of nondegenerate bilinear form. As it turned out, such a superalgebras are of vector type with respect to several derivations. Moreover, such a superalgebra is embedded into a simple superalgebra of vector type that is constructed by one derivation. In [16–18], some examples were constructed of the simple Jordan superalgebras of vector type with respect to two derivations. These examples answer the Cantarini–Kac question from [19], where the linearly compact simple Jordan superalgebras over an algebraically closed field of characteristic 0 were described. The article [20] gives some examples of prime Jordan superalgebras of vector type with respect to arbitrary number of derivations.

In contrast to the special Jordan superalgebras there exist nonspecial Jordan superalgebras with associative even part such that $(a, m, b) \neq 0$. In [21], the unital simple Jordan superalgebras with associative nil-semisimple even part such that $(a, m, b) \neq 0$ were described. The even part $A = A_0 + A_1$ of such a superalgebra is a Z_2-graded algebra. The odd part $M = M_0 + N$ is the direct sum of two projective A_0-modules of rank 1. Furthermore, $A_0 + M_0$ is a unital simple abelian subsuperalgebra. Moreover, it was shown that such a superalgebra is embedded into a simple superalgebra of the Jordan bracket.

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In this article, we study the unital simple Jordan superalgebras with associative even part such that $(a, m, b) = 0$, where $a, b \in A$ and $m \in M$. The two cases are possible. The first is $(x, a, y) = 0$, where $a \in A$ and $x, y \in M$, i.e., every pair of odd elements induces the zero derivation of the even part. In this case we show that the initial superalgebra is either a superalgebra of nondegenerate bilinear form with the even part being a two-dimensional composition algebra or a four-dimensional simple superalgebra of type $J(\mathcal{C}, v)$, where $\mathcal{C}$ is a two-dimensional composition algebra (see [22]).

The second case of $(x, a, y) \neq 0$ for some $a \in A$ and $x, y \in M$ remains open.

Note that the unital simple finite-dimensional Jordan superalgebras over an algebraically closed field were described in [3, 22–25]. The article [26] describes the unital infinite-dimensional Jordan Z_n-graded algebras such that the dimension of every homogeneous n-component has the uniformly bounded growth.

Studying nonassociative superalgebras with associative even part is of great interest, as evidenced by the series of articles [27–32].

§ 1. Preliminary Results

Let F be a field of characteristic not 2, and let $A = A_0 + A_1$ be an arbitrary $\mathbb{Z}_2$-graded algebra; i.e., $A_0^2 \subseteq A_0$, $A_1^2 \subseteq A_0$, $A_0A_1 \subseteq A_1$, and $A_1A_0 \subseteq A_1$. The algebra A is a *superalgebra.* The vector space $A_0(A_1)$ is an *even* (*odd*) *part* of the $\mathbb{Z}_2$-graded algebra A. The elements in $A_0 \cup A_1$ are *homogeneous.* We let $|x|$, where $x \in A_0 \cup A_1$, stand for the parity index of a homogeneous element x:

$$|x| = \begin{cases} 0 & \text{if } x \in A_0 \, (x \text{ is } \textit{even}), \\ 1 & \text{if } x \in A_1 \, (x \text{ is } \textit{odd}). \end{cases}$$

Let G be the Grassmann algebra over F; i.e., G is an associative algebra given by the generators $1, e_1, e_2, \dots$ and the defining relations $e_i^2 = 0$, $e_ie_j = -e_je_i$. The products $1, e_{i_1} \dots e_{i_k}$, where $i_1 < i_2 < \dots < i_k$, form a basis for G. Let G_0 and G_1 be vector subspaces generated by the products of even or odd length respectively. Then $G = G_0 + G_1$ is a $\mathbb{Z}_2$-graded algebra. Let $A = A_0 + A_1$ be an arbitrary $\mathbb{Z}_2$-graded algebra. Then $G(A) = G_0 \otimes A_0 + G_1 \otimes A_1$ is a subalgebra of $G \otimes A$ (the tensor product over F), and $G(A)$ is called the *Grassmann enveloping of A.*

A superalgebra $J = J_0 + J_1$ is a *Jordan superalgebra* if and only if the Grassmann enveloping $G(J)$ of J is a Jordan algebra; i.e., in $G(J)$ we have $xy = yx$ and $(x^2y)x = x^2(yx)$. Denote by A and M the even and the odd parts of J, respectively. The following identities for homogeneous elements hold in every Jordan superalgebra J:

$$ab = (-1)^{|a||b|}ba, \tag{1}$$

$$\begin{aligned}&[(ab)c]d + (-1)^{|b||c|+|b||d|+|c||d|}[(ad)c]b + (-1)^{|a|(|b|+|c|)+(|a|+|b|+|c|)|d|}[(db)c]a \\ &\quad = (ab)(cd) + (-1)^{|b||c|}(ac)(bd) + (-1)^{|b||d|+|c||d|}(ad)(bc),\end{aligned} \tag{2}$$

$$\begin{aligned}&[(ab)c]d + (-1)^{|b||c|+|b||d|+|c||d|}[(ad)c]b + (-1)^{|a|(|b|+|c|)+(|a|+|b|+|c|)|d|}[(db)c]a \\ &\quad = [a(bc)]d + (-1)^{|c||d|}[a(bd)]c + (-1)^{|b||d|+|c||d|}(a(dc)]b,\end{aligned} \tag{3}$$

$$(ab, c, d) + (-1)^{|b||c|+|c||d|+|d||b|}(ad, c, b) + (-1)^{|a|(|b|+|c|+|d|)+|d||c|}(bd, c, a) = 0, \tag{4}$$

where $(x, y, z) = (xy)z - x(yz)$ is the associator of x, y, and z. From (1)–(4) it follows that

$$(a, bc, d) = (-1)^{|a||b|}b(a, c, d) + (-1)^{|c||d|}(a, b, d)c. \tag{5}$$

Also,

$$(a, b, c) + (-1)^{|a||b|+|a||c|}(b, c, a) + (-1)^{|a||c|+|b||c|}(c, a, b) = 0, \tag{6}$$

$$(a, b, c) = -(-1)^{|a||b|+|a||c|+|b||c|}(c, b, a). \tag{7}$$

Note that in every algebra we have

$$a(b,c,d)-(ab,c,d)-(a,b,cd)+(a,bc,d)+(a,b,c)d=0. \tag{8}$$

Given $a\in J$, denote by R_a the right multiplication by a. Put $D_x=R_x^2$ for $x\in M$. Then D_x is a derivation of J. Indeed, $D_x(y)=\frac{(-1)^{|x||y|}}{2}(x,y,x)$ for $y\in A\cup M$. Therefore, if $y,z\in A\cup M$ then

$$\begin{aligned}D_x(yz)&=\frac{(-1)^{|x||yz|}}{2}(x,yz,x)\overset{\text{by (5)}}{=}\frac{(-1)^{|x||yz|}}{2}((-1)^{|x||y|}y(x,z,x)+(-1)^{|x||z|}(x,y,x))z\\&=(-1)^{|x||yz|}((-1)^{|x||y|}(-1)^{|x||z|}yD_x(z)+(-1)^{|x||z|}(-1)^{|x||y|}D_x(y))z=yD_x(z)+D_x(y)z.\end{aligned}$$

Hence, $R_xR_y+R_yR_x$ is a derivation of J for all $x,y\in M$.

Let us provide some examples of simple Jordan superalgebras.

1. The superalgebra of a bilinear form. Let $V=V_0\oplus V_1$ be a Z_2-graded vector space over F with a supersymmetric bilinear form $f(x,y)$ (i.e., f is symmetric on V_0, f is skew-symmetric on V_1, and $f(V_0,V_1)=0$). Consider the direct sum of vector spaces $J=F\cdot 1+V$. Define the product on J by putting $1\cdot 1=1$, $1\cdot v=v\cdot 1=v$, $v_1\cdot v_2=f(v_1,v_2)\cdot 1$, where $v,v_1,v_2\in V$. Then J is a Jordan superalgebra with the even part $A=F\cdot 1+V_0$ and the odd part $M=V_1$. If f is nondegenerate then J is a simple superalgebra, except for the case $V_1=0$, $V_0=F\cdot e$, and $f(e,e)=\alpha^2$.

2. The superalgebras of type $J(\mathcal{C},v)$. Let $A=Fe_1+Fe_2$ be the direct sum of two fields, and let $M=Fx+Fy$ be a two-dimensional vector space over F. Define a product on the vector space $A+M$ putting

$$e_1e_2=0,\quad e_i^2=e_i,\quad e_ix=xe_i=\frac{1}{2}x,\quad e_iy=ye_i=\frac{1}{2}y,\quad i=1,2,$$
$$x^2=y^2=0,\quad xy=-yx=e_1+te_2,$$

where $t\in F$. Denote the so-obtained algebra by D_t. Then D_t is a Jordan superalgebra. The superalgebra D_t is simple if and only if $t\neq 0$.

Let $A=\mathcal{C}=F+Fv$ be a two-dimensional composition algebra over F, $v^2\in F$, and $v^2\neq 0$. Define the product on the vector space $A+M$ by putting

$$vx=xv=yv=vy=0,\quad xy=-yx=\alpha+v\beta,$$

where $\alpha,\beta\in F$ cannot be zero simultaneously. Denote the so-obtained algebra by $J(\mathcal{C},v)$ (see [22]). A basis for $J(\mathcal{C},v)$ can be chosen in such a way that either $\alpha=1$, $\beta=0$, or $\alpha=0$, $\beta=1$, or $\alpha=1$, $\beta=1$.

Let $\overline{F}$ be the algebraical closure of F. Consider the tensor product $J(\mathcal{C},v)\otimes\overline{F}$. Identify $v\otimes 1$, $x\otimes 1$, and $y\otimes 1$ with v, x, and y. Then

$$\mathcal{C}\otimes\overline{F}=\overline{F}+v\overline{F}=\overline{F}e_1+\overline{F}e_2,$$

$v=\gamma s$, where $\gamma\in\overline{F}$, $s=e_1-e_2$ and $e_i^2=e_i$, $i=1,2$.

If $\alpha=1$ and $\beta=0$ then $J(\mathcal{C},v)\otimes\overline{F}$ is isomorphic to D_1. If $\alpha=0$ and $\beta=1$ then $J(\mathcal{C},v)\otimes\overline{F}$ is isomorphic to D_{-1}. It follows from here that $J(\mathcal{C},v)$ is a simple Jordan superalgebra in these two cases.

If $\alpha=1$ and $\beta=1$ then $xy=1+\gamma s=(1+\gamma)e_1+(1-\gamma)e_2$ in $J(\mathcal{C},v)\otimes\overline{F}$. Hence, $J(\mathcal{C},v)\otimes\overline{F}$ is a simple Jordan superalgebra when $\gamma\neq\pm 1$. Therefore, if $v^2\neq 1$ then $J(\mathcal{C},v)$ is a simple Jordan superalgebra.

Lemma 1. *Given an arbitrary Jordan superalgebra $J=A+M$, we have*

$$[((M,A,A)+A(M,A,A))M]M\subseteq(M,A,A)+A(M,A,A).$$

Furthermore, let N be an A-submodule of M such that $(NM)M\subseteq N$. Then $I=(A,N,N)+NN$ is an ideal of A, and $IM\subseteq N$.

Proof. Note that $(M,A,A)=(A,A,M)$ by (7), and $(A,M,A)\subseteq(M,A,A)$ by (6). Write $x\equiv y$ if $x-y\in(M,A,A)+A(M,A,A)$. Let $a,b,c\in A$ and $m,n,k\in M$.

By (5) $D_{x,y}=R_xR_y-(-1)^{|x||y|}R_yR_x$ is a superderivation of J for all homogeneous x and y. Hence, for $D=D_{m,b}$ we have

$$((m,a,b)n)k\equiv((m,a,b),n,k)=(D_{m,b}(a),n,k)=(D(a),n,k)$$
$$=-(a,D(n),k)+(a,n,D(k))+D((a,n,k))\subseteq(A,A,M)+(A,M,A)\subseteq(A,A,M).$$

Assume that $m'=(m,a,b)$ and $c\in A$. Then

$$((m'c)n)k=(m'c)(nk)+(m'c,n,k)=m'(c(nk))+(m',c,nk)+(m'c,n,k)\equiv(m'c,n,k).$$

By (4), (7), and the above

$$(m'c,n,k)=(m',n,kc)+(c,n,m'k)\equiv((m,a,b),n,kc)\equiv 0,$$

whence the first inclusion follows.

Prove the second. Let $I=(A,N,N)+NN$. By (5), I is an ideal of A. Also, by (5)

$$(A,N,N)M\subseteq(A,NM,N)+(A,M,N)N\subseteq N.$$

Furthermore, $(NN)M\subseteq N$. Therefore, $IM\subseteq N$.

Corollary 1. *Let $J=A+M$ be a unital simple Jordan superalgebra with associative even part. Then one of the equalities $(M,A,A)=0$ and $M=(M,A,A)$ holds in J.*

Proof. Let $N=(M,A,A)$ and $I=(NM)A+N$. Show that I is an ideal of J. By (8)

$$(m,a,b)c=-m(a,b,c)+(ma,b,c)+(m,a,bc)-(m,ab,c)\in(M,A,A)$$

for $a,b,c\in A$ and $m\in M$. Therefore, $AN\subseteq N$, and $AI\subseteq I$. Clearly, $NM\subseteq I$. By Lemma 1,

$$[(NM)A]M\subseteq(NM,A,M)+(NM)M\subseteq N.$$

Hence, I is an ideal of J. By the simplicity of J we have either $(M,A,A)=0$ or $M=(M,A,A)$.

Let $J=A+M$ be a unital simple Jordan superalgebra with associative even part such that $(A,A,M)\neq 0$. Then $M=(A,A,M)$. Assume that $(A,M,A)=0$ as well. Recall that $(A,M,A)=0$, when J is a special superalgebra (see [16]). In [21], the unital simple Jordan superalgebras with associative nil-semisimple even part such that $(A,M,A)\neq 0$ were described.

Lemma 2. *Let I be an ideal of A such that $(IM)M\subseteq I$. Then either $I=A$ or $I=0$. In particular, $A=(A,M,M)+MM$, and $A=A(MM)$.*

Proof. Let $K=I+IM$. Since $(A,M,A)=0$, K is an ideal of J. Therefore, either $I=A$ or $I=0$.

It follows from here that $A=(A,M,M)+MM$. Indeed, let $I=(A,M,M)+MM$. Then I is an ideal of A, and $(IM)M\subseteq I$ by (5). Hence, $I=A$. Analogously, $A=A(MM)$.

Lemma 3. *We have*

$$(a,y,z)x-(a,z,y)x+(a,z,x)y-(a,x,z)y=(a,y,x)z-(a,x,y)z$$

in J for all $a\in A$ and $x,y,z\in M$.

Proof. Since $(A,M,A)=0$,

$$0=(a,b,x)+(b,x,a)+(x,a,b)=(a,b,x)+(x,a,b)$$

for all $a,b\in A$ and $x\in M$ by (6). Therefore, $(a,b,x)=(b,a,x)$ by (7), whence

$$(a,xy,z)=(xy,a,z)\overset{\text{by (4)}}{=}(xz,a,y)+(zy,a,x)$$

$$=(a,xz,y)+(a,zy,x)\overset{\text{by (5)}}{=}x(a,z,y)-(a,x,y)z+z(a,y,x)-(a,z,x)y.$$

On the other hand,

$$(xy,a,z)=(a,xy,z)\overset{\text{by (5)}}{=}x(a,y,z)-(a,x,z)y.$$

Hence,

$$(a,y,z)x-(a,z,y)x+(a,z,x)y-(a,x,z)y=(a,y,x)z-(a,x,y)z.$$

Lemma 4. *For the superalgebra J, either $A = (A, M, M)$ or $(M, A, M) = 0$.*

Proof. Let $A \neq (A, M, M)$. By (6) and (7) $(M, A, M) \subseteq (A, M, M)$. By (8)

$$A(M, A, M) \subseteq (AM, A, M) + (A, MA, M) + (A, M, AM) + (A, M, A)M \subseteq (A, M, M).$$

Therefore, the vector subspace (A, M, M) includes the ideal $A(M, A, M)$ of A. If $(M, A, M) \neq 0$ then (A, M, M) includes the greatest ideal I of A. Then

$$(IM)M \subseteq (I, M, M) + I(MM) \subseteq (A, M, M).$$

We may assume that $(IM)M \not\subseteq I$. By (2)

$$[(IM)M]A \subseteq [(IA)M]M + I[(AM)M] + IA(MM) + (IM)(AM) \subseteq I + (IM)M,$$

whence $I+(IM)M$ is an ideal of A which lies in (A, M, M). By the choice of I we infer that $I+(IM)M = (A, M, M)$. Hence, (A, M, M) is an ideal of A. Since

$$[(A, M, M)M]M \subseteq ((A, M, M), M, M) + (A, M, M)(MM) \subseteq (A, M, M),$$

we have $(A, M, M) = A$. Consequently, $(M, A, M) = 0$.

Put $(M, A, M) = 0$ in $J = A + M$.

Lemma 5. *For all $a \in A$ and $x, y \in M$ we have*

$$(ax)y = -(ay)x, \quad R_x^3 = 0, \quad R_xR_yR_x + R_x^2R_y + R_yR_x^2 = 0.$$

Proof. Take $a \in A$. Then $(ax)x = \frac{1}{2}(x, a, x) = 0$, whence $(ax)y = -(ay)x$.

Let $y \in M$. Then $yR_x^3 = (yx)R_x^2 = 0$, since $yx \in A$. Linearizing $R_x^3 = 0$ by x, we get

$$R_xR_yR_x + R_x^2R_y + R_yR_x^2 = 0.$$

By Lemmas 3 and 5

$$(a, y, z)x + (a, z, x)y + (a, x, y)z = 0 \tag{9}$$

for all $a \in A$ and $x, y, z \in M$.

Lemma 6. *Let I be a nonzero ideal of A. Then $A = I + (IM)M$, and $M = IM$.*

Proof. By (2)

$$[(IM)M]A \subseteq [(IA)M]M + I[(AM)M] + IA(MM) + (IM)(AM) \subseteq I + (IM)M.$$

Therefore, $I + (IM)M$ is an ideal of A.

Let $x, y \in M$. Then $IR_xR_yR_x \subseteq IR_yR_x^2 + IR_x^2R_y$ by Lemma 5. Since R_x^2 is a derivation and $IR_x^2 = 0$; therefore, $IR_xR_yR_x \subseteq IM$. Linearizing this inclusion by x, we get $I(R_xR_yR_z + R_zR_yR_x) \subseteq IM$ for all $x, y, z \in M$. By (3)

$$I(R_xR_yR_z - R_zR_yR_x) \subseteq IM.$$

Therefore, $IR_xR_yR_z \subseteq IM$, whence

$$[I + (IM)M]M \cdot M \subseteq (IM)M \subseteq I + (IM)M.$$

Hence, $A = I + (IM)M$ by Lemma 2. Since $M = AM$, $M = IM$.

Denote by $Z(J)$ the center of $J = J_0 + J_1$, i.e., $Z(J) = Z_0 + Z_1$, where

$$Z_i = \{z \in J_i \mid (z, J, J) = 0,\ zx = (-1)^{|z||x|}xz\ \forall x \in J_0 \cup J_1\}.$$

Lemma 7. *Let*

$$\mathcal{C}_0 = \{a \in A \mid (a, x, y) = 0 \ \forall x, y \in M\} \ \text{ and } \ \mathcal{C}_1 = \{a \in A \mid ax = 0 \ \forall x \in M\}.$$

Then $\mathcal{C}_0 \neq 0$, $\mathcal{C}_0 \subseteq Z(J)$. In particular, $\mathcal{C}_0$ is a field. If $\mathcal{C}_1 \neq 0$ then $\mathcal{C}_1 = \mathcal{C}_0 v$, v is invertible, and $\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1$ is a two-dimensional composition algebra over $\mathcal{C}_0$ with the norm $n(\alpha + \beta v) = \alpha^2 - \beta^2 v^2$.

Proof. Take $a, b \in \mathcal{C}_0$. Then

$$(ab, M, M) \subseteq (aM, M, b) + (bM, M, a) = 0$$

by (4). Therefore, $\mathcal{C}_0$ is a subalgebra. Show that $(\mathcal{C}_0, A, M) = 0$. Since $(A, M, A) = 0$, $(\mathcal{C}_0, A, M)A \subseteq (\mathcal{C}_0, A, M)$. By (5)

$$(\mathcal{C}_0, A, M)M \subseteq (\mathcal{C}_0, AM, M) + A(\mathcal{C}_0, M, M) = 0.$$

Hence, $(\mathcal{C}_0, A, M)$ is an ideal of J. Consequently, $(\mathcal{C}_0, A, M) = 0$, whence $\mathcal{C}_0 \subseteq Z(J)$. Since J is a simple superalgebra, $\mathcal{C}_0$ is a field.

Take $a, b \in \mathcal{C}_1$. Then $(ab, M, M) = 0$ by (4), i.e., $ab \in \mathcal{C}_0$. By the above

$$(\mathcal{C}_0 a)M \subseteq \mathcal{C}_0(aM) + (\mathcal{C}_0, a, M) = 0.$$

Thus, $\mathcal{C}_0\mathcal{C}_1 \subseteq \mathcal{C}_1$. Since

$$(aA, M, M) \subseteq (aM, M, A) + (AM, M, a) \subseteq (MM)a \subseteq aA,$$

aA is an ideal of A and $(aA)M \cdot M \subseteq aA$. Hence, either $aA = A$ or $aA = 0$. Consequently, if $a \neq 0$ then a is invertible in A.

Let $c \in \mathcal{C}_0 \cap \mathcal{C}_1$ and $c \neq 0$. Then c is invertible in $\mathcal{C}_0$, $cM = 0$, and $(\mathcal{C}_0, c, M) = 0$, whence $c = 0$. Hence, $\mathcal{C}_0 \cap \mathcal{C}_1 = 0$.

Let $v \in \mathcal{C}_1$ and $v \neq 0$. As it was shown above, v is invertible in A. Then $v^2 \in \mathcal{C}_0$ and $v^2 \neq 0$. Therefore, $a = (v^2)^{-1}(av)v \in \mathcal{C}_0 v$ for $a \in \mathcal{C}_1$. So $\mathcal{C}_1 = \mathcal{C}_0 v$, whence $\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_0 v$ is a two-dimensional composition algebra over $\mathcal{C}_0$ with the norm $n(\alpha + \beta v) = \alpha^2 - \beta^2 v^2$.

Let $R^M(A)$ be the subalgebra of $\mathrm{End}_F(M)$ generated by $R_a : M \mapsto M$, $a \in A$. Then $R^M(A)$ is an associative commutative algebra, and M is an associative $R^M(A)$-module. Furthermore,

$$(A, x, y)M \subseteq xR^M(A) + yR^M(A)$$

for all $x, y \in M$ by (9).

§ 2. The Odd Part M Is Not a 2-Generated Module over $R^M(A)$

In this section we consider the case that the odd part of Jordan superalgebra is not a two-generated module over $R^M(A)$.

Lemma 8. *Let $x, y, u, v \in M$. Then either $M = xR^M(A) + yR^M(A)$ for some $x, y \in M$ or $(A, x, y)((A, u, v)M) = 0$ and $(A, M, M)(A, MM, M) = 0$.*

Proof. Let $N = (A, x, y)M$. Clearly, $NA \subseteq N$. Put $I = (A, N, N) + NN$. Then I is an ideal of A by (5), and

$$(A, N, N)M \subseteq (A, NM, N) + (A, M, N)N \subseteq N \subseteq xR^M(A) + yR^M(A)$$

by (5) as well. Moreover,

$$(NN)M \subseteq (N, N, M) + N(NM) \subseteq (A, x, y)(N, M, M) + N \subseteq xR^M(A) + yR^M(A).$$

Hence, $IM \subseteq N$. If $I \neq 0$ then

$$M = IM \subseteq xR^M(A) + yR^M(A)$$

by Lemma 6. Therefore, we may assume that $NN = 0$. Then by $(M, A, M) = 0$ we have

$$(A, x, y)((A, x, y)M) \cdot M \subseteq (A, x, y)M \cdot (A, x, y)M \subseteq NN = 0.$$

Since $(A, x, y)((A, x, y)M)$ is a Jordan A-submodule of M; therefore, $(A, x, y)((A, x, y)M)$ is an ideal of J. Hence, $(A, x, y)((A, x, y)M) = 0$. Linearizing the last equality by x, y and applying $(A, M, A) = 0$, we deduce $(A, x, y)((A, u, v)M) = 0$, i.e., $(A, M, M)((A, M, M)M) = 0$. Then $(A, M, M)(A, MM, M) = 0$ by (5).

Assume that $M \neq xR^M(A) + yR^M(A)$ for all $x, y \in M$.

Lemma 9. *We have $(A, MM, M) = 0$, $(A, M, M)M = 0$, and $(MM, M, M) = 0$. Furthermore, $(A, M, M) \neq 0$.*

PROOF. Let $N = (A, MM, M)$. Then $NA \subseteq N$ by $(A, M, A) = 0$.

Put $I = (A, N, M) + NM$. Then I is an ideal, and

$$(A, N, M)M \subseteq (A, NM, M) + (A, M, M)N \subseteq N$$

by (5). Put $D = D_{a,m}$ for $a \in A$ and $m \in M$. By (5) we have

$$\begin{aligned}(D(MM), M, M) &\subseteq (MM, D(M), M) + (MM, M, D(M)) + D((MM, M, M))\\ &\subseteq (MM, (A, M, M), M) + (MM, M, (A, M, M)) + (A, (MM, M, M), M)\\ &\subseteq N + (A, MM, M)(MM) \subseteq N.\end{aligned}$$

Hence, $(N, M, M) \subseteq N$ and $(NM)M \subseteq (N, M, M) + N(MM) \subseteq N$. Therefore, $IM \subseteq N$.

If $I \neq 0$ then $M = IM \subseteq N$ by Lemma 6. Then $(A, M, M)M = 0$ by Lemma 8. Therefore, $(A, MM, M) = 0$ by (5). Hence, we may assume that $I = 0$. Thus, $N = (A, MM, M) = 0$. By (9)

$$(a, y, z)x + (a, z, x)y + (a, x, y)z = 0.$$

Therefore,

$$(a, y, z)x - (a, y, z)x + (a, z, y)x = 0$$

for all $a \in A$, $x, y, z \in M$ by (5) and Lemma 5. Hence, $(A, M, M)M = 0$.

Since $M = (A, A, M)$ and $A = (A, M, M) + MM$; therefore,

$$\begin{aligned}MM &= (A, A, M)M = (A, (A, M, M), M)M\\ &\overset{\text{by (5)}}{=} (A, M, M)(A, M, M) + (A, (A, M, M)M, M) = (A, M, M)(A, M, M).\end{aligned}$$

Consequently,

$$(MM, M, M) = ((A, M, M)^2, M, M) \subseteq ((A, M, M)M, M, (A, M, M)) = 0$$

by (4), whence $(A, M, M) \neq 0$.

Lemma 10. *The algebra $A = MM + (A, M, M)$ is a $\mathbb{Z}_2$-graded algebra with even part MM and odd part (A, M, M).*

PROOF. By Lemma 9 $(MM)^2 \subseteq MM + (MM, M, M) = MM$. Therefore, MM is a subalgebra of A. It was shown in Lemma 9 that $(A, M, M)^2 = MM$. By (5) and Lemma 9

$$(MM)(A, M, M) \subseteq (A, (MM)M, M) + (A, MM, M)M \subseteq (A, M, M).$$

Put $I = MM \cap (A, M, M)$. Then $I(MM) \subseteq I$ and $(A, M, M)I \subseteq I$ by the proved above. Hence, I is an ideal of A. Furthermore, $IM = 0$ by Lemma 9. Therefore, I is an ideal of J. Thus, $I = 0$, whence $A = MM + (A, M, M)$ is a $\mathbb{Z}_2$-graded algebra with even part MM and odd part (A, M, M).

Put $A_0 = MM$ and $A_1 = (A, M, M)$. Then $A = A_0 + A_1$ is a $\mathbb{Z}_2$-graded algebra by Lemma 10, and $A_1M = 0$ by Lemma 9. Clearly, $1 \in A_0$, whence $J_0 = A_0 + M$ is a subsuperalgebra of J.

Lemma 11. *The superalgebra $J_0 = A_0 + M$ is a simple Jordan superalgebra of nondegenerate supersymmetric bilinear form over the field A_0. Furthermore, A is a two-dimensional composition algebra over A_0.*

PROOF. Let $I = I_0 + I_1$ be an ideal of J_0. Then $I + IA_1 = I + I_0A_1$, and $I + I_0A_1$ is an ideal of J by Lemmas 9 and 10. Therefore, either $I = 0$ or $J = I + I_0A_1$ and $A = I_0 + I_0A_1$. The last two equalities imply $I_0 = A_0$ and $I = J_0$. Hence, J_0 is simple.

By Lemma 9

$$(A_0, A_0, M) = (A_0, M, M) = (M, M, A_0) = (M, A_0, M) = 0.$$

Therefore, $A_0 \subseteq Z(J)$. Since J_0 is simple, A_0 is a field. Hence, J_0 is the superalgebra of nondegenerate supersymmetric bilinear form $f(x, y) = xy$ over the field A_0, where $x, y \in M$. Moreover, $A_1 = \{a \in A \mid aM = 0\} \neq 0$. By Lemma 7 every nonzero element in A_1 is invertible.

Let $a \in A_1$ and $(a, M, M) = 0$. Then $a(MM) = 0$, whence $a = 0$. Therefore, $A_0 = \{a \in A \mid (a, M, M) = 0\}$. Hence, $A = A_0 + A_0 v$ is a two-dimensional composition algebra over A_0 by Lemma 7.

Thus, we have

Theorem 1. *Let $J = A + M$ be a unital simple Jordan superalgebra with associative even part. Assume that $(A, M, A) = (M, A, M) = 0$ and M is not a two-generated $R^M(A)$-module. Then J is a superalgebra of nondegenerate supersymmetric bilinear form given on a superspace $V = V_0 + V_1$ over the field A_0. Furthermore, $V_0 = A_0 v$ is a one-dimensional vector space over A_0, and $A = A_0 + A_0 v$ is a two-dimensional composition algebra over A_0, and $M = V_1$.*

§ 3. The Odd Part M Is a 2-Generated Module over $R^M(A)$

Here we consider the case that the odd part of the Jordan superalgebra M is $xR^M(A) + yR^M(A)$.

Lemma 12. *The following hold:*
(1) $mR^M(A) \cdot mR^M(A) = 0$ for every $m \in M$. In particular, $MM = xR^M(A) \cdot y = yR^M(A) \cdot x$.
(2) $xR^M(A) \cap yR^M(A) = 0$.
(3) If $x\phi = 0$ for $\phi \in R^M(A)$ then $y\phi = 0$.
(4) If $y\phi = 0$ for $\phi \in R^M(A)$ then $x\phi = 0$.

Proof. Let $\phi, \psi \in R^M(A)$ and $m \in M$. Since $(A, M, A) = (M, A, M) = 0$, we have

$$m\phi \cdot m\psi = (m\phi)\psi \cdot m = (m\psi)\phi \cdot m = m\psi \cdot m\phi.$$

On the other hand, $m\phi \cdot m\psi = -m\psi \cdot m\phi$. Thus, $mR^M(A) \cdot mR^M(A) = 0$.

Show that $xR^M(A) \cap yR^M(A) = 0$. Put $N = xR^M(A) \cap yR^M(A)$. Then $NA \subseteq N$, and $N \cdot xR^M(A) = N \cdot yR^M(A) = 0$. Therefore,

$$NM = N(xR^M(A) + yR^M(A)) \subseteq N \cdot xR^M(A) + N \cdot yR^M(A) = 0.$$

Thus, N is an ideal of J, whence $N = 0$.

Let $x\phi = 0$ as well. Then

$$xR^M(A) \cdot yR^M(A)\phi \subseteq x\phi R^M(A) \cdot yR^M(A) = 0.$$

Therefore, $yR^M(A)\phi$ is an ideal of J, and $M \cdot yR^M(A)\phi = 0$. Hence, $y\phi = 0$.

Item (4) may be proved analogously.

Lemma 13. *The algebra $R^M(A)$ is a field. Moreover, A is nil-semisimple.*

Proof. Take $\phi, \psi, \tau \in R^M(A)$. Then by Item (1) of Lemma 12

$$(x\phi, y\psi, y\tau) = (x\phi \cdot y\psi) \cdot y\tau = [(x \cdot y\psi\phi)y]\tau = (x, y\psi\phi, y)\tau \overset{\text{by (5)}}{=} (x, y, y)\psi\phi\tau.$$

Analogously, $(y\phi, x\psi, x\tau) = (y, x, x)\psi\phi\tau$. Therefore,

$$(M\phi, M\psi, M\tau) \subseteq (M, M, M)\psi\phi\tau.$$

Let $\phi \in R^M(A)$, $\phi \neq 0$, and $N = M\phi$. Then $NA \subseteq N$, and

$$(NM)M \subseteq (M\phi, M, M) + M\phi \cdot (MM) \subseteq M\phi = N.$$

Therefore, $I = (A, N, N) + NN$ is an ideal of A, and $IM \subseteq N$ by Lemma 1.

If $I \neq 0$ then $M = IM \subseteq N = M\phi$ by Lemma 6. Since M is a finitely generated $R^M(A)$-module, the mapping $m \in M \mapsto m\phi \in M$ is an isomorphism of $R^M(A)$-modules. Then ϕ is invertible in $R^M(A)$. Indeed, let $\alpha : M\phi \mapsto M$ be the inverse mapping to $m \in M \mapsto m\phi \in M$ in $\mathrm{End}_{R^M(A)}(M)$. Then

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix},$$

where $\alpha_{ij} \in R^M(A)$. Since $m\phi\alpha = m$ for every $m \in M$; therefore, $\alpha_{11} = \alpha_{22}$, $\alpha_{12} = \alpha_{21} = 0$, and $\phi\alpha_{11} = 1$ by Lemma 12.

If $I = 0$ then $M\phi \cdot M\phi = 0$, whence $M\phi^2 \cdot M = M\phi \cdot M\phi = 0$. Since $(M\phi^2)A \subseteq M\phi^2$, $M\phi^2$ is an ideal of J. Therefore, $M\phi^2 = 0$.

Put $K = (A, N, M) + NM$. Then K is an ideal of A by (5). Given $u, w \in M$, we write $u \equiv_{\mathrm{mod}\, N} w$ provided that $u - w \in N$.

Show that $KM \subseteq N$. Indeed,

$$(NM)M \subseteq (M\phi, M, M) + N(MM) \subseteq (M, M, M)\phi + N \subseteq N.$$

Let $a \in A$ and $u, w, z \in M$. Then

$$u(a, w\phi, z) = -u(a, z, w\phi) \overset{\text{by (5)}}{=} -(a, uz, w\phi) - (a, u, w\phi)z \equiv_{\mathrm{mod}\, N} z(a, w\phi, u)$$

by Lemma 5. Since $(M, A, M) = 0$, $(a, w\phi, z) = -(a, z\phi, w)$. Therefore,

$$u(a, w\phi, u) = -u(a, u\phi, w) \overset{\text{by (5)}}{=} -(a, u \cdot u\phi, w) - (a, u, w)(u\phi) = -((a, u, w)u)\phi \in N$$

by Lemma 12. Linearizing this relation by u, we get $M(A, N, M) \subseteq N$. Hence, $KM \subseteq N = M\phi$. Since $M \neq M\phi$; therefore, $N = 0$, i.e., $M\phi = 0$. We arrive at a contradiction with $\phi \neq 0$.

Thus, $R^M(A)$ is a field.

Let $a \in A$ and $a^2 = 0$. Then

$$2R_a^3 = -R_{a^3} + 3R_{a^2}R_a = 0$$

in $R^M(A)$. Since $R^M(A)$ is a field, $R_a = 0$. Hence, $a \in \mathcal{C}_1 = \{b \in A \mid bm = 0\, \forall m \in M\}$. By Lemma 7 a is either invertible or zero. Therefore, A lacks nonzero nilpotent elements.

Lemma 14. *Let $R_{MM} = \left\{\sum_i R_{u_i w_i} \mid u_i, w_i \in M\right\}$ and $\mathcal{C}_1 = \{a \in A \mid aM = 0\}$. Then*
(1) $R_{uw}R_a = R_{ua \cdot w}$ for all $u, w \in M$ and $a \in A$.
(2) $R_{MM} = 0$ or $R_{MM} = R^M(A)$.
(3) $A \neq MM$.
(4) If $R_{MM} = R^M(A)$ then $A = MM + \mathcal{C}_1$ and $\mathcal{C}_1 \neq 0$.
(5) $\mathcal{C}_1 = \{a \in A \mid aM = 0\} \neq 0$.

Proof. Let $a \in A$ and $\phi \in R^M(A)$. Then

$$xR_{(x\phi \cdot y)}R_a = -(x, x\phi, y)a \overset{\text{by (5)}}{=} -(x, x\phi a, y) = xR_{(x\phi a \cdot y)}$$

by Lemma 12. Analogously, $yR_{(x\phi \cdot y)}R_a = yR_{(x\phi a \cdot y)}$. Therefore, $R_{(x\phi \cdot y)}R_a = R_{(x\phi a \cdot y)}$. Hence, Item (1) holds.

Since $MM = xR^M(A)y$, R_{MM} is an ideal of $R^M(A)$ as well. Then by Lemma 13 either $R_{MM} = 0$ or $R^M(A) = R_{MM}$, i.e., Item (2) holds.

Assume that $A = MM$. Then $1 = x\phi \cdot y$, where $\phi \in R^M(A)$. By Lemma 12 we may assume that $xy = 1$. Therefore,

$$0 = (xy, a, u) = (xu, a, y) + (uy, a, x)$$

for all $a \in A$ and $u \in M$ by (4). Hence, $(xu, a, y) \in xR^M(A) \cap yR^M(A)$, and $(xu, a, y) = 0$ for all $a \in A$ and $u \in M$ by Lemma 12. Since $MM = x \cdot yR^M(A)$, $(A, A, yR^M(A)) = 0$. Analogously, $(A, A, xR^M(A)) = 0$. Therefore, $(A, A, M) = 0$; a contradiction. Hence, $A \neq MM$, i.e., Item (3) holds.

Let $R^M(A) = R_{MM}$. Then $A \subseteq MM + \mathcal{C}_1$. Since $MM \neq A$, $\mathcal{C}_1 \neq 0$.

If $R_{MM} = 0$ then $MM \subseteq \mathcal{C}_1$, whence $\mathcal{C}_1 \neq 0$.

Lemma 15. *In the notations of Lemma 7 if $MM \cap \mathcal{C}_1 \neq 0$ or $MM \cap \mathcal{C}_0 \neq 0$ then $A = \mathcal{C}_0 + \mathcal{C}_1$ is a two-dimensional composition algebra over $\mathcal{C}_0$. In the first case $\mathcal{C}_1 = MM$ and $\mathcal{C}_0 = (MM)^2$. In the second case $MM = \mathcal{C}_0$.*

Proof. Let $MM \cap \mathcal{C}_1 \neq 0$. Then $x\phi \cdot y \in \mathcal{C}_1$ and $x\phi \cdot y \neq 0$, where $\phi \in R^M(A)$. We may assume that $xy \in \mathcal{C}_1$. Since $R_{xa\cdot y} = R_{xy}R_a = 0$ in $R^M(A)$, we have $R_{MM} = 0$.

Thus, $(MM)M = 0$, i.e., $MM \in \mathcal{C}_1$. By Lemma 7 $\mathcal{C}_0 \subseteq Z(J)$, $\mathcal{C}_1 = \mathcal{C}_0 v$, where v is invertible and $v^2 \in \mathcal{C}_0$. Therefore, $vM^2 \subseteq \mathcal{C}_0$ and $(MM)^2 \subseteq \mathcal{C}_0$. By (4)

$$\begin{aligned}(A, M, M) = (v(v^{-1}A), M, M) &\subseteq ((v^{-1}A)M, M, v) + (vM, M, A)\\ &\subseteq (M, M, v) = vM^2 \subseteq \mathcal{C}_0.\end{aligned}$$

Since $A = (A, M, M) + MM$; therefore, $A = \mathcal{C}_0 + \mathcal{C}_1$ and $MM = \mathcal{C}_1$, $(MM)^2 = \mathcal{C}_0$.

Let $\mathcal{C}_0 \cap MM \neq 0$ and $\mathcal{C}_1 \cap MM = 0$. Then $x\phi \cdot y = b$, where $\phi \in R^M(A)$, $b \in \mathcal{C}_0$, and $b \neq 0$. By Lemma 7 $x\phi \cdot yb^{-1} = 1$. We may assume that $xy = 1$. Repeating the argument of Item (3) of Lemma 14, we get $(MM, A, M) = 0$. Hence, $(MM, \mathcal{C}_1, M) = 0$. By Lemma 7 $\mathcal{C}_1 = \mathcal{C}_0 v$. Then $0 = (MM, v, M) = (MM)v \cdot M$. Therefore, $(MM)v \subseteq \mathcal{C}_1$ and $MM \subseteq \mathcal{C}_0$. Since $\mathcal{C}_1 \cap MM = 0$, $R_{MM} = R^M(A)$. Then $A = MM + \mathcal{C}_1 = \mathcal{C}_0 + \mathcal{C}_1$ by Lemma 14, whence $MM = \mathcal{C}_0$.

Assume that $MM \cap \mathcal{C}_1 = 0$ and $MM \cap \mathcal{C}_0 = 0$. Then $1 = x\phi \cdot y + v\alpha$ by Lemmas 14 and 7, where $\phi \in R^M(A)$, $\alpha \in \mathcal{C}_0$, $\alpha \neq 0$, v is invertible in A, and $vM = 0$. Without loss of generality we may assume that $xy = 1 + v$.

Lemma 16. *Let $A_0 = (1 - v)M^2$. Then*
(1) A_0 is a subalgebra of A;
(2) $v^2 \neq 1$, $A = A_0 + vA_0$ is a $\mathbb{Z}_2$-graded algebra, and $\mathcal{C}_0 \subseteq A_0$;
(3) $A_0 = \mathcal{C}_0$.

Proof. Proceed in steps:

1. Prove firstly that $(MM)^2 \subseteq (1 + v)M^2$. Let $x\phi \cdot y, x\psi \cdot y \in MM$, where $\phi, \psi \in R^M(A)$. Since $x\phi \cdot y = x \cdot y\phi$, we have

$$\begin{aligned}(x\phi \cdot y)(x\psi \cdot y) &= (x, y\phi, x\psi \cdot y) + x \cdot y\phi(x\psi \cdot y) \overset{\text{by } (4)}{=} (x\psi, y\phi, xy) + x(y\phi, y\psi, x)\\ &\overset{\text{by } (5)}{=} (x\psi, y\phi, v) + x \cdot (y\phi, y, x)\psi = (x\psi \cdot y\phi)v + x \cdot y\phi\psi = (1 + v)(x\phi\psi \cdot y).\end{aligned}$$

Consequently, $(MM)^2 \subseteq (1 + v)M^2$, whence

$$A_0^2 = (1 - v)M^2 \cdot (1 - v)M^2 \subseteq (1 - v)^2(1 + v)M^2 \subseteq (1 - v)(1 - v^2)M^2.$$

By Lemma 7, $1 - v^2 \in \mathcal{C}_0 \subseteq Z(J)$. Hence,

$$A_0^2 \subseteq (1 - v)(1 - v^2)M^2 \subseteq (1 - v)M^2 = A_0.$$

2. By Lemma 7 v is invertible in A. Therefore,

$$(A, M, M) = (v(v^{-1}A), M, M) \overset{\text{by } (4)}{\subseteq} ((v^{-1}A)M, M, v) \subseteq (v, M, M) = vM^2.$$

Since $A = MM + (A, M, M)$; therefore, $A = M^2 + vM^2$, whence

$$A = (1 - v)M^2 + (1 + v)M^2.$$

By the above, $((1 - v)M^2)^2 \subseteq (1 - v)(1 - v^2)M^2$. If $v^2 = 1$ then $(1 - v)M^2 = 0$ by Lemma 13. Then $A = (1 + v)M^2$, whence $1 + v$ is invertible in A. Hence, $v = 1$, and $A = MM$. By Lemma 14 $A \neq MM$. Therefore, $v^2 \neq 1$. Then $1 - v^2$ is invertible in A, and

$$(1 + v)(1 - v^2)^{-1} = (1 - v^2)^{-1}x \cdot y \in M^2.$$

Consequently, $1 \in A_0$, whence $\mathcal{C}_0 \subseteq A_0$.

Since $A = M^2 + vM^2$ and $1 - v$ is invertible in A; therefore,

$$A = (1 - v)A = (1 - v)M^2 + v(1 - v)M^2 = A_0 + vA_0.$$

Let $I = A_0 \cap vA_0$. Then I is an ideal of A. If $I \neq 0$ then $A = I + MM$ by Lemma 6, whence

$$A = (1 - v)A = (1 - v)I + (1 - v)M^2 = I + A_0.$$

Hence, $A = A_0$. Then

$$A = (1 + v)A = (1 + v)A_0 = (1 + v)(1 - v)M^2 = (1 - v^2)M^2 \subseteq M^2,$$

which contradicts Item (3) of Lemma 14. It follows from here that $I = 0$. Therefore, $A = A_0 + vA_0$ is a $\mathbb{Z}_2$-graded algebra.

3. Firstly, we prove that A_0 is a field. Let I be a nonzero ideal of A_0. Then $I + vI$ is a nonzero ideal of A. By Lemma 6 $A = I + vI + MM$. Then

$$A = (1 - v)A = (1 - v)(I + vI + MM) = I + vI + A_0 = A_0 + vI,$$

whence $vI = vA_0$ by Item (2). Hence, $I = A_0$. Thus, A_0 is a field.

Show that $A_0 = \mathcal{C}_0$. Since $A_0 \cap vA_0 = 0$, $M^2 \cap vM^2 = 0$. Let $A_0 \setminus \mathcal{C}_0 \neq \varnothing$ and $a, b \in A_0 \setminus \mathcal{C}_0$. Then by Item (4) of Lemma 14 we get

$$v(xa \cdot y) = x\phi \cdot y + v\alpha, \quad v(xb \cdot y) = x\psi \cdot y + v\beta,$$

where $\phi, \psi \in R^M(A)$, and $\alpha, \beta \in \mathcal{C}_0$, whence

$$v(x(a\beta - b\alpha) \cdot y) = v(xa \cdot y)\beta - v(xb \cdot y)\alpha = (x\phi \cdot y)\beta - (x\psi \cdot y)\alpha \in MM.$$

Since $M^2 \cap vM^2 = 0$, $v(x(a\beta - b\alpha) \cdot y) = 0$. Hence, $x(a\beta - b\alpha) \cdot y = 0$. By Item (1) of Lemma 14

$$0 = R_{x(a\beta - b\alpha)\cdot y} = R_{xy}R_{(a\beta - b\alpha)} = R_{(a\beta - b\alpha)}$$

in $R^M(A)$. Therefore, $a\beta - b\alpha \in \mathcal{C}_1 = \mathcal{C}_0 v \subseteq vA_0$ by Lemma 7. Since $a\beta - b\alpha \in A_0$, $a\beta = b\alpha$. If $xa \cdot y = 0$ then by analogy $a \in \mathcal{C}_0 v \in vA_0$, whence $a = 0$. Therefore, $xa \cdot y \neq 0$ and $xb \cdot y \neq 0$. Since $M^2 \cap vM^2 = 0$, $\alpha \neq 0$ and $\beta \neq 0$. Then $b = \alpha^{-1}\beta a$.

If $a^2 \notin \mathcal{C}_0$ then $a^2 = \gamma a$ for some $\gamma \in \mathcal{C}_0$ by the proved above. Since A_0 is a field, $a \in \mathcal{C}_0$. Hence, $a^2 \in \mathcal{C}_0$ for every $a \in A_0 \setminus \mathcal{C}_0$.

Let $a \in A_0 \setminus \mathcal{C}_0$. Then $1 + a \in A_0 \setminus \mathcal{C}_0$. Thus, $a^2 \in \mathcal{C}_0$, and $(1 + a)^2 \in \mathcal{C}_0$, whence $a \in \mathcal{C}_0$. Therefore, $A_0 = \mathcal{C}_0$.

Theorem 2. *Let $J = A + M$ be a unital simple Jordan superalgebra with associative even part. Assume that (A, M, A)=(M, A, M)=0 and M is a two-generated $R^M(A)$-module. Then J is the superalgebra $J(\mathcal{C}, v)$. More precisely, $A = \mathcal{C} = \mathcal{C}_0 + \mathcal{C}_0 v$ is a two-dimensional composition algebra over the field $\mathcal{C}_0$, while $\mathcal{C}_0$ is the center of J, v is invertible, and $vM = 0$. The odd part M is a vector space of dimension 2 over $\mathcal{C}_0$. Moreover, one of the following holds*:

(1) *M has a basis x, y such that $xy = 1$;*

(2) *M has a basis x, y such that $xy = v$;*

(3) *M has a basis x, y such that $xy = 1 + v$ and v is not a solution to $t^2 - 1$.*

Proof. Let $\mathcal{C}_0 = \{a \in A \mid (a, M, M) = 0\}$ and $\mathcal{C}_1 = \{a \in A \mid aM = 0\}$. By Lemma 14 $\mathcal{C}_1 \neq 0$. Then $\mathcal{C}_0 \subseteq Z(J)$ by Lemma 7, $\mathcal{C}_1 = \mathcal{C}_0 v$, and $\mathcal{C}_0 + \mathcal{C}_0 v$ is a composition algebra over $\mathcal{C}_0$.

If $\mathcal{C}_0 \cap MM \neq 0$ then $A = \mathcal{C}_0 + \mathcal{C}_0 v$ and $\mathcal{C}_0 = MM$ by Lemma 15. Since M is a two-generated $R^M(A)$-module, $M = \mathcal{C}_0 x + \mathcal{C}_0 y$ where $x, y \in M$. Hence, $x\alpha \cdot y = 1$ for some $\alpha \in \mathcal{C}_0$. By Lemma 12 $x\alpha, y$ is a basis for M. Then we may assume that $xy = 1$, and Item (1) is proved.

If $\mathcal{C}_1 \cap MM \neq 0$ then $A = \mathcal{C}_0 + \mathcal{C}_0 v$ and $\mathcal{C}_1 = MM$ by Lemma 15. Since M is a two-generated $R^M(A)$-module, $M = \mathcal{C}_0 x + \mathcal{C}_0 y$ where $x, y \in M$. Hence, $x\alpha \cdot y = v$ for some $\alpha \in \mathcal{C}_0$. By Lemma 12 $x\alpha, y$ is a basis for M. Then we may assume that $xy = v$, and Item (2) is proved.

Let $\mathcal{C}_0 \cap MM = 0$ and $\mathcal{C}_1 \cap MM = 0$. By Lemma 16 $A = \mathcal{C}_0 + \mathcal{C}_0 v$ and $v^2 \neq 1$. Hence, $M = \mathcal{C}_0 x + \mathcal{C}_0 y$, where $x, y \in M$. Moreover, a basis x, y may be chosen such that $xy = 1 + v$. Thus, Item (3) is proved.

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