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BOUNDARY VALUE PROBLEMS FOR ODD ORDER FORWARD-BACKWARD-TYPE DIFFERENTIAL EQUATIONS WITH TWO TIME VARIABLES

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Abstract: We study solvability of boundary value problems for odd order differential equations in time variables. The presence of a discontinuous alternating coefficient is a peculiarity of these equations. We prove existence and uniqueness theorems for the regular solutions of such an equation, i.e. those that have all Sobolev generalized derivatives entering the equation under study.

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Introduction

We study the ultraparabolic equation with two time variables t and a of the form

$$h(x)u_t + u_a - u_{xx} + c(x,t,a)u = f(x,t,a), \tag{1}$$

$$h(x)(u_t + u_a) - u_{xx} + c(x,t,a)u = f(x,t,a), \tag{2}$$

and also the equations

$$h(x)u_t - u_{aaa} - u_{xx} + c(x,t,a)u = f(x,t,a), \tag{3}$$

$$h(x)(u_t - u_{aaa}) - u_{xx} + c(x,t,a)u = f(x,t,a), \tag{4}$$

which can be called quasiultraparabolic; we consider the case of the sign-changing function $h(x)$ with a discontinuity of the first kind at the point of the sign change.

Forward-backward differential equations (first of all parabolic and ultraparabolic equations) arise in mathematical modeling of many physical processes [1–7]. It seems possible that at present forward-backward parabolic equations are studied better (see [7–15]). As for forward-backward ultraparabolic equations, we can mention only the article [16] where some model equations are studied but of another type of forward-backwardness than that in (1)–(4). Equations (3) and (4) (quasiultraparabolic) with this type of forward-backwardness were not studied before.

We can note also the following fact. Recently, essential attention has been paid to the study of higher order nonclassical forward-backward differential equations [13–15, 17–20] but equations of the forms (3) and (4) were not considered earlier.

Let us describe the structure of the article. Section 1 presents the results on solvability of boundary value problems for the ultraparabolic equations (1) and (2). Section 2 studies the solvability of boundary value problems for (3) and (4). Section 3 deals with the possible refinements of the results.

In what follows, the derivatives are understood in the Sobolev sense (unless otherwise stated). The solvability of boundary value problems for (1)–(4) is studied in spaces with these generalized derivatives.

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1. Boundary Value Problems for Forward-Backward Ultraparabolic Equations

Let a function $h(x)$ on $[-1,1]$ satisfy the condition

$$h(x) \in C([0,1]), \quad h(x) > 0 \quad \text{for } x \in [0,1], \quad h(x) \in C([-1,0)),$$
$$\exists \lim_{x\to -0} h(x) = h(-0), \quad h(x) < 0 \quad \text{for } x \in [-1,0), \quad h(-0) < 0. \tag{5}$$

Put $Q^+ = (0,1) \times (0,T) \times (0,A)$, $Q^- = (-1,0) \times (0,T) \times (0,A)$ (T and A are positive numbers), $Q_1 = Q^+ \cup Q^-$, and $Q = (-1,1) \times (0,T) \times (0,A)$. Assume that $c(x,t,a)$ and $f(x,t,a)$ are defined for $(x,t,a) \in \overline{Q}$, while α and β are given reals.

Boundary Value Problem I. *Find a solution $u(x,t,a)$ to* (1) *on Q_1 satisfying the boundary conditions*

$$u(x,t,a)|_{x=-1} = u(x,t,a)|_{x=1} = 0, \quad (t,a) \in (0,T) \times (0,A), \tag{6}$$
$$u(x,t,a)|_{t=T} = 0, \quad (x,a) \in (-1,0) \times (0,A), \tag{7}$$
$$u(x,t,a)|_{t=0} = 0, \quad (x,a) \in (0,1) \times (0,A), \tag{8}$$
$$u(x,t,a)|_{a=0} = 0, \quad (x,t) \in [(-1,0) \times (0,T)] \cup [(0,1) \times (0,T)], \tag{9}$$

and the conjugation conditions

$$u(x,t,a)|_{x=+0} = \alpha u(x,t,a)|_{x=-0}, \quad (t,a) \in (0,T) \times (0,A), \tag{10}$$
$$u_x(x,t,a)|_{x=-0} = \beta u_x(x,t,a)|_{x=+0}, \quad (t,a) \in (0,T) \times (0,A). \tag{11}$$

Boundary Value Problem II. *Find a solution $u(x,t,a)$ to* (2) *on Q_1 satisfying* (6)–(8), (10), (11), *and*

$$u(x,t,a)|_{a=A} = 0, \quad (x,t) \in (-1,0) \times (0,T), \tag{12}$$
$$u(x,t,a)|_{a=0} = 0, \quad (x,t) \in (0,1) \times (0,T). \tag{13}$$

The boundary conditions of Problems I and II correspond to the Gevrey conditions [21] (and the Fichera conditions as well; cp. [22, 23]); the presence of the conjugation conditions in these problems is defined by the discontinuity of $h(x)$ at $x = 0$.

Let G be a bounded domain in $\mathbb{R}^3$ of the variables (x,t,a) with piecewise smooth boundary. Put

$$V_0(G) = \{v(x,t,a) : v(x,t,a) \in W_2^1(G),\ v_{xx}(x,t,a) \in L_2(G)\},$$
$$V_0 = \{v(x,t,a) : v(x,t,a) \in V_0(Q^-),\ v(x,t,a) \in V_0(Q^+)\}.$$

Obviously, V_0 is a vector space that can be endowed with the norm

$$\|v\|_{V_0} = \big(\|v\|^2_{W_2^1(Q^-)} + \|v\|^2_{W_2^1(Q^+)} + \|v_{xx}\|^2_{L_2(Q^-)} + \|v_{xx}\|^2_{L_2(Q^+)}\big)^{\frac{1}{2}}.$$

It is also obvious that V_0 under this norm is a Banach space.

Theorem 1. *Assume* (5) *and the conditions*

$$c(x,t,a) \in C(\overline{Q}), \quad c(x,t,a) \geq 0 \quad \text{for } (x,t,a) \in \overline{Q}, \tag{14}$$
$$\alpha\beta \geq 0. \tag{15}$$

Then Boundary Value Problem I *has at most one solution in V_0.*

Proof. Assume that Boundary Value Problem I has a solution $u(x,t,a)$ in V_0. First, we consider the case of $\alpha \neq 0$ and $\beta \neq 0$. Let $\gamma = \frac{\beta}{\alpha}$. Multiply (1) by $u(x,t,a)$ and integrate over the cylinder Q^-; next, we multiply this equation by $\gamma u(x,t,a)$ and integrate the result over Q^+. Summing the two equalities obtained, we arrive at

$$\int\limits_{Q^-} u_x^2\,dxdtda + \gamma\int\limits_{Q^+} u_x^2\,dxdtda + \int\limits_{Q^-} cu^2\,dxdtda + \gamma\int\limits_{Q^+} cu^2\,dxdtda$$
$$-\frac{1}{2}\int\limits_0^A\int\limits_{-1}^0 h(x)u^2(x,0,a)\,dxda + \frac{\gamma}{2}\int\limits_0^A\int\limits_0^1 h(x)u^2(x,T,a)\,dxda$$
$$+\frac{1}{2}\int\limits_0^T\int\limits_{-1}^0 u^2(x,t,A)\,dxdt + \frac{\gamma}{2}\int\limits_0^T\int\limits_0^1 u^2(x,t,A)\,dxdt$$
$$= \int\limits_{Q^-} fu\,dxdtda + \gamma\int\limits_{Q^+} fu\,dxdtda.$$

Since γ is positive, the last relation justifies the a priori estimate

$$\int\limits_{Q^-}\left(u_x^2+u^2\right)dxdtda + \int\limits_{Q^+}\left(u_x^2+u^2\right)dxdtda \le N_0\left[\int\limits_{Q^-} f^2\,dxdtda + \int\limits_{Q^+} f^2\,dxdtda\right], \tag{16}$$

where the real N_0 is defined only by α and β.

If $\alpha = 0$ or $\beta = 0$ then Boundary Value Problem I reduces to two independent problems in the cylinders Q^- and Q^+; estimate (16) obviously holds for these problems.

Estimate (16) valid for all numbers α and β such that $\alpha\beta \ge 0$ implies that Boundary Value Problem I has at most one solution in V_0.

The theorem is proven.

Let us proceed to the study of solvability of Boundary Value Problem I.

Theorem 2. *Assume* (5), (14), (15) *and the additional condition*

$$c(x,t,a) \in C^1(\overline{Q}). \tag{14$'$}$$

Then, for every function $f(x,t,a)$ such that $f(x,t,a) \in L_2(Q)$, $f_t(x,t,a) \in L_2(Q^-)$, $f_a(x,t,a) \in L_2(Q^-)$, $f_t(x,t,a) \in L_2(Q^+)$, $f_a(x,t,a) \in L_2(Q^+)$, $f(x,T,a) = 0$ for $(x,a) \in (-1,0)\times(0,A)$, $f(x,0,a) = 0$ for $(x,a) \in (0,1)\times(0,A)$, and $f(x,t,0) = 0$ for $(x,t) \in [(-1,0)\times(0,T)]\cup[(0,1)\times(0,T)]$, Boundary Value Problem I *has a solution $u(x,t,a)$ in V_0.*

Proof. We reduce the conjugation problem to auxiliary boundary value problems for "loaded" equations (see, for instance, [24, 25] and [19, 20]).

Let G be a bounded domain with $(x,t,a) \in G$. Denote

$$V_1(G) = \{v(x,t,a) : v(x,t,a) \in V_0(G),\ v_{xxt}(x,t,a) \in L_2(G),\ v_{xxa}(x,t,a) \in L_2(G)\},$$
$$V_2(G) = \{v(x,t,a) : v(x,t,a) \in V_1(G),\ v_{xxtt}(x,t,a) \in L_2(G),\ v_{xxaa}(x,t,a) \in L_2(G)\}.$$

Endow these spaces with the norms

$$\|v\|_{V_1(G)} = \left(\|v\|^2_{V_0(G)} + \|v_{xxt}\|^2_{L_2(G)} + \|v_{xxa}\|^2_{L_2(G)}\right)^{\frac{1}{2}},$$
$$\|v\|_{V_2(G)} = \left(\|v\|^2_{V_1(G)} + \|v_{xxtt}\|^2_{L_2(G)} + \|v_{xxaa}\|^2_{L_2(G)}\right)^{\frac{1}{2}}.$$

In what follows, we assume that $w(x,t,a)$ and $z(x,t,a)$ are functions on Q^+ and Q^-, respectively.

Consider the following problem: *Find* $w(x,t,a)$ *and* $z(x,t,a)$ *such that*

$$h(x)w_t + w_a - w_{xx} + cw + \frac{\alpha(1-x)}{1+\alpha\beta}\{h(x)[\beta w_{xt}(+0,t,a) + z_t(-0,t,a)]$$
$$+[\beta w_{xa}(+0,t,a) + z_a(-0,t,a)] + c[\beta w_x(+0,t,a) + z(-0,t,a)]\} = f(x,t,a), \tag{17}$$
$$h(x)z_t + z_a - z_{xx} + cz + \frac{\beta(1+x)}{1+\alpha\beta}\{h(x)[w_{xt}(+0,t,a) - \alpha z_t(-0,t,a)]$$
$$+[w_{xa}(+0,t,a) - \alpha z_a(-0,t,a)] + c[w_x(+0,t,a) - \alpha z(-0,t,a)]\} = f(x,t,a) \tag{18}$$

in Q^+ *and* Q^- *satisfying the conditions*

$$w(x,t,a)|_{x=+0} = w(x,t,a)|_{x=1} = 0, \quad (t,a) \in (0,T)\times(0,A), \tag{19}$$

$$z_x(x,t,a)|_{x=-0} = z(x,t,a)|_{x=-1} = 0, \quad (t,a) \in (0,T)\times(0,A), \tag{20}$$

$$w(x,t,a)|_{t=0} = 0, \quad (x,a) \in (0,1)\times(0,A), \tag{21}$$

$$w(x,t,a)|_{a=0} = 0, \quad (x,t) \in (0,1)\times(0,T), \tag{22}$$

$$z(x,t,a)|_{t=T} = 0, \quad (x,a) \in (-1,0)\times(0,A), \tag{23}$$

$$z(x,t,a)|_{a=0} = 0, \quad (x,t) \in (-1,0)\times(0,T). \tag{24}$$

Equations (17) and (18) are "loaded." Demonstrate that under the conditions of the theorem problem (17)–(24) has a solution $(w(x,t,a),\, z(x,t,a))$ such that $w(x,t,a) \in V_1(Q^+)$ and $z(x,t,a) \in V_1(Q^-)$. We use regularization and continuation in a parameter.

Let ε be a positive number. Consider the following problem: *Find* $w(x,t,a)$ *and* $z(x,t,a)$ *satisfying*

$$\varepsilon(w_{xxtt} + w_{xxaa}) - w_{xx} + h(x)w_t + w_a + cw + \frac{\alpha(1-x)}{1+\alpha\beta}\{h(x)[\beta w_{xt}(+0,t,a)$$
$$+z_t(-0,t,a)] + [\beta w_{xa}(+0,t,a) + z_a(-0,t,a)]$$
$$+c[\beta w_x(+0,t,a) + z(-0,t,a)]\} = f(x,t,a), \tag{17_ε}$$
$$\varepsilon(z_{xxtt} + z_{xxaa}) - z_{xx} + h(x)z_t + z_a + cz + \frac{\beta(1+x)}{1+\alpha\beta}\{h(x)[w_{xt}(+0,t,a)$$
$$-\alpha z_t(-0,t,a)] + [w_{xa}(+0,t,a) - \alpha z_a(-0,t,a)]$$
$$+c[w_x(+0,t,a) - \alpha z(-0,t,a)]\} = f(x,t,a) \tag{18_ε}$$

in Q^+ *and* Q^-, *respectively,* (19)–(24), *and such that*

$$w_t(x,t,a)|_{t=T} = 0, \quad (x,a) \in (0,1)\times(0,A), \tag{25}$$

$$w_a(x,t,a)|_{a=A} = 0, \quad (x,t) \in (0,1)\times(0,T), \tag{26}$$

$$z_t(x,t,a)|_{t=0} = 0, \quad (x,a) \in (-1,0)\times(0,A), \tag{27}$$

$$z_a(x,t,a)|_{a=A} = 0, \quad (x,t) \in (-1,0)\times(0,T). \tag{28}$$

Show that, for every fixed number ε, there is a solution $(w(x,t,a), z(x,t,a))$ to this problem such that $w(x,t,a) \in V_2(Q^+)$ and $z(x,t,a) \in V_2(Q^-)$ if the conditions of the theorem hold and $f(x,t,a)$ belongs to $L_2(Q)$. We apply continuation in a parameter.

Let λ be a number in $[0,1]$. Consider the following problem: *Find solutions* $w(x,t,a)$ *and* $z(x,t,a)$ *to the equations*

$$\varepsilon(w_{xxtt}+w_{xxaa})-w_{xx}+\lambda\Big\{h(x)w_t+w_a+cw$$
$$+\frac{\lambda\alpha(1-x)}{1+\lambda^2\alpha\beta}[\lambda\beta h[w_{xt}(+0,t,a)+z_t(-0,t,a)]+[\lambda\beta w_{xa}(+0,t,a)$$
$$+z_a(-0,t,a)]+c[\lambda\beta w_x(+0,t,a)+z(-0,t,a)]]\Big\}=f(x,t,a), \qquad (17_{\varepsilon,\lambda})$$

$$\varepsilon(z_{xxtt}+z_{xxaa})-z_{xx}+\lambda\Big\{h(x)z_t+z_a+cz$$
$$+\frac{\lambda\beta(1+x)}{1+\lambda^2\alpha\beta}[h[w_{xt}(+0,t,a)-\lambda\alpha z_t(-0,t,a)]$$
$$+[w_{xa}(+0,t,a)-\lambda\alpha z_a(-0,t,a)]+c[w_x(+0,t,a)$$
$$-\lambda\alpha z(-0,t,a)]]\Big\}=f(x,t,a) \qquad (18_{\varepsilon,\lambda})$$

in Q^+ *and* Q^-, *respectively, satisfying* (19)–(28). Show that the theorem [26, Chapter III, Section 14] on continuation in a parameter is applicable to the family of problems $(17_{\varepsilon,\lambda})$, $(18_{\varepsilon,\lambda})$, (19)–(28).

To this end, it suffices to demonstrate that

(a) the boundary value problem $(17_{\varepsilon,0})$, $(18_{\varepsilon,0})$, (19)–(28) for a fixed ε under the conditions of the theorem and the membership of $f(x,t,a)$ in $L_2(Q^+)$ and $L_2(Q^-)$ has a solution $(w(x,t,a),z(x,t,a))$ such that $w(x,t,a)\in V_2(Q^+)$ and $z(x,t,a)\in V_2(Q^-)$;

(b) the solutions $(w(x,t,a),z(x,t,a))$ to problem $(17_{\varepsilon,\lambda})$, $(18_{\varepsilon,\lambda})$, (19)–(24) such that $w(x,t,a)\in V_2(Q^+)$ and $z(x,t,a)\in V_2(Q^-)$ under the conditions of the theorem and a fixed ε satisfy the a priori estimate

$$\|w\|_{V_2(Q^+)}+\|z\|_{V_2(Q^-)}\le R_0(\|f\|_{L_2(Q^+)}+\|f\|_{L_2(Q^-)}) \qquad (29)$$

with the constant R_0 defined by the functions $h(x)$ and $c(x,t,a)$, the numbers α, β, T, A, and ε.

The validity of (a) is obvious. Verify (b).

Define the function $u(x,t,a)$ by the equalities

$$u(x,t,a)=w(x,t,a)+\frac{\lambda\alpha(1-x)[\lambda\beta w_x(+0,t,a)+z(-0,t,a)]}{1+\lambda^2\alpha\beta},$$

$$u(x,t,a)=z(x,t,a)+\frac{\lambda\beta(1+x)[w_x(+0,t,a)-\lambda\alpha z(-0,t,a)]}{1+\lambda^2\alpha\beta}$$

in Q^+ and Q^-, respectively. This function satisfies the equation

$$\varepsilon(u_{xxtt}+u_{xxaa})-u_{xx}+\lambda[h(x)u_t+u_a+cu]=f(x,t,a) \qquad (30_\lambda)$$

in Q_1, the conditions

$$u(x,t,a)|_{x=+0}=\lambda\alpha u(x,t,a)|_{x=-0},\quad (t,a)\in(0,T)\times(0,A), \qquad (31_\lambda)$$

$$u_x(x,t,a)|_{x=-0}=\lambda\beta u_x(x,t,a)|_{x=+0},\quad (t,a)\in(0,T)\times(0,A), \qquad (32_\lambda)$$

as well as (6)–(9), (21)–(28).

First, we consider the case of $\alpha \neq 0$ and $\beta \neq 0$. Multiply (30_λ) by $-(u_{tt} + u_{aa})$ and integrate over Q^-; next, multiply this equation by $-\gamma(u_{tt} + u_{aa})$ and integrate the result over Q^+. Summing the two equalities obtained and integrating by parts, we infer that

$$\varepsilon \int\limits_{Q^-} \left(u_{xtt}^2 + 2u_{xta}^2 + u_{xaa}^2\right) dxdtda + \varepsilon\gamma \int\limits_{Q^+} \left(u_{xtt}^2 + 2u_{xta}^2 + u_{xaa}^2\right) dxdtda$$

$$+ \int\limits_{Q^-} \left(u_{xt}^2 + u_{xa}^2\right) dxdtda + \gamma \int\limits_{Q^+} \left(u_{xt}^2 + u_{xa}^2\right) dxdtda - \frac{\lambda}{2} \int\limits_0^A \int\limits_{-1}^0 h(x) u_t^2(x, T, a)\, dxda$$

$$-\frac{\lambda}{2} \int\limits_0^A \int\limits_{-1}^0 h(x) u_a^2(x, 0, a)\, dxda + \frac{\lambda}{2} \int\limits_0^T \int\limits_{-1}^0 u_t^2(x, t, A)\, dxdt$$

$$+\frac{\lambda}{2} \int\limits_0^T \int\limits_{-1}^0 u_a^2(x, t, 0)\, dxdt + \frac{\gamma\lambda}{2} \int\limits_0^A \int\limits_0^1 h(x) u_t^2(x, 0, a)\, dxda$$

$$+\frac{\gamma\lambda}{2} \int\limits_0^A \int\limits_0^1 h(x) u_a^2(x, T, a)\, dxda + \frac{\gamma\lambda}{2} \int\limits_0^A \int\limits_0^1 h(x) u_t^2(x, 0, a)\, dxda$$

$$+\frac{\gamma\lambda}{2} \int\limits_0^T \int\limits_0^1 u_t^2(x, t, A)\, dxdt + \frac{\gamma\lambda}{2} \int\limits_0^T \int\limits_0^1 u_a^2(x, t, 0)\, dxdt$$

$$+ \int\limits_{Q^-} c\left(u_t^2 + u_a^2\right) dxdtda + \gamma \int\limits_{Q^+} c\left(u_t^2 + u_a^2\right) dxdtda$$

$$+ \int\limits_{Q^-} (c_t u u_t + c_a u u_a)\, dxdtda + \gamma \int\limits_{Q^+} (c_t u u_t + c_a u u_a)\, dxdtda$$

$$= - \int\limits_{Q^-} f(u_{tt} + u_{aa})\, dxdtda - \gamma \int\limits_{Q^+} f(u_{tt} + u_{aa})\, dxdtda. \tag{33_λ}$$

The Young inequality, estimate (16), the simplest integral inequalities (namely, the estimates of the functions u_{tt} and u_{aa} in L_2 through u_{xtt} and u_{xaa}), and (33_λ) ensure that

$$\int\limits_{Q^-} \left(u_{xtt}^2 + u_{xta}^2 + u_{xaa}^2 + u_{xt}^2 + u_{xa}^2\right) dxdtda + \int\limits_{Q^+} \left(u_{xtt}^2 + u_{xta}^2 + u_{xaa}^2 + u_{xt}^2 + u_{xa}^2\right) dxdtda$$

$$\leq N_1 \biggl(\int\limits_{Q^-} f^2\, dxdtda + \int\limits_{Q^+} f^2\, dxdtda \biggr), \tag{34}$$

where the constant N_1 depends only on α, β, and ε.

Multiplying (30_λ) by $u_{xxtt} + u_{xxaa}$, integrating over the cylinders Q^- and Q^+, and employing the Young inequality and (34), we easily derive that

$$\int\limits_{Q^-} \left(u_{xxtt}^2 + u_{xxta}^2 + u_{xxaa}^2\right) dxdtda + \int\limits_{Q^+} \left(u_{xxtt}^2 + u_{xxta}^2 + u_{xxaa}^2\right) dxdtda$$

$$\leq N_2 \biggl(\int\limits_{Q^-} f^2\, dxdtda + \int\limits_{Q^+} f^2\, dxdtda \biggr), \tag{35_λ}$$

where the constant N_2 depends on $h(x)$, α, β, and ε.

We have

$$\begin{aligned} w(x,t,a) &= u(x,t,a) - \lambda\alpha(1-x)u(-0,t,a), \\ z(x,t,a) &= u(x,t,a) - \lambda\beta(1+x)u_x(+0,t,a). \end{aligned} \tag{36}$$

These equalities, together with (34) and (35) for the function $u(x,t,a)$, validate (29) for $w(x,t,a)$ and $z(x,t,a)$ in the case of $\alpha \neq 0$ and $\beta \neq 0$.

If either $\alpha = 0$ or $\beta = 0$ then one of the equations of $(17_{\varepsilon,\lambda})$, $(18_{\varepsilon,\lambda})$ becomes independent (it does not contain $w(x,t,a)$ or $z(x,t,a)$), and so the required estimate for one of the functions is obvious; as a consequence the second function satisfies the estimate as well.

Hence, for a fixed ε, the solutions $w(x,t,a)$ and $z(x,t,a)$ to $(17_{\varepsilon,\lambda})$, $(18_{\varepsilon,\lambda})$, (19)–(28) under the conditions of the theorem satisfy (29) uniformly in λ. Together with the obvious solvability of $(17_{\varepsilon,0})$, $(18_{\varepsilon,0})$, (19)–(28) in $V_2(Q^+)$ and $V_2(Q^-)$, the last fact means that (17_ε), (18_ε), (19)–(28) has a solution $(w(x,t,a), z(x,t,a))$ such that $w(x,t,a) \in V_2(Q^+)$, $z(x,t,a) \in V_2(Q^-)$.

The above arguments ensure that the family of the problems (17_ε), (18_ε), (19)–(28) gives rise a family of solutions $(w^\varepsilon(x,t,a), z^\varepsilon(x,t,a))$ such that $w^\varepsilon(x,t,a) \in V_2(Q^+)$ and $z^\varepsilon(x,t,a) \in V_2(Q^-)$. Given $(w^\varepsilon(x,t,a), z^\varepsilon(x,t,a))$, demonstrate that we can extract a sequence that converges to a solution $(w(x,t,a), z(x,t,a))$ to (17)–(24).

First of all, we establish that the family of solutions to (17_ε), (18_ε), (19)–(28) satisfies a priori estimates uniformly in ε provided that $f(x,t,a)$ meets the conditions of the theorem.

We omit the index ε for solutions to (17_ε), (18_ε), (19)–(28) in the proof of a priori estimates below.

As before, we first consider the case of $\alpha \neq 0$ and $\beta \neq 0$.

Given a solution $(w(x,t,a), z(x,t,a))$ to (17_ε), (18_ε), (19)–(28), define $u(x,t,a)$ by the equalities

$$\begin{aligned} u(x,t,a) &= w(x,t,a) + \frac{\alpha(1-x)[\beta w_x(+0,t,a) + z(-0,t,a)]}{1+\alpha\beta}, \\ u(x,t,a) &= z(x,t,a) + \frac{\beta(1+x)[w_x(+0,t,a) - \alpha z(-0,t,a)]}{1+\alpha\beta} \end{aligned} \tag{37}$$

in Q^+ and Q^- respectively. The function $u(x,t,a)$ satisfies (30_1). Equality (33_1) corresponds to this equation. Integrate by parts on the right-hand side of this equality. Applying the Young inequality and the simplest integral inequalities, we find that the function $u(x,t,a)$ determined by a solution $(w(x,t,a), z(x,t,a))$ to (17_ε), (18_ε), (19)–(28) satisfies the estimate

$$\begin{aligned} \varepsilon \int\limits_{Q^-} \left(u_{xtt}^2 + u_{xta}^2 + u_{xaa}^2\right) dxdtda + \varepsilon \int\limits_{Q^+} \left(u_{xtt}^2 + u_{xta}^2 + u_{xaa}^2\right) dxdtda \\ + \int\limits_{Q^-} \left(u_{xt}^2 + u_{xa}^2\right) dxdtda + \int\limits_{Q^+} \left(u_{xt}^2 + u_{xa}^2\right) dxdtda \\ \leq M_1 \Bigg[\int\limits_{Q^-} \left(f_t^2 + f_a^2\right) dxdtda + \int\limits_{Q^+} \left(f_t^2 + f_a^2\right) dxdtda \Bigg], \end{aligned} \tag{38}$$

where the constant M_1 depends only on α and β.

Multiply (30_1) by $-u_{xx}(x,t,a)$ and integrate over the cylinders Q^- and Q^+. Summing the two equalities obtained, we infer

$$\varepsilon \int\limits_{Q^-} \left(u_{xxt}^2 + u_{xxa}^2\right) dxdtda + \varepsilon \int\limits_{Q^+} \left(u_{xtt}^2 + u_{xxa}^2\right) dxdtda$$
$$+ \int\limits_{Q^-} u_{xx}^2 \, dxdtda + \int\limits_{Q^+} u_{xx}^2 \, dxdtda$$
$$= \int\limits_{Q^-} (hu_t + u_a + cu - f)u_{xx} \, dxdtda + \int\limits_{Q^+} (hu_t + u_a + cu - f)u_{xx} \, dxdtda.$$

This relation, the Young inequality, and (38) yield

$$\varepsilon \int\limits_{Q^-} \left(u_{xxt}^2 + u_{xxa}^2\right) dxdtda + \varepsilon \int\limits_{Q^+} \left(u_{xtt}^2 + u_{xxa}^2\right) dxdtda$$
$$+ \int\limits_{Q^-} u_{xx}^2 \, dxdtda + \int\limits_{Q^+} u_{xx}^2 \, dxdtda$$
$$\leq M_2 \biggl(\int\limits_{Q^-} \left(f^2 + f_t^2 + f_a^2\right) dxdtda + \int\limits_{Q^+} \left(f^2 + f_t^2 + f_a^2\right) dxdtda \biggr), \tag{39}$$

where the constant M_2 is defined by $c(x,t,a)$, $h(x)$, α, and β.

The last estimate

$$\varepsilon^2 \int\limits_{Q^-} \left(u_{xxtt}^2 + u_{xxaa}^2\right) dxdtda + \varepsilon^2 \int\limits_{Q^+} \left(u_{xxtt}^2 + u_{xxaa}^2\right) dxdtda$$
$$\leq M_3 \biggl(\int\limits_{Q^-} \left(f^2 + f_t^2 + f_a^2\right) dxdtda + \int\limits_{Q^+} \left(f^2 + f_t^2 + f_a^2\right) dxdtda \biggr) \tag{40}$$

is an obvious consequence of (30_1), (38), and (39); the constant M_3 in this estimate depends on $c(x,t,a)$, $h(x)$, α, and β.

In the case of $\alpha = 0$ or$\beta = 0$, the validity of (38)–(40) is obvious.

The above estimates ensure passage to the limit. Note that it suffices to justify a possibility of passage to the limit for the family $\{u^\varepsilon(x,t,a)\}$.

Choose a sequence $\{\varepsilon_n\}$ of positive numbers such that $\varepsilon_n \to 0$ as $n \to \infty$. To every ε_n, there correspond the function $u_n(x,t,a)$ that is a solution to the equation

$$\varepsilon_n(u_{nxxtt} + u_{nxxaa}) - u_{nxx} + hu_{nt} + u_{na} = f(x,t,a)$$

on Q_1 satisfying (10), (11), and (19)–(28). The family $\{u_n(x,t,a)\}$ satisfies (38)–(40) uniformly in n. These estimates and the reflexivity of Hilbert space imply that there exist sequences $\{\varepsilon_{n_k}\}$, $\{u_{n_k}(x,t,a)\}$, and a function $u(x,t,a)$ such that

$$\varepsilon_{n_k} \to 0,$$
$$u_{n_k xx}(x,t,a) \to u_{xx}(x,t,a) \quad \text{weakly in } L_2(Q^-) \text{ and } L_2(Q^+),$$
$$u_{n_k t}(x,t,a) \to u_t(x,t,a) \quad \text{weakly in } L_2(Q^-) \text{ and } L_2(Q^+),$$
$$u_{n_k a}(x,t,a) \to u_a(x,t,a) \quad \text{weakly in } L_2(Q^-) \text{ and } L_2(Q^+),$$
$$\varepsilon_{n_k} u_{n_k xxtt}(x,t,a) \to 0 \quad \text{weakly in } L_2(Q^-) \text{ and } L_2(Q^+),$$
$$\varepsilon_{n_k} u_{n_k xxaa}(x,t,a) \to 0 \quad \text{weakly in } L_2(Q^-) \text{ and } L_2(Q^+)$$

as $k \to \infty$. These convergences imply that the limit function $u(x,t,a)$ satisfies (1). It is obvious also that $u(x,t,a)$ belongs to V_0 and satisfies (6)–(9), (10), and (11). Hence, $u(x,t,a)$ is a solution to Boundary Value Problem I.

The theorem is proven.

The existence and uniqueness theorems similar to Theorems 1 and 2 are valid for Boundary Value Problem II as well.

Theorem 3. *Let conditions* (5), (14), (14′), *and* (15) *hold. Then Boundary Value Problem* II *has at most one solution in* V_0.

Proof. Demonstration is similar to that of Theorem 1.

Theorem 4. *Assume that conditions* (5) *and* (14)–(16) *hold. Then, for every function* $f(x,t,a)$ *such that* $f(x,t,a) \in L_2(Q)$, $f_t(x,t,a) \in L_2(Q^-)$, $f_a(x,t,a) \in L_2(Q^-)$, $f_t(x,t,a) \in L_2(Q^+)$, $f_a(x,t,a) \in L_2(Q^+)$, $f(x,T,a) = 0$ *for* $(x,a) \in (-1,0) \times (0,A)$, $f(x,0,a) = 0$ *for* $(x,a) \in (0,1) \times (0,A)$, $f(x,t,A) = 0$ *for* $(x,t) \in (-1,0) \times (0,T)$, *and* $f(x,t,0) = 0$ *for* $(0,1) \times (0,T)$, *Boundary Value Problem* II *has a solution* $u(x,t,a)$ *in* V_0.

Proof. Let ε be a positive number. Consider the following problem: *Find solutions* $w(x,t,a)$ *and* $z(x,t,a)$ *to the equations*

$$\varepsilon(w_{xxtt} + w_{xxaa}) - w_{xx} + h(x)[w_t + w_a] + cw + \frac{\alpha(1-x)}{1+\alpha\beta}\{h(x)[\beta w_{xt}(+0,t,a) + z_t(-0,t,a)]$$
$$+h(x)[\beta w_{xa}(+0,t,a) + z_a(-0,t,a)] + c[\beta w_x(+0,t,a) + z(-0,t,a)]\} = f(x,t,a), \tag{41_ε}$$
$$\varepsilon(z_{xxtt} + z_{xxaa}) - z_{xx} + h(x)[z_t + z_a] + cz + \frac{\beta(1+x)}{1+\alpha\beta}\{h(x)[w_{xt}(+0,t,a) - \alpha z_t(-0,t,a)]$$
$$+h(x)[w_{xa}(+0,t,a) - \alpha z_a(-0,t,a)] + c[w_x(+0,t,a) - \alpha z(-0,t,a)]\} = f(x,t,a) \tag{42_ε}$$

in Q^+ *and* Q^-, *respectively, satisfying* (19)–(23), (25)–(27), *and the conditions*

$$z(x,t,a)|_{a=A} = 0, \quad (x,t) \in (-1,0) \times (0,T), \tag{43}$$

$$z_a(x,t,a)|_{a=0} = 0 \quad (x,t) \in (-1,0) \times (0,T). \tag{44}$$

It is easy to establish by continuation in a parameter (with the use of an equation of the form $(17_{\varepsilon,\lambda})$, $(18_{\varepsilon,\lambda})$) that (41_ε), (42_ε), (19)–(23), (25)–(27), (43), (44) for a fixed ε has a solution $(w(x,t,a), z(x,t,a))$ such that $w(x,t,a) \in V_2(Q^+)$ and $z(x,t,a) \in V_2(Q^-)$ provided that $f(x,t,a)$ belongs to $L_2(Q)$. Next, repeating the proof of (38)–(40), it is easy to show that the family $\{u^\varepsilon(x,t,a)\}$ of functions constructed with the use of $w^\varepsilon(x,t,a)$ and $z^\varepsilon(x,t,a)$ by means of (36) and (37) contains a sequence that converges to a solution to Boundary Value Problem II. The solution belongs to the required class.

The theorem is proven.

2. Boundary Value Problems for Forward-Backward Quasiultraparabolic Equations

We study boundary value problems for (3) and (4) with different boundary conditions in the variable a. The scheme of the study of these problems is similar to that for Boundary Value Problems I and II. More exactly, the boundary value problems in question are reduced to auxiliary problems for the functions $w(x,t,a)$ and $z(x,t,a)$ whose solvability is established by regularization and continuation in a parameter, while the passage to the limit in the parameter of regularization is realized with the help of "appropriate" a priori estimates.

Let us proceed to a detailed exposition of the results.

In what follows, (5) is assumed fulfilled.

Boundary Value Problem III. *Find a solution $u(x,t,a)$ to* (3) *in Q_1 satisfying the boundary conditions*

$$u(x,t,a)|_{x=-1} = u(x,t,a)|_{x=1} = 0, \quad (t,a) \in (0,T) \times (0,A), \tag{45}$$

$$u(x,t,a)|_{t=T} = 0, \quad (x,a) \in (-1,0) \times (0,A), \tag{46}$$

$$u(x,t,a)|_{t=0} = 0, \quad (x,a) \in (0,1) \times (0,A), \tag{47}$$

$$u(x,t,a)|_{a=0} = u_a(x,t,a)|_{a=0} = 0, \quad (x,t) \in [(-1,0) \times (0,T)] \cup [(0,1) \times (0,T)], \tag{48}$$

$$u(x,t,a)|_{a=A} = 0, \quad (x,t) \in [(-1,0) \times (0,T)] \cup [(0,1) \times (0,T)], \tag{49}$$

and the conjugation conditions (10) *and* (11).

Boundary Value Problem IV. *Find a solution $u(x,t,a)$ to* (3) *in Q_1 satisfying* (45)–(48), (10), (11), *and the condition*

$$u_{aa}(x,t,a)|_{a=A} = 0, \quad (x,t) \in [(-1,0) \times (0,T)] \cup [(0,1) \times (0,T)]. \tag{50}$$

Boundary Value Problem V. *Find a solution $u(x,t,a)$ to* (4) *in Q_1 satisfying* (45)–(47), (10), (11), *and the conditions*

$$u(x,t,a)|_{a=0} = u_a(x,t,a)|_{a=0} = u(x,t,a)|_{a=A} = 0, \quad (x,t) \in (0,1) \times (0,T), \tag{51}$$

$$u(x,t,a)|_{a=0} = u(x,t,a)|_{a=A} = u_a(x,t,a)|_{a=A} = 0, \quad (x,t) \in (-1,0) \times (0,T). \tag{52}$$

Boundary Value Problem VI. *Find a solution $u(x,t,a)$ to* (4) *in Q_1 satisfying* (45)–(47), (10), (11), *and the conditions*

$$u(x,t,a)|_{a=0} = u_a(x,t,a)|_{a=0} = u_{aa}(x,t,a)|_{a=A} = 0, \quad (x,t) \in (0,1) \times (0,T), \tag{53}$$

$$u_{aa}(x,t,a)|_{a=0} = u(x,t,a)|_{a=A} = u_a(x,t,a)|_{a=A} = 0, \quad (x,t) \in (-1,0) \times (0,T). \tag{54}$$

Define the spaces $V_{0,1}$, $V_{1,1}$, and $V_{2,1}$ and the norms on them as follows:

$$V_{0,1} = \{v(x,t,a) : v(x,t,a) \in V_0,\ v_{aaa}(x,t,a) \in L_2(Q^-),\ v_{aaa}(x,t,a) \in L_2(Q^+)\},$$
$$V_{1,1} = \{v(x,t,a) : v(x,t,a) \in V_{0,1},\ v_{xxt}(x,t,a) \in L_2(Q^-),\ v_{xxt}(x,t,a) \in L_2(Q^+),$$
$$v_{xxaaa}(x,t,a) \in L_2(Q^-),\ v_{xxaaa}(x,t,a) \in L_2(Q^+)\},$$
$$V_{2,1} = \{v(x,t,a) : v(x,t,a) \in V_{1,1},\ v_{xxtt}(x,t,a) \in L_2(Q^-),$$
$$v_{xxtt}(x,t,a) \in L_2(Q^+),\ v_{xxaaaaaa}(x,t,a) \in L_2(Q^-),\ v_{xxaaaaaa}(x,t,a) \in L_2(Q^+)\},$$

$$\|v\|_{V_{0,1}} = \left(\|v\|_{V_0}^2 + \int\limits_{Q^-} v_{aaa}^2\, dxdtda + \int\limits_{Q^+} v_{aaa}^2\, dxdtda \right)^{\frac{1}{2}},$$

$$\|v\|_{V_{1,1}} = \left(\|v\|_{V_{0,1}}^2 + \int\limits_{Q^-} (v_{xxt}^2 + v_{xxaaa}^2)\, dxdtda + \int\limits_{Q^+} (v_{xxt}^2 + v_{xxaaa}^2)\, dxdtda \right)^{\frac{1}{2}},$$

$$\|v\|_{V_{2,1}} = \left(\|v\|_{V_{1,1}}^2 + \int\limits_{Q^-} (v_{xxtt}^2 + v_{xxaaaaaa}^2)\, dxdtda + \int\limits_{Q^+} (v_{xxtt}^2 + v_{xxaaaaaa}^2)\, dxdtda \right)^{\frac{1}{2}}.$$

Obviously, $V_{0,1}$, $V_{1,1}$, and $V_{2,1}$ with the above norms are Banach spaces.

The proofs of uniqueness of solutions to Boundary Value Problems III–VI coincide and we state only one uniqueness theorem.

Theorem 5. *Let* (5), (14), *and* (15) *hold. Then each of the Problems* III–VI *has at most one solution in* $V_{0,1}$.

Proof. Demonstration is similar to that of Theorem 1.

Theorem 6. *Let* (5), (14), (15), *and the condition*

$$c(x,t,a) \in C^3(\overline{Q}) \tag{14''}$$

hold. Then, for every function $f(x,t,a)$ *such that* $f(x,t,a) \in L_2(Q)$, $f_t(x,t,a) \in L_2(Q^-)$, $f_a(x,t,a) \in L_2(Q^-)$, $f_{aa}(x,t,a) \in L_2(Q^-)$, $f_{aaa}(x,t,a) \in L_2(Q^-)$, $f_t(x,t,a) \in L_2(Q^+)$, $f_a(x,t,a) \in L_2(Q^+)$, $f_{aa}(x,t,a) \in L_2(Q^+)$, $f_{aaa}(x,t,a) \in L_2(Q^+)$, $f(x,T,a) = 0$ *for* $(x,a) \in (-1,0) \times (0,A)$, $f(x,0,a) = 0$ *for* $(x,a) \in (0,1) \times (0,A)$, *and* $f(x,t,0) = f(x,t,A) = f_a(x,t,0) = 0$ *for* $(x,t) \in [(-1,0) \times (0,T)] \cup [(0,1) \times (0,T)]$, *Boundary Value Problem* III *has a solution* $u(x,t,a)$ *such that* $u(x,t,a) \in V_{0,1}$.

Proof. Consider the following problem: *Find a solutions* $w(x,t,a)$ *and* $z(x,t,a)$ *to the equations*

$$\begin{aligned} h(x)w_t - w_{aaa} - w_{xx} + cw + \frac{\alpha(1-x)h(x)}{1+\alpha\beta}[\beta w_{xt}(+0,t,a) + z_t(-0,t,a)] \\ -\frac{\alpha(1-x)}{1+\alpha\beta}[\beta w_{xaaa}(+0,t,a) + z_{aaa}(-0,t,a)] \\ +c[\beta w_x(+0,t,a) + z(-0,t,a)] = f(x,t,a), \end{aligned} \tag{55}$$

$$\begin{aligned} h(x)z_t - z_{aaa} - z_{xx} + cz + \frac{\beta(1+x)h(x)}{1+\alpha\beta}[w_{xt}(+0,t,a) - \alpha z_t(-0,t,a)] \\ -\frac{\beta(1+x)}{1+\alpha\beta}[w_{xaaa}(+0,t,a) - \alpha z_{aaa}(-0,t,a)] \\ +c[w_x(+0,t,a) - \alpha z(-0,t,a)] = f(x,t,a) \end{aligned} \tag{56}$$

in Q^+ *and* Q^-, *respectively, satisfying the conditions*

$$w(+0,t,a) = w(1,t,a) = 0, \quad (t,a) \in (0,T) \times (0,A), \tag{57}$$

$$z_x(-0,t,a) = z(-1,t,a) = 0, \quad (t,a) \in (0,T) \times (0,A), \tag{58}$$

$$w(x,0,a) = 0, \quad (x,a) \in (0,1) \times (0,A), \tag{59}$$

$$w(x,t,0) = w_a(x,t,0) = w(x,t,A) = 0, \quad (x,t) \in (0,1) \times (0,T), \tag{60}$$

$$z(x,T,a) = 0, \quad (x,a) \in (-1,0) \times (0,A), \tag{61}$$

$$z(x,t,0) = z_a(x,t,0) = z(x,t,A) = 0, \quad (x,t) \in (-1,0) \times (0,T). \tag{62}$$

To prove solvability of this problem, we employ regularization and continuation in a parameter.

Let ε be a positive number. Consider the following problem: *Find functions* $w(x,t,a)$ *and* $z(x,t,a)$ *satisfying the equations*

$$\begin{aligned} \varepsilon(w_{xxtt} + w_{xxaaaaaa}) - w_{xx} + h(x)w_t - w_{aaa} + cw + \frac{\alpha(1-x)h(x)}{1+\alpha\beta}[\beta w_{xt}(+0,t,a) \\ +z_t(-0,t,a)] - \frac{\alpha(1-x)}{1+\alpha\beta}[\beta w_{xaaa}(+0,t,a) + z_{aaa}(-0,t,a)] \\ +c[\beta w_x(+0,t,a) + z(-0,t,a)] = f(x,t,a), \end{aligned} \tag{55_ε}$$

$$\begin{aligned} \varepsilon(z_{xxtt} + z_{xxaaaaaa}) - z_{xx} + h(x)z_t - z_{aaa} + cz + \frac{\beta(1+x)h(x)}{1+\alpha\beta}[w_{xt}(+0,t,a) \\ -\alpha z_t(-0,t,a)] - \frac{\beta(1+x)}{1+\alpha\beta}[w_{xaaa}(+0,t,a) - \alpha z_{aaa}(-0,t,a)] \\ +c[w_x(+0,t,a) - \alpha z(-0,t,a)] = f(x,t,a), \end{aligned} \tag{56_ε}$$

in Q^+ and Q^-, respectively, (57)–(62), *and also the conditions*

$$w_t(x,T,a)=0, \quad (x,a)\in(0,1)\times(0,A), \tag{63}$$

$$w_{aaa}(x,t,0)=w_{aaa}(x,t,A)=w_{aaaa}(x,t,A)=0, \quad (x,t)\in(0,1)\times(0,T), \tag{64}$$

$$z_t(x,0,a)=0, \quad (x,a)\in(-1,0)\times(0,A), \tag{65}$$

$$z_{aaa}(x,t,0)=z_{aaa}(x,t,A)=z_{aaaa}(x,t,A)=0, \quad (x,t)\in(-1,0)\times(0,T). \tag{66}$$

Solvability of this problem is established by continuation in a parameter. Since all arguments are similar to those in the proof of existence of a solution to problem (17_ε), (18_ε), (19)–(28), we demonstrate only how to derive necessary a priori estimates.

Define the function $u(x,t,a)$ by the equalities

$$u(x,t,a)=w(x,t,a)+\frac{\alpha(1-x)[\beta w_x(+0,t,a)+z(-0,t,a)]}{1+\alpha\beta},$$

$$u(x,t,a)=z(x,t,a)+\frac{\beta(1+x)[w_x(+0,t,a)-\alpha z(-0,t,a)]}{1+\alpha\beta}$$

in Q^+ and Q^-, respectively. The function satisfies the equation

$$\varepsilon(u_{xxtt}+u_{xxaaaaaa})-u_{xx}+h(x)u_t-u_{aaa}=f(x,t,a) \tag{67}$$

in Q_1, as well as (10), (11), and (45)–(49), (63)–(66).

Assume that $\alpha\neq 0$ and $\beta\neq 0$. Multiply (67) by $-u_{tt}-u_{aaaaaa}$ and integrate over Q^-; next, multiply this equation by $\gamma(-u_{tt}-u_{aaaaaa})$ and integrate over Q^+. Summing the results, integrating by parts, and applying the Young inequality, we see that solutions to (67), (10), (11), (45)–(49), (57)–(62) satisfy the first a priori estimate

$$\begin{aligned}\int\limits_{Q^-}&\left(u_{xtt}^2+u_{xtaaa}^2+u_{xaaaaaa}^2+u_{xt}^2+u_{xaaa}^2\right)dxdtda\\ &+\int\limits_{Q^+}\left(u_{xtt}^2+u_{xtaaa}^2+u_{xaaaaaa}^2+u_{xt}^2+u_{xaaa}^2\right)dxdtda\\ &\qquad\le N_3\Bigg(\int\limits_{Q^-}f^2\,dxdtda+\int\limits_{Q^+}f^2\,dxdtda\Bigg),\end{aligned} \tag{68}$$

where the constant N_3 depends only on α, β, and ε.

The second a priori estimate is derived after the multiplication of (67) by $u_{xxtt}+u_{xxaaaaaa}$ and the integration; this estimate is of the form

$$\begin{aligned}\int\limits_{Q^-}\left(u_{xxtt}^2+u_{xxtaaa}^2+u_{xxaaaaaa}^2\right)dxdtda&+\int\limits_{Q^+}\left(u_{xxtt}^2+u_{xxtaaa}^2+u_{xxaaaaaa}^2\right)dxdtda\\ &\le N_4\Bigg(\int\limits_{Q^-}f^2\,dxdtda+\int\limits_{Q^+}f^2\,dxdtda\Bigg),\end{aligned} \tag{69}$$

where the constant N_4 depends only on $h(x)$, α, β, and ε.

The validity of (68) and (69) in the cases of $\alpha=0$ or $\beta=0$ is obvious.

Estimates (68) and (69) for $u(x,t,a)$ imply similar estimates for the functions $w(x,t,a)$ and $z(x,t,a)$. In turn, the estimates for $w(x,t,a)$ and $z(x,t,a)$ and continuation in a parameter ensure that, for a fixed ε and $f(x,t,a)$ from $L_2(Q)$, (55_ε), (56_ε), (57)–(66) has a solution $(w(x,t,a), z(x,t,a))$ such that the corresponding function $u(x,t,a)$ belongs to $V_{2,1}$ and is a solution to the problem (67), (45)–(49), (63)–(66). Verify that under additional conditions on $f(x,t,a)$ the function $u(x,t,a)$ has a priori estimates uniformly in ε.

As before, we first consider the case of $\alpha \neq 0$ and $\beta \neq 0$.

Multiply (67) by $-u_{tt} - u_{aaaaaa}$ and integrate over the cylinder Q^-; next, multiply this equation by $\gamma(-u_{tt} - u_{aaaaaa})$ and integrate over the cylinder Q^+; summing the results, integrating by parts, and applying the Young inequality and integral inequalities, we obtain the estimate

$$\varepsilon \int\limits_{Q^-} \left(u_{xtt}^2 + u_{xtaaa}^2 + u_{xaaaaaa}^2\right) dxdtda + \varepsilon \int\limits_{Q^+} \left(u_{xtt}^2 + u_{xtaaa}^2 + u_{xaaaaaa}^2\right) dxdtda$$
$$+ \int\limits_{Q^-} \left(u_{xt}^2 + u_{xaaa}^2\right) dxdtda + \int\limits_{Q^+} \left(u_{xt}^2 + u_{xaaa}^2\right) dxdtda$$
$$\leq M_4 \Bigg[\int\limits_{Q^-} \left(f^2 + f_t^2 + f_{aaa}^2\right) dxdtda + \int\limits_{Q^+} \left(f^2 + f_t^2 + f_{aaa}^2\right) dxdtda\Bigg], \tag{70}$$

where the constant M_4 depends only on α and β.

The estimate

$$\varepsilon^2 \int\limits_{Q^-} (u_{xxtt}^2 + u_{xxaaaaaa}^2)\, dxdtda + \varepsilon^2 \int\limits_{Q^+} (u_{xxtt}^2 + u_{xxaaaaaa}^2)\, dxdtda$$
$$\leq M_5 \Bigg(\int\limits_{Q^-} (f^2 + f_t^2 + f_{aaa}^2)\, dxdtda + \int\limits_{Q^+} (f^2 + f_t^2 + f_{aaa}^2)\, dxdtda\Bigg) \tag{71}$$

is an obvious consequence of (70); the constant M_5 here depends only on $h(x)$, α, and β.

Estimates (70) and (71) allows us to organize the passage to the limit (see the arguments at the end of the proof of Theorem 2). The limit function $u(x,t,a)$ belongs to $V_{1,1}$ and is a solution to Boundary Value Problem III.

The theorem is proven.

Theorem 7. *Let* (5), (14), (14″), *and* (15) *hold. Then, for every function* $f(x,t,a)$ *such that* $f(x,t,a) \in L_2(Q)$, $f_t(x,t,a) \in L_2(Q^-)$, $f_a(x,t,a) \in L_2(Q^-)$, $f_{aa}(x,t,a) \in L_2(Q^-)$, $f_{aaa}(x,t,a) \in L_2(Q^-)$, $f_t(x,t,a) \in L_2(Q^+)$, $f_a(x,t,a) \in L_2(Q^+)$, $f_{aa}(x,t,a) \in L_2(Q^+)$, $f_{aaa}(x,t,a) \in L_2(Q^+)$, $f(x,T,a) = 0$ *for* $(x,a) \in (-1,0) \times (0,A)$, $f(x,0,a) = 0$ *for* $(x,a) \in (0,1) \times (0,A)$, *and* $f(x,t,0) = f_a(x,t,0) = f_{aa}(x,t,A) = 0$ *for* $(x,t) \in [(-1,0) \times (0,T)] \cup [(0,1) \times (0,T)]$, *Boundary Value Problem* IV *has a solution* $u(x,t,a)$ *such that* $u(x,t,a) \in V_{0,1}$.

Proof. Given $\varepsilon > 0$, consider the following problem: *Find functions* $w(x,t,a)$ *and* $z(x,t,a)$ *satisfying* (55_ε) *and* (56_ε) *in* Q^+ *and* Q^-, *respectively, and the boundary conditions*

$$w(+0,t,a) = w(1,t,a) = 0, \quad (t,a) \in (0,T) \times (0,A),$$
$$z_x(-0,t,a) = z(-1,t,a) = 0, \quad (t,a) \in (0,T) \times (0,A),$$
$$w(x,0,a) = 0, \quad (x,a) \in (0,1) \times (0,A),$$
$$w_t(x,T,a) = 0, \quad (x,a) \in (0,1) \times (0,A),$$

$$w(x,t,0) = w_a(x,t,0) = w_{aaa}(x,t,0) = w_{aa}(x,t,A) = w_{aaaa}(x,t,A)$$
$$= w_{aaaaa}(x,t,A) = 0, \quad (x,t) \in (0,1) \times (0,T),$$
$$z_t(x,0,a) = 0, \quad (x,a) \in (-1,0) \times (0,A),$$
$$z(x,t,0) = z_a(x,t,0) = z_{aaa}(x,t,0) = z_{aa}(x,t,A) = z_{aaaa}(x,t,A)$$
$$= z_{aaaaa}(x,t,A) = 0, \quad (x,t) \in (-1,0) \times (0,T).$$

The existence of a solution $(w(x,t,a), z(x,t,a))$ to this problem such that $w(x,t,a) \in V_{2,1}(Q^+)$ and $z(x,t,a) \in V_{2,1}(Q^-)$ is established by continuation in a parameter. Given functions $w(x,t,a)$ and $z(x,t,a)$, construct a function $u(x,t,a)$ as before and show that this function satisfies a priori estimates uniformly in ε under the conditions of the theorem for $f(x,t,a)$. These estimates allow us to justify that we can pass to the limit and the limit function is a solution to Boundary Value Problem IV.

The theorem is proven.

Theorem 8. *Let* (5), (14), (14″), *and* (15) *hold. Then, for every function* $f(x,t,a)$ *such that* $f(x,t,a) \in L_2(Q)$, $f_t(x,t,a) \in L_2(Q^-)$, $f_a(x,t,a) \in L_2(Q^-)$, $f_{aa}(x,t,a) \in L_2(Q^-)$, $f_{aaa}(x,t,a) \in L_2(Q^-)$, $f_t(x,t,a) \in L_2(Q^+)$, $f_a(x,t,a) \in L_2(Q^+)$, $f_{aa}(x,t,a) \in L_2(Q^+)$, $f_{aaa}(x,t,a) \in L_2(Q^+)$, $f(x,T,a) = 0$ *for* $(x,a) \in (-1,0) \times (0,A)$, $f(x,0,a) = 0$ *for* $(x,a) \in (0,1) \times (0,A)$, $f(x,t,0) = f_a(x,t,0) = f(x,t,A) = 0$ *for* $(x,t) \in (0,1) \times (0,T)$, *and* $f(x,t,0) = f(x,t,A) = f_a(x,t,A) = 0$ *for* $(x,t) \in (-1,0) \times (0,T)$, *Boundary Value Problem* V *has a solution* $u(x,t,a)$ *such that* $u(x,t,a) \in V_{0,1}$.

Theorem 9. *Let* (5), (14), (14″), *and* (15) *hold. Then, for every function* $f(x,t,a)$ *such that* $f(x,t,a) \in L_2(Q)$, $f_t(x,t,a) \in L_2(Q^-)$, $f_a(x,t,a) \in L_2(Q^-)$, $f_{aa}(x,t,a) \in L_2(Q^-)$, $f_{aaa}(x,t,a) \in L_2(Q^-)$, $f_t(x,t,a) \in L_2(Q^+)$, $f_a(x,t,a) \in L_2(Q^+)$, $f_{aa}(x,t,a) \in L_2(Q^+)$, $f_{aaa}(x,t,a) \in L_2(Q^+)$, $f(x,T,a) = 0$ *for* $(x,a) \in (-1,0) \times (0,A)$, $f(x,0,a) = 0$ *for* $(x,a) \in (-1,0) \times (0,A)$, $f(x,t,A) = f_a(x,t,A) = f_{aa}(x,t,0) = 0$ *for* $(x,t) \in (-1,0) \times (0,T)$, *and* $f(x,t,0) = f_a(x,t,0) = f_{aa}(x,t,A) = 0$ *for* $(x,t) \in (0,1) \times (0,T)$, *Boundary Value Problem* VI *has a solution* $u(x,t,a)$ *such that* $u(x,t,a) \in V_{0,1}$.

Proof. Demonstrations of Theorems 8 and 9 proceed along the lines of Theorems 6 and 7: we reduce the problem to a system of "loaded" equations for $w(x,t,a)$ and $z(x,t,a)$, employ the regularization of the system obtained by a system of the form (55_ε), (56_ε), apply continuation in a parameter, and pass to the limit.

3. Remarks and Complements

1. Boundary conditions (6) (or (45)) can be replaced with the conditions of the second, third, or mixed boundary value problems, we note only that in the case of the second boundary value problem we need the additional condition
$$c(x,t,a) \geq c_0 > 0 \quad \text{for } (x,t,a) \in \overline{Q}.$$

2. Conjugation condition (11) can be replaced with the condition
$$u_x(x,t,a)|_{x=-0} = \beta u_x(x,t,a)|_{x=+0} + \gamma u(x,t,a)|_{x=+0}. \tag{11′}$$
Moreover, the coefficients α and β in (10) and (11) and α, β, and γ in (10) and (11′) can be functions of t and a. The corresponding conditions on $\alpha(t,a)$, $\beta(t,a)$, and $\gamma(t,a)$ are easy to point out.

3. The symmetric segment $[-1,1]$, which is the domain of variation of x, can be replaced with the segment $[d_1, d_2]$, where $d_1 < 0$ and $d_2 > 0$.

4. It is possible to add lower-order terms to (1)–(4), for example, the summands $b(x,t)u_x$ and $c_1(x,t,a)u_{aa}$ can be added to equations (1), (2) and (3), (4), respectively, and so on. The additional conditions are easy to describe.

5. Forward-backward differential equations with two time variables t and a (for example, t is the astronomic time and a (age) is a biological time- or the age), arise in biology, neutron transport, and in the description of counter flows of gases (see, for instance, [6, 7, 24, 27]).

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