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Maps preserving $A^*A + AA^*$ on C^*-algebras

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Abstract. Let $\mathcal{A}$ be a C^*-algebra of real-rank zero and $\mathcal{B}$ be a C^*-algebra with unit I. It is shown that the mapping $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ which preserves arithmetic mean and satisfies

$$\Phi(A^*A) = \frac{\Phi(A)^*\Phi(A) + \Phi(A)\Phi(A)^*}{2},$$

for all normal elements $A \in \mathcal{A}$, is an $\mathbb{R}$-linear continuous Jordan $*$-homomorphism provided that $0 \in \operatorname{Ran} \Phi$. Also, Φ is the sum of a linear Jordan $*$-homomorphism and a conjugate-linear Jordan $*$-homomorphism. This result also presents an application of maps which preserve the square absolute value.

Keywords. C^*-algebra; $\mathbb{C}$-linear; $\mathbb{C}$-antilinear; homomorphism; linear preserver problem; real rank zero.

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1. Introduction and preliminaries

Linear preserver problems have been the main subject of many researches done by mathematicians in recent years (see [1–9,14–28]).

Historically, many mathematicians devoted their studies to additive or linear Jordan or Lie product preservers between rings or operator algebras. Such maps are always called Jordan homomorphism or Lie homomorphism [13,14]. By a $*$-homomorphism, we mean a map $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ which preserves the ring structure and for which $\Phi(A^*) = \Phi(A)^*$ for every $A \in \mathcal{A}$. A map $\Phi : \mathcal{A} \to \mathcal{B}$ is said to be a Jordan $*$-homomorphism if it is $\mathbb{R}$-linear, $\Phi(A^*) = \Phi(A)^*$ and $\Phi(A)^2 = \Phi(A^2)$ for all $A \in \mathcal{A}$.

Let $\mathcal{R}$ be a $*$-ring. For $A, B \in \mathcal{R}$, $A \bullet B = AB + BA^*$ and $[A, B]_* = AB - BA^*$, which are two different kinds of new products. These products are found to be playing an important role in some research topics, and its study has recently attracted the attention of many authors (for example, see [15,25]). A natural problem is to study whether the map Φ preserving the new product on ring or algebra $\mathcal{R}$ is a ring or algebraic isomorphism. In [5], Cui and Li proved a bijective map Φ on factor von Neumann algebras which preserves $[A, B]_*$ must be a $*$-isomorphism. Moreover, in [10], Li discussed that the non-linear bijective map preserving $A \bullet B$ is also a $*$-ring isomorphism. Also, in [28], it is shown that a bijective unital map (not necessarily linear) which preserves $AP \pm PA^*$ for projection operators P must be $*$-additive (i.e., additive and $*$-preserving) on prime C^*-algebra. Recently, Liu and Ji [11] proved that a bijective map Φ on factor von Neumann algebras preserves $A^*B + BA^*$ if and only if Φ is a $*$-isomorphism.

It is interesting to study the mappings that preserve $A^*A + AA^*$, but this can be related to preserving absolute value maps. Thus it is necessary to know about the background of this work.

Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C^*-algebras. We say that an additive mapping $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ preserves absolute value (resp. square absolute value) if $\Phi(|A|) = |\Phi(A)|$ (resp. $\Phi(|A|^2) = |\Phi(A)|^2$) for every $A \in \mathcal{A}$, where $|A|^2 = A^*A$. We also say a map $\Phi : \mathcal{A} \to \mathcal{B}$ preserves the arithmetic mean if $\Phi(\frac{A+B}{2}) = \frac{\Phi(A)+\Phi(B)}{2}$ for all $A, B \in \mathcal{A}$. The class of all self-adjoint, skew self-adjoint and normal elements in $\mathcal{A}$ will be denoted by $\mathcal{A}_s$, $\mathcal{A}_{sk}$ and $\mathcal{A}_N$, respectively.

Let H and K be Hilbert spaces and let $B(H)$ and $B(K)$ denote the algebras of all bounded linear operators on H and K, respectively. In [16], it is shown that if an additive mapping $\Phi : B(H) \longrightarrow B(K)$ satisfies $|\Phi(A)| = \Phi(|A|)\ \forall A \in B(H)$, $\Phi(iI)K \subset \overline{\Phi(I)K}$ and $\Phi(I)$ is a projection, then Φ is the sum of two $*$-homomorphisms, one which is $\mathbb{C}$-linear and the other is $\mathbb{C}$-antilinear. Moreover, in [26], it is proved that if $\mathcal{A}$ and $\mathcal{B}$ are two unital C^*-algebras, $\Phi : \mathcal{A} \to \mathcal{B}$ is an additive surjective mapping satisfying $\Phi(|A|) = |\Phi(A)|$ for every $A \in \mathcal{A}$ and $\Phi(I)$ a projection, then the restriction of mapping Φ to both $\mathcal{A}_s$ and $\mathcal{A}_{sk}$ is a Jordan $*$-homomorphism onto the corresponding set in $\mathcal{B}$. Furthermore, if $\mathcal{B}$ is a C^*-algebra of real-rank zero, then Φ is a $\mathbb{C}$-linear or $\mathbb{C}$-antilinear $*$-homomorphism.

The aim of this paper is to continue this work by studying mappings $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ that preserve the arithmetic mean and $A^*A + AA^*$ on $\mathcal{A}_N$, which in fact, satisfies $\Phi(A^*A) = \frac{\Phi(A)^*\Phi(A)+\Phi(A)\Phi(A)^*}{2}$ for all $A \in \mathcal{A}_N$, where $\mathcal{A}$ be a C^*-algebra of real-rank zero and $\mathcal{B}$ be a C^*-algebra with identity I. It is shown that such a map $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ on C^*-algebras is an $\mathbb{R}$-linear Jordan $*$-homomorphism provided that $0 \in \mathrm{Ran}\ \Phi$. Also, Φ is the sum of linear Jordan $*$-homomorphism and a conjugate-linear Jordan $*$-homomorphism. Also, in the last part our paper, we apply this result to obtain a description of maps on C^*-algebras which preserve the arithmetic mean and square absolute value.

2. Main results

We need the following lemma for proving our main theorems.

Lemma 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras. If $\Phi : \mathcal{A} \to \mathcal{B}$ is a map which preserves positivity and the arithmetic mean on $\mathcal{A}_s$, then*

(*i*) *Φ preserves self-adjoint operators.*
(*ii*) *Φ is an order preserving map.*
(*iii*) $\Phi(\lambda A) = \lambda\Phi(A) + (1 - \lambda)\Phi(0)$ *for all $A \in \mathcal{A}_s$ and $\lambda \in \mathbb{R}$.*

Proof.

(i) Let $A \in \mathcal{A}_s$. We have $\Phi(A) = \Phi(\frac{2A+0}{2}) = \frac{\Phi(2A)+\Phi(0)}{2}$, that is,

$$\Phi(2A) = 2\Phi(A) - \Phi(0). \tag{2.1}$$

It implies that

$$\Phi(A+B) = \Phi\left(\frac{2A+2B}{2}\right) = \frac{\Phi(2A)+\Phi(2B)}{2} = \Phi(A)+\Phi(B)-\Phi(0), \tag{2.2}$$

for all $A, B \in \mathcal{A}_s$.

Let $A \in \mathcal{A}$ be a self-adjoint operator. We can write $A = A_+ - A_-$, where A_+, A_- are positive operators. Hence we obtain $\Phi(A) = \Phi(A_+ - A_-) = \Phi(A_+) - \Phi(A_-) + \Phi(0)$. This implies that Φ preserves self-adjoint operators.

(ii) Assume that $A \leq B$. We have

$$\Phi(B) = \Phi\Big(\frac{2B-2A+2A}{2}\Big) = \frac{\Phi(2B-2A)}{2} + \frac{\Phi(2A)}{2}. \tag{2.3}$$

This follows that $\Phi(2A) \leq 2\Phi(B)$ which implies that $\frac{1}{2}\Phi(0) \leq \Phi(B)$ for every positive operator B. By using this inequality and (2.3), we obtain

$$\Phi(2A) \leq 2\Phi(B) - \frac{1}{2}\Phi(0). \tag{2.4}$$

Replacing A with 0 in (2.4), we have $\frac{3}{4}\Phi(0) \leq \Phi(B)$ for every positive operator. Again by applying (2.3), we have $\Phi(2A) \leq 2\Phi(B) - \frac{3}{4}\Phi(0)$. It is then clear that $\frac{n-1}{n}\Phi(0) \leq \Phi(B)$ and $\Phi(2A) \leq 2\Phi(B) - \frac{n-1}{n}\Phi(0)$ for positive integers n and hence

$$\Phi(0) \leq \Phi(B), \quad \Phi(2A) \leq 2\Phi(B) - \Phi(0). \tag{2.5}$$

By using (2.1) and (2.5), we obtain $\Phi(A) \leq \Phi(B)$, that is, Φ is order preserving.

(iii) By applying (2.2), we can compute

$$\Phi(rA) = r\Phi(A) + (1-r)\Phi(0), \tag{2.6}$$

for all rational number r and for all $A \in \mathcal{A}_s$.

Now, let $A \in \mathcal{A}$ be a positive operator and $\lambda \in \mathbb{R}$ be fixed for the moment and consider arbitrary rational numbers r, s with $r < \lambda < s$. Since Φ is an order preserving, by using (2.5) and (2.6), we have

$$\begin{aligned} 2r\Phi(A) + (1-2r)\Phi(0) + \Phi(0) &= \Phi(2rA) + \Phi(0) \\ &\leq 2\Phi(\lambda A) \\ &\leq 2\Phi(sA) \\ &= 2s\Phi(A) + 2(1-s)\Phi(0). \end{aligned}$$

This gives $\Phi(\lambda A) = \lambda\Phi(A) + (1-\lambda)\Phi(0)$. □

We are now in a position to state our main results.

Theorem 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras with identity I. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a map which $0 \in \operatorname{Ran} \Phi$ preserves the arithmetic mean and satisfies $\Phi(A^*A) = \frac{\Phi(A)^*\Phi(A)+\Phi(A)\Phi(A)^*}{2}$ for all $A \in \mathcal{A}_N$. Then, Φ is a continuous map and the restriction of the mapping is Φ to $\mathcal{A}_s$ Jordan homomorphism. Furthermore, if $I \in \operatorname{Ran} \Phi$, then Φ is unital and $*$-preserving.*

Proof. We prove our theorem in several steps.

Step 1. Φ preserves self-adjoint operators and $\Phi(S^2) = \Phi(S)^2$ for all $S \in \mathcal{A}_s$.

Let $A \in \mathcal{A}$ be a positive operator. Then there exists a unique positive operator $B \in \mathcal{A}$ such that $A = B^2$. We have $\Phi(A) = \Phi(B^2) = \frac{\Phi(B)^*\Phi(B)+\Phi(B)\Phi(B)^*}{2} \geq 0$. By Lemma 2.1, Φ preserves self-adjoint operators. Hence by the hypothesis we reach the statement.

Step 2. $\Phi(S)\Phi(0) = \Phi(0)\Phi(S) = \Phi(0)$ for all $S \in \mathcal{A}_N$ such that $\Phi(S)$ is a self-adjoint element.

Let S be a normal operator such that $\Phi(S)$ be a self-adjoint element by (2.2). Note that according to the hypothesis, (2.2) holds for all $A, B \in \mathcal{A}3$. We have

$$\begin{aligned} 3\Phi(0) + \Phi(S^*S) &= 2\Phi(0) + 2\Phi\Big(\frac{S^*S}{2}\Big) \\ &= 2\Phi(0) + 2\Big[2\Phi\Big(\frac{S^*S}{4}\Big) - \Phi(0)\Big] \\ &= 4\Phi\Big(\frac{S^*S}{4}\Big) = \Big(2\Phi\Big(\frac{S}{2}\Big)\Big)^*\Big(2\Phi\Big(\frac{S}{2}\Big)\Big) \\ &= [\Phi(S) + \Phi(0)]^*[\Phi(S) + \Phi(0)] \\ &= \Phi(S^*S) + \Phi(0) + \Phi(0)\Phi(S) + \Phi(S)^*\Phi(0). \end{aligned}$$

This equation follows

$$\Phi(S)\Phi(0) + \Phi(0)\Phi(S) = 2\Phi(0).$$

Multiply the above equation by $\Phi(0)$ from left and right, then we obtain

$$\Phi(0)\Phi(S)\Phi(0) + \Phi(0)\Phi(S) = 2\Phi(0).$$
$$\Phi(S)\Phi(0) + \Phi(0)\Phi(S)\Phi(0) = 2\Phi(0).$$

These equations imply

$$\Phi(S)\Phi(0) = \Phi(0)\Phi(S) = \Phi(0).$$

Step 3. Φ is an additive map.

Note that we have $\Phi(A+B) = \Phi(A)+\Phi(B)-\Phi(0)$ for all $A, B \in \mathcal{A}$, so it is enough to show that $\Phi(0) = 0$. By the hypothesis, we can find an element $A \in \mathcal{A}$ such that $\Phi(A) = 0$. We have $\Phi(A) = \Phi(A_1 + iA_2) = \Phi(A_1) + \Phi(iA_2) - \Phi(0) = 0$, where $A_1 = \operatorname{Re} A = (A + A^*)/2$ and $A_2 = \operatorname{Im} A = (A - A^*)/(2i)$. By multiplying the above equation by $\Phi(0)$ from left and by using Step 2, we get $0 = \Phi(0)\Phi(A_1) + \Phi(0)\Phi(iA_2) - \Phi(0) = \Phi(0)$. (Note that $\Phi(iA_2) = -\Phi(A_1) + \Phi(0)$ is a self-adjoint element.) It follows that Φ is an additive map.

Step 4. If self-adjoint operators S, T commute, then

$$\Phi(|S|)\Phi(iT)^* = -\Phi(|S|)\Phi(iT).$$

$|S| + iT$ is a normal operator because $|S|, T$ commute. Hence we have

$$\begin{aligned} 2\Phi(S^2 + T^2) &= \Phi(|S| + iT)^*\Phi(|S| + iT) + \Phi(|S| + iT)\Phi(|S| + iT)^* \\ &= 2\Phi(|S|)^2 + 2\Phi(T)^2 \\ &\quad + \Phi(|S|)[\Phi(iT)^* + \Phi(iT)] \\ &\quad + [\Phi(iT)^* + \Phi(iT)]\Phi(|S|). \end{aligned}$$

This follows that $\Phi(|S|)[\Phi(iT)^* + \Phi(iT)] = -[\Phi(iT)^* + \Phi(iT)\Phi(|S|)$ and hence $\Phi(|S|)^2[\Phi(iT)^*+\Phi(iT)] = [\Phi(iT)^*+\Phi(iT)]\Phi(|S|)^2$. It follows that $\Phi(|S|)^2|\Phi(iT)^*+\Phi(iT)| = |\Phi(iT)^* + \Phi(iT)|\Phi(|S|)^2$. Since by Step 1, $\Phi(|S|)$ is positive, it follows that $\Phi(|S|)|\Phi(iT)^* + \Phi(iT)| = |\Phi(iT)^* + \Phi(iT)|\Phi(|S|)$ and thus $\Phi(|S|)(\Phi(iT)^* + \Phi(iT)) = (\Phi(iT)^* + \Phi(iT))\Phi(|S|)$ and so $\Phi(|S|)(\Phi(iT)^* + \Phi(iT)) = 0$. In fact, $\Phi(|S|)\Phi(iT)^* = -\Phi(|S|)\Phi(iT)$.

Step 5. Φ is an $\mathbb{R}$-linear continuous map.

By Step 3, Φ is additive and by Lemma 2.1, $\mathbb{R}$ is a linear map.

To prove continuity, assume that $A \in \mathcal{A}_N$. Since Φ is order preserving, by the assumption and inequality $|A| \leq ||A||I$, we have

$$||\Phi(A)||^2 = |||\Phi(A)|^2|| \leq 2||\Phi(A^*A)|| \leq 2||A||^2||\Phi(I)||.$$

Taking square root, we obtain $||\Phi(A)|| \leq \sqrt{2||\Phi(I)||}||A||$.

Now, assume A is an arbitrary element, then we have

$$\begin{aligned} ||\Phi(A)|| &= ||\Phi(A_1 + iA_2)|| \leq ||\Phi(A_1)|| + ||\Phi(iA_2)|| \\ &\leq \sqrt{2||\Phi(I)||}||A_1|| + \sqrt{2||\Phi(I)||}||A_2|| \\ &\leq 2\sqrt{2||\Phi(I)||}||A||. \end{aligned}$$

Hence Φ is a continuous map.

Step 6. Restriction of the map Φ to $\mathcal{A}_s$ is a Jordan homomorphism.

By Steps 1 and 5, we reach the statement.

From now on, to prove the last part of the theorem, we assume that $I \in \operatorname{Ran} \Phi$.

Step 7. Φ is a unital map.

Since $I \in \operatorname{Ran} \Phi$, we can find an operator $U \in \mathcal{A}$ such that $\Phi(U) = I$. We can write $\Phi(U) = \Phi(U_1 + iU_2) = \Phi(U_1) + \Phi(iU_2) = I$, and so,

$$\Phi(iU_2) = I - \Phi(U_1) \tag{2.7}$$

is a self-adjoint operator. Hence by applying Step 4, we get $\Phi(I)\Phi(iU_2) = -\Phi(I)\Phi(iU_2)^* = -\Phi(I)\Phi(iU_2)$ and so $\Phi(I)\Phi(iU_2) = 0$. Hence by multiplying equation (2.7) by $\Phi(I)$ from left and right, we obtain

$$\Phi(I)\Phi(U_1) = \Phi(U_1)\Phi(I) = \Phi(I). \tag{2.8}$$

Also we have $\Phi((I+U_1)^2) = \Phi(I+U_1)^2$. It follows that $\Phi(U_1) = \Phi(I)$ by (2.8) and so $\Phi(U_1)$ is a projection element. Also, since $\Phi(iU)$ is self-adjoint, we have $\Phi(U_2)^2 = \Phi(iU_2)^2$. Therefore, by equation (2.7), we have $\Phi(U_2)^2 = \Phi(iU_2)^2 = I + \Phi({U_1}^2) - 2\Phi(U_1) = I - \Phi({U_1}^2)$. So $\Phi({U_1}^2 + {U_2}^2) = I$. This means that there exists a positive operator U such that $\Phi(U) = I$.

Finally, since Φ is Jordan homomorphism on $\mathcal{A}_s$, we have $2I = \Phi(2UI) = \Phi(U)\Phi(I) + \Phi(I)\Phi(U) = 2\Phi(I)$, that is, $\Phi(I) = I$.

Step 8. Φ is $*$-preserving.

Steps 4 and 7 imply that Φ is $*$-preserving. □

COROLLARY 2.3

Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras with identity I. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a map which preserves the arithmetic mean and square absolute value on $\mathcal{A}_N$. Then, Φ is a continuous map and the restriction of the mapping Φ to $\mathcal{A}_s$ is a Jordan homomorphism provided that $0 \in \operatorname{Ran} \Phi$. *Furthermore, if $I \in \operatorname{Ran} \Phi$, then Φ is unital and $*$-preservivg.*

Proof. According to Theorem 2.2, it is enough to show that Φ preserves normal elements. Firstly, by a similar argument as in the proof of Steps 1, 2 and 3 of Theorem 2.2, one can show that these steps are valid in this result, in fact, the restriction of Φ to $\mathcal{A}_S$ is a Jordan homomorphism.

As in the proof of Step 2 of [22], we prove that if $A = A_1 + iA_2$ is a normal operator, then

$$\Phi(iA_2)^*\Phi(A_1) = -\Phi(A_1)\Phi(iA_2). \tag{2.9}$$

Recall that A_1 and A_2 commute.

As in Step 7, we can show that Φ is a unital map. Substituting $A_1 = I$ in equation (2.9) implies that Φ is $*$-preserving and hence we have $\Phi(iA_2)\Phi(A_1) = \Phi(A_1)\Phi(iA_2)$. Therefore, $\Phi(A)$ is a normal operator. The statement follows from Theorem 2.2. □

Theorem 2.4. *Let $\mathcal{A}$ be a C^*-algebra of real-rank zero and $\mathcal{B}$ be a C^*-algebra with identity I. Let $\Phi : \mathcal{A} \subseteq B(H) \to \mathcal{B} \subseteq B(K)$ be a map which* $0 \in \operatorname{Ran} \Phi$ *preserves the arithmetic mean on $\mathcal{A}_N$ and satisfies $\Phi(A^*A) = \frac{\Phi(A)^*\Phi(A)+\Phi(A)\Phi(A)^*}{2}$ for all $A \in \mathcal{A}_N$. Then, Φ is an $\mathbb{R}$-linear Jordan $*$-homomorphism. Also, there exist orthogonal projections Q_1 and Q_2 in $\mathcal{B}$ such that*

(*i*) $\Phi(A) = \Phi_1(A) \oplus \Phi_2(A)$ *for all $A \in \mathcal{A}$, where*
(*ii*) $\Phi_1 : \mathcal{A} \longrightarrow Q_1\mathcal{B}Q_1$ *is a linear Jordan $*$-homomorphism.*
(*iii*) $\Phi_2 : \mathcal{A} \longrightarrow Q_2\mathcal{B}Q_2$ *is a conjugate-linear Jordan $*$-homomorphism.*

Proof. As above, we prove our theorem in several steps.

Step 1. Φ is $*$-preserving.

Let P be a projection. By the hypothesis, we have both $\Phi(iP)\Phi(iP)^* \leq 2\Phi(P)^2$ and $\Phi(iP)^*\Phi(iP) \leq 2\Phi(P)^2$. By Theorem 1 of [6], it implies that $\operatorname{Ran} \Phi(iP) \subset \operatorname{Ran} \Phi(P)$

and Ran $\Phi(iP)^* \subset$ Ran $\Phi(P)$, respectively and so $\Phi(P)\Phi(iP) = \Phi(iP)$ and $\Phi(iP)^* = \Phi(P)\Phi(iP)^*$. Now, by using Step 4 of Theorem 2.2, we get $\Phi(iP)^* = \Phi(P)\Phi(iP)^* = -\Phi(P)\Phi(iP) = -\Phi(iP)$. If S is of the form $A = \sum_{j=1}^{n} \lambda_j P_j$ for some scalars $\lambda_j \in \mathcal{C}$ and finitely many mutually orthogonal projections P_j, then we have

$$\Phi(A)^* = \Phi\Big(\sum_{j=1}^{n} \lambda_j P_j\Big)^* = \sum_{j=1}^{n} \Phi(\lambda_j P_j)^* = \sum_{j=1}^{n} \Phi(\overline{\lambda_j} P_j) = \Phi(A^*).$$

Since by Theorem 2.6 of [2] every element can be approximated by elements of the above form; hence, the continuity of Φ entails $\Phi(A)^* = \Phi(A^*)$ for every $A \in \mathcal{A}$. That is, Φ is $*$-preserving.

Step 2. $\Phi(|A|) = |\Phi(A)|$ for all $A \in \mathcal{A}_s \cup \mathcal{A}_{sk}$ (i.e. Φ preserves absolute values) on $\mathcal{A}_s$ and $\mathcal{A}_{sk}$.

By Step 1, Φ is $*$-preserving and so by the hypothesis and Theorem 2.2, we can obtain $|\Phi(A)|^2 = \Phi(|A|^2) = \Phi(|A|)^2$ for all $A \in \mathcal{A}_s \cup \mathcal{A}_{sk}$. It follows the result.

Step 3. Φ preserves orthogonality projection and

$$\Phi(I)\Phi(S) = \Phi(S)\Phi(I) = \Phi(S), \quad \forall S \in \mathcal{A}_s.$$

Clearly, Φ preserves projections. Let P and Q be two orthogonal projections. By Theorem 2.2, the restriction of the mapping Φ to $\mathcal{A}_s$ is a Jordan homomorphism and thus, $\Phi(P)\Phi(Q) + \Phi(Q)\Phi(P) = \Phi(PQ + QP) = \Phi(0) = 0$. Hence

$$\begin{aligned}\Phi(P)\Phi(Q) &= -\Phi(Q)\Phi(P) = -\Phi(Q)^2\Phi(P)\\ &= \Phi(Q)\Phi(P)\Phi(Q) = -\Phi(P)\Phi(Q),\end{aligned}$$

which implies that $\Phi(Q)\Phi(P) = \Phi(P)\Phi(Q) = 0$.

In particular, it follows that $0 = \Phi(P)\Phi(I - P) = \Phi(I - P)\Phi(P)$, and consequently, $\Phi(I)\Phi(P) = \Phi(P)\Phi(I) = \Phi(P)$. Since every self-adjoint element with finite spectra are dense in $\mathcal{A}_s$, the continuity of Φ entails $\Phi(I)\Phi(S) = \Phi(S)\Phi(I) = \Phi(S)$ for all $S \in \mathcal{A}_s$.

Step 4. There exists a partial isometry $V \in B(K)$ with initial space Ran $\Phi(I)$ such that $\Phi(A)V = V\Phi(A)$ and

$$\Phi(A + iB) = \Phi(A) + V\Phi(B), \qquad \forall A, B \in \mathcal{A}_s.$$

Let $\Phi(iI) = V\Phi(I)$ be the polar decomposition of $\Phi(iI)$. (Note that $|\Phi(iI)| = \Phi(|iI|) = \Phi(I)$.) Then V is a partial isometry with initial space Ran $\Phi(I)$. Also, we have $\Phi(I)V^* = \Phi(iI)^* = -\Phi(iI) = -V\Phi(I)$. By Theorem 2.2, $\Phi(I)^2 = \Phi(I)$ and so, we can conclude that $\Phi(I)V^*\Phi(I) = V\Phi(I)^2 = -V\Phi(I) = \Phi(I)V^* = \Phi(I)^2V^* = -\Phi(I)V\Phi(I)$. It follows that $\langle (V^* + V)\Phi(I)x, \Phi(I)y\rangle = \langle \Phi(I)(V^* + V)\Phi(I)x, y\rangle = 0$ for all $x, y \in H$ and so $V^* + V = 0$, that is, $V^* = -V$. Moreover, assume that $P \in \mathcal{A}$ is a projection and, $\Phi(iP) = V_P\Phi(P)$ and $\Phi(i(I - P)) = V_{I-P}\Phi(I - P)$ are the polar decomposition of $\Phi(iP)$ and $\Phi(i(I - P))$ respectively. Then by Step 3, it follows that

$$\begin{aligned}\Phi(iP) &= V_P\Phi(P) = V_P\Phi(P)^2\\ &= [V_P\Phi(P) + V_{I-P}\Phi(I-P)]\Phi(P)\\ &= \Phi(iI)\Phi(P) = V\Phi(I)\Phi(P)\\ &= V\Phi(P).\end{aligned}$$

If S is of the form $A = \sum_{j=1}^{n} \lambda_j P_j$ for some scalers $\lambda_j \in \mathcal{R}$ and finitely many mutually orthogonal projections P_j, then we have

$$\Phi(iA) = \Phi\Big(\sum_{j=1}^{n} i\lambda_j P_j\Big) = \sum_{j=1}^{n} \lambda_j \Phi(iP_j) = V\Phi\Big(\sum_{j=1}^{n}\lambda_j P_j\Big) = V\Phi(A).$$

Since by Theorem 2.6 of [2] every element can be approximated by elements of the above form; hence, the continuity of Φ entails $\Phi(iA) = V\Phi(A)$ for every self-adjoint $A \in \mathcal{A}_s$, and also $V\Phi(A) = \Phi(iA) = -\Phi(iA)^* = -\Phi(A)V^* = \Phi(A)V$. This completes the proof.

Step 5. Φ is a Jordan $*$-homomorphism.

Let $A \in \mathcal{A}$ be an arbitrary element. By Step 4 and Theorem 2.2, we can compute

$$\begin{aligned}\Phi(A^2) &= \Phi((A_1 + iA_2)^2)\\ &= \Phi(A_1)^2 - \Phi(A_2)^2 + \Phi(i(A_1A_2 + A_2A_1))\\ &= \Phi(A_1)^2 - \Phi(A_2)^2 + V\Phi(A_1)\Phi(A_2) + V\Phi(A_2)\Phi(A_1)\\ &= \Phi(A_1)(\Phi(A_1) + V\Phi(A_2)) + V\Phi(A_2)(\Phi(A_1) + V\Phi(A_2))\\ &= (\Phi(A_1) + V\Phi(A_2))^2 = (\Phi(A_1) + \Phi(iA_2))^2\\ &= \Phi(A_1 + iA_2)^2 = \Phi(A)^2,\end{aligned}$$

which means that Φ is a Jordan $*$-homomorphism.

Step 6. Φ is the sum of a linear Jordan $*$-homomorphism and a conjugate-linear Jordan $*$-homomorphism.

Let $A \in \mathcal{A}$. By using Steps 3 and 4, we obtain

$$\begin{aligned}\Phi(I)\Phi(A) &= \Phi(I)\Phi(A_1 + iA_2)\\ &= \Phi(I)\Phi(A_1) + \Phi(I)\Phi(iA_2)\\ &= \Phi(A_1) + \Phi(I)V\Phi(A_2)\\ &= \Phi(A_1) + V\Phi(I)\Phi(A_2)\\ &= \Phi(A_1) + V\Phi(A_2) = \Phi(A_1) + \Phi(iA_2)\\ &= \Phi(A_1 + iA_2) = \Phi(A).\end{aligned}$$

Similarly, we can get $\Phi(A) = \Phi(A)\Phi(I)$ for all $A \in \mathcal{A}$.

Since $\Phi(iI)$ is skew self-adjoint, $\Phi(iI) = iW$ for some self-adjoint element $W \in \mathcal{B}$. It is easy to see that $Q = \frac{W+\Phi(I)}{2}$ is a projection and $\Phi(iI) = i(2Q - \Phi(I))$.

Now, assume that $Q = Q_1$ is a nontrivial projection in $\mathcal{B}$ and $Q_2 = I - Q_1$ (it is obvious that if Q is a trivial projection, then Φ is a linear or conjugate-linear Jordan $*$-homomorphism). Denote $\mathcal{B}_{ij} = Q_i\mathcal{B}Q_j,\ i, j = 1, 2$, then $\mathcal{B} = \sum_{i,j=1}^{2} \mathcal{B}_{ij}$. By using

Steps 3 and 4, we obtain for $A \in \mathcal{A}$

$$\begin{aligned}\Phi(iI)\Phi(A) &= \Phi(iI)\Phi(A_1 + iA_2)\\ &= \Phi(iI)\Phi(A_1) + \Phi(iI)\Phi(iA_2)\\ &= V\Phi(I)\Phi(A_1) + V\Phi(I)V\Phi(A_2)\\ &= \Phi(A_1)V\Phi(I) + V\Phi(A_2)V\Phi(I)\\ &= \Phi(A_1)\Phi(iI) + V\Phi(A_2)\Phi(iI)\\ &= \Phi(A_1 + iA_2)\Phi(iI) = \Phi(A)\Phi(iI).\end{aligned}$$

It follows that Q commutes with $\Phi(A)$ for all $A \in \mathcal{A}$. Consequently, we write $\Phi(\mathcal{A}) = \Phi(\mathcal{A})_{11} + \Phi(\mathcal{A})_{22} = B_{11} + B_{22}$. It follows that $B_{ij} \in \mathcal{B}_{ij}$ when we write B_{ij}. It follows that

$$\Phi(A) = Q_1\Phi(A)Q_1 + Q_2\Phi(A)Q_2 = Q_1\Phi(A) + Q_2\Phi(A),$$

for all $A \in \mathcal{A}$. Now, define Φ_1 and Φ_2 by $\Phi_i(A) = Q_i\Phi(A), \quad i = 1, 2$. Then by Step 5, $\Phi_i : \mathcal{A} \longrightarrow \mathcal{B}_i$ is the Jordan $*$-homomorphism, $i = 1, 2$. In addition, as $\Phi_1(iI) = Q_1\Phi(iI)$, we see that $\Phi_1(iI) = iQ_1$ and $\Phi_2(iI) = -iQ_2$. By using Step 4, we have

$$\begin{aligned}\Phi_1(iA) &= Q_1\Phi(iA) = Q_1V\Phi(A)\\ &= Q_1V\Phi(I)\Phi(A) = Q_1\Phi(iI)\Phi(A)\\ &= iQ_1(Q_1 - Q_2)\Phi(A) = iQ_1\Phi(A)\\ &= i\Phi_1(A)\end{aligned}$$

for all $A \in \mathcal{A}_s$. Hence Φ_1 is a linear map.

Similarly, we can show that $\Phi_2 : \mathcal{A} \longrightarrow \mathcal{B}_2$ is a conjugate-linear Jordan $*$-homomorphism. □

Our last result gives a characterization of maps preserving square absolute values and arithmetics by using Theorem 2.4.

COROLLARY 2.5

Let $\mathcal{A}$ be a C^*-algebra of real-rank zero and $\mathcal{B}$ be a C^*-algebra with identity I. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a map which preserves the arithmetic mean and square absolute value on $\mathcal{A}_N$. Then, Φ is an $\mathbb{R}$-linear Jordan homomorphism on $\mathcal{A}_s$ provided $0 \in \operatorname{Ran}\Phi$. *Furthermore, if $I \in \operatorname{Ran}\Phi$, then there exist orthogonal projections Q_1 and Q_2 in $\mathcal{B}$ such that*

(*i*) $\Phi(A) = \Phi_1(A) \oplus \Phi_2(A)$ *for all* $A \in \mathcal{A}$.
(*ii*) $\Phi_1 : \mathcal{A} \longrightarrow Q_1\mathcal{B}Q_1$ *is a linear Jordan $*$-homomorphism.*
(*iii*) $\Phi_2 : \mathcal{A} \longrightarrow Q_2\mathcal{B}Q_2$ *is a conjugate-linear Jordan $*$-homomorphism.*

Proof. Similarly to Corollary 2.3, we can show that Φ preserves normal elements which together by Theorem 2.4 implies the statement. □

The following examples show that the condition of $0 \in \operatorname{Ran}\Phi$ is needed in the above theorems and corollaries and also the condition of $I \in \operatorname{Ran}\Phi$ is needed in the above corollaries.

Example 2.6. Define an additive mapping $\Phi : \mathbb{C} \longrightarrow B(\mathbb{C}^2)$ by

$$\Phi(a + ib) = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$$

for all $a, b \in \mathbb{R}$, where $\mathbb{C}$ is the C^*-algebra all of complex numbers. Obviously, Φ satisfies the assumptions in the above corollaries except the condition of $I \in \operatorname{Ran} \Phi$ but not their conclusions.

Example 2.7. The constant map $A \to I$, $A \in \mathcal{A}$ satisfies the assumptions in the above theorems and corollaries except the condition of $0 \in \operatorname{Ran} \Phi$ but not their conclusions.

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