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Group classification, conservation laws and Painlevé analysis for Klein–Gordon–Zakharov equations in (3+1)-dimension

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Abstract. In this paper, we study Klein–Gordon–Zakharov equations which describe the propagation of strong turbulence of the Langmuir wave in a high-frequency plasma. Using the symbolic manipulation tool Maple, the classifications of symmetry algebra are carried out, and the construction of several local non-trivial conservation laws based on a direct method of Anco and Bluman is illustrated. Starting with determination of symmetry algebra, the one- and two-dimensional optimal systems are constructed, and optimality is also established using various invariant functions of full adjoint action. Apart from classification and construction of several conservation laws, the Painlevé analysis is also performed in a symbolic manner which describes the non-integrability of equations.

Keywords. Klein–Gordon–Zakharov equations; optimal systems; conservation laws; Painlevé analysis.

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1. Introduction

The symmetry analysis of partial differential equations developed by Sophus Lie is the most powerful algorithmic technique as of now while studying some types of geometric transformations that promised to have considerable relevance in the subsequent study of symmetries. It is a well-known fact [1,2] that the Lie group theory is based on an inspirational idea taken from the theory of algebraic equations given by Galois and Abel. The first step in the Lie theory involves the determination of group of point transformations. Such a group is also called a symmetry group. Any subgroup of a symmetry group is capable of reducing the differential equation into an equation with fewer number of independent variables, which corresponds to group invariant solutions, but not all subgroups generate group invariant solutions [3]. As described by Olver [4], there exist infinitely many subgroups of full symmetry group, such that most of them turn out to be equivalent through group of inner automorphism, thus leading to redundant group invariant solutions. The group of inner automorphism may be used to classify the given symmetry algebra into classes of pairwise non-conjugate subalgebras. Such a collection of pairwise non-conjugate subalgebras is called optimal system.

The classification of Lie algebra was already known to Lie himself [5] and subsequent work on the classification of Lie algebras can be seen in the excellent review carried out by Boza *et al* [6]. Apart from various methodologies developed for classification of Lie algebras, the systematic study given by Patera *et al* [7–9] in a series of papers is par excellence. The complete classification of all Lie algebras of dimension ≤ 4 into conjugacy classes can be found in their subsequent work [10]. Around the same period of time, based on global adjoint matrix, Ovsiannikov [3] developed a technique to classify Lie algebras into one-dimensional optimal system and the approach is further generalised to construct the higher dimensional optimal systems. Galas and Richter [11] made further modifications in the technique of Ovsiannikov by considering the quotient group of normalisers. Apart from sophisticated techniques using fundamentals of Lie algebra, Olver [4] developed a very elementary method of reducing the general element from the Lie algebra into conjugacy classes by applying group of inner automorphisms (or adjoint actions). He also discussed the importance of considering the Killing form for identification of representatives of each equivalence class and the work of Chou *et al* [12] and Chou and Qu [13] further strengthens the optimality of conjugacy classes by introducing numerical invariants.

In this work, our aim is to study group classification and integrability of Klein–Gordon–Zakharov (KGZ) equations in (3+1)-dimension:

$$\phi_{tt} - \Delta\phi + \phi + \phi\,\psi = 0,$$
$$\psi_{tt} - c^2\,\Delta\psi - \Delta|\phi|^2 = 0. \tag{1}$$

Here

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is the Laplacian operator in $\mathbb{R}^3$, $\phi = \phi(x, y, z, t)$, $\psi = \psi(x, y, z, t)$ and c is the propagation speed of a wave. System (1) describes the propagation of strong turbulence of the Langmuir wave (rapid oscillations of the electron density in conducting media such as plasmas or metals) in a high-frequency plasma [14]. The function ϕ denotes the fast time scale component of electric field raised by electrons and the function ψ denotes the deviation of ion density from its equilibrium. The functions ϕ and ψ are originally real vector valued and real scalar valued, respectively. KGZ equations have similar shape to Zakharov equations and Klein–Gordon–Schrödinger equations. Ozawa *et al* [15], Tsutaya [16] and Ozawa *et al* [17] have studied the well posedness in energy space, global existence and asymptotic behaviour of solutions for the Cauchy problem of the KGZ equations. Equations (1) have been studied for exact solutions by various researchers [18–22].

The paper is organised as follows. In §2, the Lie symmetry analysis and group classification of Lie algebra into one- and two-dimensional optimal subalgebras are given, and the optimality is confirmed by the introduction of several invariants of full adjoint action. In §3, based on the direct method, we have demonstrated the construction of local non-trivial conservation laws. In §4, the package 'wkptest' is also implemented in order to analyse the Painlevé property of equations. Finally in §5, the conclusion is drawn.

2. Symmetry group of the KGZ equations

The procedure for determining symmetry algebra for partial differential equations is well known [4,23–25]. The procedure is so algorithmic that it has been successfully implemented in symbolic languages such as 'Maple' and 'Mathematica'. The Maple package 'PDEtools' written by Cheb-Terrab and Von Bülow [26] is so interactive and efficient that it becomes indispensable for researchers in the field of PDEs. The PDEtools package can be applied by setting $\phi = u + \mathrm{i}v$ in (1). We obtain

$$\begin{aligned}
\Delta_1 &= \psi u + u + u_{t,t} - u_{x,x} - u_{y,y} - u_{z,z} = 0,\\
\Delta_2 &= \psi v + v + v_{t,t} - v_{x,x} - v_{y,y} - v_{z,z} = 0,\\
\Delta_3 &= \psi_{t,t} - c^2(\psi_{x,x} + \psi_{y,y} + \psi_{z,z}) - 2u_x^2\\
&\quad -2uu_{x,x} - 2v_x^2 - 2vv_{x,x} - 2u_y^2 - 2\,uu_{y,y}\\
&\quad -2v_y^2 - 2vv_{y,y} - 2u_z^2 - 2uu_{z,z}\\
&\quad -2v_z^2 - 2vv_{z,z} = 0.
\end{aligned} \tag{2}$$

The element of symmetry algebra for (2) may be written as

$$V = \xi^x\partial_x + \xi^y\partial_y + \xi^z\partial_z + \xi^t\partial_t + \eta^u\partial_u + \eta^v\partial_v + \eta^\psi\partial_\psi. \tag{3}$$

The infinitesimals ξ's and η's can be recovered from the invariance criteria

$$\mathrm{pr}^{(2)}\,V\cdot\Delta_i|_{\Delta_i=0} = 0,\quad i = 1, 2, 3. \tag{4}$$

Here $\mathrm{pr}^{(2)}$ is the second-order prolongation of vector field V and the related detailed discussion about the prolongation of vector field can be seen in [23] (see e.g. Theorem 2.4.2-1). The Maple package PDEtools quickly gives out the nine-dimensional symmetry algebra

$$\begin{aligned}
V_1 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\\
&\quad + (-2w - 2)\frac{\partial}{\partial w},\\
V_2 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\\
V_3 &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\quad V_4 = \frac{\partial}{\partial x},\\
V_5 &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},\quad V_6 = \frac{\partial}{\partial y},\\
V_7 &= \frac{\partial}{\partial z},\quad V_8 = \frac{\partial}{\partial t},\quad V_9 = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}.
\end{aligned} \tag{5}$$

The structure of Lie algebra (5) can be properly identified from the non-zero commutation relations

$$\begin{aligned}
&[V_1, V_4] = -V_4,\quad [V_1, V_6] = -V_6,\\
&[V_1, V_7] = -V_7,\quad [V_1, V_8] = -V_8,\quad [V_2, V_3] = V_5,\\
&[V_2, V_4] = V_6,\quad [V_2, V_5] = -V_3,\quad [V_2, V_6] = -V_4,\\
&[V_3, V_4] = V_7,\quad [V_3, V_5] = V_2,\\
&[V_3, V_7] = -V_4,\quad [V_5, V_6] = V_7,\quad [V_5, V_7] = -V_6.
\end{aligned} \tag{6}$$

Notice that the Lie algebra $L = \{V_i,\ i = 1, \dots, 9\}$ defined by (5) admits the Levi decomposition

$$L = S \rtimes \mathrm{rad}(L) \tag{7}$$

with semi-simple Levi subalgebra $S = \{V_2, V_3, V_5\}$ and radical $\mathrm{rad}(L) = \{V_1, V_4, V_6, V_7, V_8, V_9\}$. Any vector field in Lie algebra (5) is capable of generating the Lie group of point transformations [4,27] through exponentiation

$$\tilde{T} = \exp(a_i V_i) T, \quad i = 1, \dots, 9, \quad \text{for } T = T(x, y, z, t, u, v, \psi). \tag{8}$$

Such transformation $\tilde{T}$ sometimes is also called as one-parameter group of infinitesimal transformation. The nine-parameter version of the Lie group of point transformation may be expressed as

$$\tilde{T} = \exp\left(\sum_{i=1}^{9} a_i V_i\right) T. \tag{9}$$

For each subgroup of symmetry group L there exists a corresponding family of group invariant solutions. Since there might be virtually infinite number of such subgroups, it would not be possible to list all such group invariant solutions. Unfortunately, most of the group invariant solutions obtained in that way would be equivalent through transformation (8). Therefore, in order to obtain inequivalent group invariant solutions under full symmetry group L, an effective technique for the classification of group invariant solution is needed. Such a technique is developed by Olver [4]. Define the adjoint transformation

$$\mathrm{Ad}_{\exp(\epsilon X)}(Y) = \exp(-\epsilon X) Y \exp(\epsilon X) = \tilde{Y}(\epsilon), \tag{10}$$

where X is the general element of L and Y from any subgroup of L. The adjoint transformation (10) ensures that the group invariant solutions under Y and $\tilde{Y}$ cannot be connected by transformation (8). The adjoint transformation (10) can be written through commutators using Campbell–Hausdorff formula as

$$\mathrm{Ad}_{\exp(\epsilon X)}(Y) = Y - \epsilon[X, Y] + \frac{\epsilon^2}{2}[X, [X, Y]] - \cdots, \tag{11}$$

where $[.\,,\,.]$ is the Lie bracket defined by (6). Let $X = \sum_{i=1}^{9} a_i V_i$, and based on Lie brackets defined at (6) and formula (11), a straightforward calculation using the Maple programming shows that

$$\begin{aligned}
&\mathrm{Ad}_{\exp(\epsilon_5 V_5)}\, \mathrm{Ad}_{\exp(\epsilon_3 V_3)}\, \mathrm{Ad}_{\exp(\epsilon_7 V_7)} \\
&\quad \times \mathrm{Ad}_{\exp(\epsilon_4 V_4)}\, \mathrm{Ad}_{\exp(\epsilon_6 V_6)}\, \mathrm{Ad}_{\exp(\epsilon_1 V_1)} \\
&\quad \times \mathrm{Ad}_{\exp(\epsilon_8 V_8)}\, \mathrm{Ad}_{\exp(\epsilon_2 V_2)}\, \mathrm{Ad}_{\exp(\epsilon_9 V_9)}\,(X) \\
&= \sum_{i=1}^{9} \tilde{a}_i V_i,
\end{aligned} \tag{12}$$

where the coefficients $\tilde{a}_i$, $i = 1, \dots, 9$, are given in Appendix B.

2.1 *Construction of invariant functions*

It is easily seen from full adjoint action and table 1 that a_1 and a_9 are invariants. However, as described in [28], the general invariant function can be obtained by solving the system of PDEs given by

$$\begin{aligned}
&a_1\phi_{a_8} = 0, \quad a_1\phi_{a_4} - a_2\phi_{a_6} - a_3\phi_{a_7} = 0, \\
&a_1\phi_{a_6} + a_2\phi_{a_4} - a_5\phi_{a_7} = 0, \\
&a_1\phi_{a_7} + a_3\phi_{a_4} + a_5\phi_{a_6} = 0, \\
&a_2\phi_{a_3} - a_3\phi_{a_2} + a_6\phi_{a_7} - a_7\phi_{a_6} = 0, \\
&-a_2\phi_{a_5} + a_4\phi_{a_7} + a_5\phi_{a_2} - a_7\phi_{a_4} = 0, \\
&a_3\phi_{a_5} + a_4\phi_{a_6} - a_5\phi_{a_3} - a_6\phi_{a_4} = 0, \\
&-a_4\phi_{a_4} - a_6\phi_{a_6} - a_7\phi_{a_7} - a_8\phi_{a_8} = 0.
\end{aligned}$$

The straightforward solution of the above system of PDEs shows that the general invariant function must be

$$\phi = F(a_1, a_2^2 + a_3^2 + a_5^2, a_9), \tag{13}$$

where F is an arbitrary function.

The particular invariant can also be obtained by the Killing form. For $X = \sum_{i=1}^{9} a_i V_i$, the straightforward calculations show that

$$\begin{aligned}
&K(X, X) = 4a_1^2 - 4a_2^2 - 4a_3^2 - 4a_5^2, \\
&K(X, X) = \text{Killing form}.
\end{aligned} \tag{14}$$

The Killing form (see e.g. [3]) is defined in the following manner:

$$K(X, X) = \mathrm{Trace}(\mathrm{ad}X \cdot \mathrm{ad}X), \tag{15}$$

where

$$\mathrm{ad}X = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -a_5 & 0 & a_3 & 0 & 0 & 0 & 0 \\
0 & a_5 & 0 & 0 & -a_2 & 0 & 0 & 0 & 0 \\
a_4 & a_6 & a_7 & -a_1 & 0 & -a_2 & -a_3 & 0 & 0 \\
0 & -a_3 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_6 & -a_4 & 0 & a_2 & a_7 & -a_1 & -a_5 & 0 & 0 \\
a_7 & 0 & -a_4 & a_3 & -a_6 & a_5 & -a_1 & 0 & 0 \\
a_8 & 0 & 0 & 0 & 0 & 0 & 0 & -a_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{16}$$

Therefore, the Killing form gives invariant $K(X) = 4(a_1^2 - a_2^2 - a_3^2 - a_5^2)$ of full adjoint action such that $K(\mathrm{Ad}g(X)) = K(X)$ for $X = \sum_{i=1}^{9} a_i V_i \in L$ and $g \in G$, the Lie group generated by L.

Various invariants may be listed as follows:

$$\eta = a_2^2 + a_3^2 + a_5^2,$$

as evident from the general invariant function (13),

Table 1. Adjoint table based on formula (11).

Ad.	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9
V_1	V_1	V_2	V_3	$e^{\epsilon} V_4$	V_5	$e^{\epsilon} V_6$	$e^{\epsilon} V_7$	$e^{\epsilon} V_8$	V_9
V_2	V_1	V_2	$V_3 \cos(\epsilon) - V_5 \sin(\epsilon)$	$V_4 \cos(\epsilon) - V_6 \sin(\epsilon)$	$V_5 \cos(\epsilon) + V_3 \sin(\epsilon)$	$V_6 \cos(\epsilon) + V_4 \sin(\epsilon)$	V_7	V_8	V_9
V_3	V_1	$V_2 \cos(\epsilon) + V_5 \sin(\epsilon)$	V_3	$V_4 \cos(\epsilon) - V_7 \sin(\epsilon)$	$V_5 \cos(\epsilon) - V_2 \sin(\epsilon)$	V_6	$V_7 \cos(\epsilon) + V_4 \sin(\epsilon)$	V_8	V_9
V_4	$V_1 - \epsilon V_4$	$V_2 + \epsilon V_6$	$V_3 + \epsilon V_7$	V_4	V_5	V_6	V_7	V_8	V_9
V_5	V_1	$V_2 \cos(\epsilon) - V_3 \sin(\epsilon)$	$V_3 \cos(\epsilon) + V_2 \sin(\epsilon)$	V_4	V_5	$V_6 \cos(\epsilon) - V_7 \sin(\epsilon)$	$V_7 \cos(\epsilon) + V_6 \sin(\epsilon)$	V_8	V_9
V_6	$V_1 - \epsilon V_6$	$V_2 - \epsilon V_4$	V_3	V_4	$V_5 + \epsilon V_7$	V_6	V_7	V_8	V_9
V_7	$V_1 - \epsilon V_7$	V_2	$V_3 - \epsilon V_4$	V_4	$V_5 - \epsilon V_6$	V_6	V_7	V_8	V_9
V_8	$V_1 - \epsilon V_8$	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9
V_9	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9

$$A = a_1,\quad B = a_9,\quad C = \begin{cases} 1, \; a_2^2 + a_3^2 + a_5^2 \neq 0, \\ 0 \;\; \text{otherwise.} \end{cases} \tag{17}$$

The last invariant C is clear from the full adjoint action given in Appendix B. It is obvious that

$$\tilde{a}_2^2 + \tilde{a}_3^2 + \tilde{a}_5^2 = a_2^2 + a_3^2 + a_5^2,$$

where $\tilde{a}_2 = \tilde{a}_3 = \tilde{a}_5 = 0$ if and only if $a_2 = a_3 = a_5 = 0$. In a similar way,

$$D = \begin{cases} 1, \; a_4^2 + a_6^2 + a_7^2 + a_8^2 \neq 0, \;\; a_1 = a_2 = a_3 = a_5 = 0, \\ 0 \;\; \text{otherwise} \end{cases} \tag{18}$$

is also an invariant of full adjoint action given by Appendix B. Further, it is pertinent to mention that the construction of invariants obtained here has been discussed in detail in [12].

2.2 *Construction of optimal system*

The detection of invariants of full adjoint action is very helpful in the classification of Lie algebra L into one-dimensional optimal subalgebras. The invariant η is always positive but still various restrictions on coefficients a_2, a_3 and a_5 can be discussed. For example, for $a_2 \neq a_3 \neq a_5 \neq 0$. A direct simplification in full adjoint action (12) gives

$$\mathrm{Ad}_{\exp(\epsilon_7 V_7)}\, \mathrm{Ad}_{\exp(\epsilon_4 V_4)}\, \mathrm{Ad}_{\exp(\epsilon_6 V_6)}\, \mathrm{Ad}_{\exp(\epsilon_8 V_8)}\,(X) = \sum_{i=1}^{9} \tilde{a}_i V_i, \tag{19}$$

where

$$\tilde{a}_4 = -a_1\epsilon_4 - a_2\epsilon_6 - a_3\epsilon_7 + a_4,$$
$$\tilde{a}_6 = -a_1\epsilon_6 + a_2\epsilon_4 - a_5\epsilon_7 + a_6,$$
$$\tilde{a}_7 = -a_1\epsilon_7 + a_3\epsilon_4 + a_5\epsilon_6 + a_7.$$
$$\tilde{a}_8 = -a_1\epsilon_8 + a_8. \tag{20}$$

These coefficients vanish on choosing

$$
\begin{aligned}
\epsilon_4 &= \frac{a_1^2a_4 - a_1a_2a_6 - a_1a_3a_7 + a_2a_5a_7 - a_3a_5a_6 + a_4a_5^2}{a_1(a_1^2+a_2^2+a_3^2+a_5^2)},\\
\epsilon_6 &= \frac{a_1^2a_6 + a_1a_2a_4 - a_1a_5a_7 - a_2a_3a_7 + a_3^2a_6 - a_3a_4a_5}{a_1(a_1^2+a_2^2+a_3^2+a_5^2)},\\
\epsilon_7 &= \frac{a_1^2a_7 + a_1a_3a_4 + a_1a_5a_6 + a_2^2a_7 - a_2a_3a_6 + a_2a_4a_5}{a_1(a_1^2+a_2^2+a_3{}^2+a_5^2)},\\
\epsilon_8 &= \frac{a_8}{a_1}.
\end{aligned}
\tag{21}
$$

Further actions by $\mathrm{Ad}_{\exp(\epsilon_2 V_2)}$ and $\mathrm{Ad}_{\exp(\epsilon_3 V_3)}$ and for suitable selection of ϵ_2 and ϵ_3, X becomes

$$P_1 = a_1V_1 + a_2V_2 + a_9V_9,$$

for some constants a_1, a_2 and a_9. No further simplifications are possible.

Proceeding in a similar fashion, the other members of optimal system can be listed as follows:

Case 1: $\eta \neq 0$.

$$
\begin{aligned}
&P_2 = a_2V_2 + a_3V_3 \pm V_6 + a_8V_8 + a_9V_9,\\
&\quad a_1 = 0,\ a_2 \neq 0,\ a_3 \neq 0,\ a_5 \neq 0,\\
&P_3 = a_1V_1 + a_3V_3 + a_9V_9,\\
&\quad a_2 = 0,\ a_3 \neq 0, a_5 \neq 0,\\
&P_4 = a_2V_2 + a_5V_5 \pm V_7 + a_8V_8 + a_9V_9,\\
&\quad a_1 = 0,\ a_2 \neq 0,\ a_3 = 0,\ a_5 \neq 0,\\
&P_5 = \pm V_4 + a_5V_5 + a_8V_8 + a_9V_9,\\
&\quad a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_5 \neq 0.
\end{aligned}
$$

Case 2: $\eta = 0$.

$$
\begin{aligned}
&P_6 = 1V_1 + a_9V_9, \quad a_2 = 0,\ a_3 = 0,\ a_5 = 0,\\
&P_7 = \pm V_7 + a_8V_8 + a_9V_9,\\
&\qquad a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_5 = 0.
\end{aligned}
$$

Case 3: Since a_1 and a_9 are invariants of full adjoint action as also cleared from table 1. So remaining members of optimal system are given by

$$P_8 = V_1, \quad P_9 = V_9.$$

The values of invariants obtained in §2.1 are calculated for every member of the optimal system $\{P_i,\ i = 1,\ldots,9\}$. These values of invariants are depicted in table 2 and the mutual inequivalence among P_i's can be easily seen from table 2.

2.3 *Two-dimensional optimal system*

In order to construct the two-dimensional optimal subalgebra, we follow the procedure developed by Ovsiannikov [3] (see e.g. p. 199) and its implementation by various researchers [11,27,29,30]. We list a pair of two-dimensional subalgebras of the form $\langle X, Y\rangle$, such that $[X, Y] = \lambda X + \mu Y$, for $X = P_i,\ i = 1,\ldots,9$. For precise selection, Y can be selected from the normaliser $\mathrm{Nor}_L(X) = \{Y \in L \mid [X, Y] \in X\}$, i.e. $[X, Y] = \lambda X$. As suggested by Galas and Richter [11], one can avoid the occurrence of X in Y by selecting Y from the factor algebra $\mathrm{Nor}_L(X)/X$, where the normaliser $\mathrm{Nor}_L(X)$ can be obtained by setting

$$\left[X, \sum_{j=1}^{9} a_jV_j\right] = \lambda X, \tag{22}$$

where λ is an arbitrary constant. From relation (22) we can find all possible non-zero a_i's that constitute $\mathrm{Nor}_L(X)$ and $\mathrm{Nor}_L(X)/X$ can be constructed by removing all linear combinations of X in the normaliser. In the following, we give a list of two-dimensional subalgebras with this construction:

$$
\begin{aligned}
M_1 &= \langle a_1V_1 + a_2V_2,\ V_9\rangle, \quad a_2 \neq 0, \quad a_9 = 0,\\
M_2 &= \langle a_1V_1 + a_9V_9,\ b_1V_2 + b_2V_3 + b_3V_5\rangle,\\
&\ a_2 = 0, a_9 \neq 0,\\
M_3 &= \langle a_2V_2 + a_3V_3 \pm V_6 + a_8V_8 + a_9V_9,\ a_2a_3V_2\\
&\qquad + a_3^2V_3 + a_2V_7\rangle,\\
M_4 &= \langle a_2V_2 + a_3V_3 \pm V_6 + a_8V_8 + a_9V_9,\ a_2V_2\\
&\qquad + a_3V_3 + V_6\rangle,\\
M_5 &= \langle a_1V_1 + a_3V_3,\ V_9\rangle, \quad a_9 = 0,
\end{aligned}
$$

Table 2. Invariants of full adjoint action (12).

	η	A	B	C	D
P_1	a_2^2	a_1	a_9	1	0
P_2	$a_2^2+a_3^2$	0	a_9	1	0
P_3	a_3^2	a_1	a_9	1	0
P_4	$a_2^2+a_5^2$	0	a_9	1	0
P_5	a_5^2	0	a_9	1	0
P_6	0	a_1	a_9	0	0
P_7	0	0	a_9	0	1
P_8	0	a_1	0	0	0
P_9	0	0	a_9	0	0

Table 3. Invariants of the general vector $\lambda X_1 + \mu X_2$ in $M = \langle X_1, X_2\rangle$.

	η	A	B	C	D
M_1	$(\lambda a_2)^2$	λa_1	μ	1	0
M_2	$\mu^2(b_1^2+b_2^2+b_3^2)$	λa_1	λa_9	1	0
M_3	$(\lambda+\mu a_3)^2(a_2^2+a_3^2)$	0	λa_9	1	0
M_4	$(\lambda+\mu)^2(a_2^2+a_3^2)$	0	λa_9	1	0
M_5	$(\lambda a_3)^2$	λa_1	μ	1	0
M_6	$(\lambda a_2)^2+(\lambda+\mu)^2 a_5^2$	0	λa_9	1	0
M_7	$(\lambda+\mu a_2)^2(a_2^2+a_5^2)$	0	λa_9	1	0
M_8	$(\lambda a_5)^2$	0	μ	1	0
M_9	$(\lambda a_5)^2$	0	λa_9	1	0
M_{10}	$(\lambda a_5)^2$	0	μb_2	1	0
M_{11}	$(\mu b_1)^2$	0	λa_9	1	0
M_{12}	$\mu^2(b_1^2+b_2^2+b_3^2)$	λ	μa_9	1	0

$$
\begin{aligned}
M_6 &= \langle a_2V_2 + a_5V_5 \pm V_7 + a_8V_8 \\
&\quad + a_9V_9, a_5V_5 + a_2V_7\rangle,\\
M_7 &= \langle a_2V_2 + a_5V_5 \pm V_7 + a_8V_8 + a_9V_9, a_2^2V_2 \\
&\quad - a_5V_4 + a_2a_5V_5\rangle,\\
M_8 &= \langle V_4 + a_5V_5 + a_8V_8, V_9\rangle, \quad a_9 = 0,\\
M_9 &= \langle V_4 + a_5V_5 + a_9V_9, V_8\rangle, \quad a_8 = 0,\\
M_{10} &= \langle V_4 + a_5V_5, b_1V_8 + b_2V_9\rangle, \quad a_8 = 0, \quad a_9 = 0,\\
M_{11} &= \langle V_7 + a_8V_8 + a_9V_9, b_1V_2 + b_2V_4 + b_3V_6\rangle,\\
M_{12} &= \langle V_1, b_1V_2 + b_2V_3 + b_3V_5 + b_4V_9\rangle,\\
M_{13} &= \langle V_9, P_i^*\rangle.
\end{aligned}
$$

Here

$P_i^* = P_i$ (without centre V_9, $i \neq 1, 8, 9$).

It remains to show that M_i's are mutually inequivalent. In view of the well-known fact [3,12,13], any two-dimensional subalgebra $\langle X_1, X_2\rangle$ can be written as

$X = \lambda X_1 + \mu X_2$.

The subalgebra $\langle X_1, X_2\rangle$ is inequivalent to subalgebra $\langle Y_1, Y_2\rangle$ if it is impossible to write $Y_i = \lambda_i X_1 + \mu_i X_2$ for any λ_i, μ_i, $i = 1, 2$. As an illustration, we show that M_1 and M_8 are inequivalent. On the contrary, let $V_4 + a_5V_5 + a_8V_8$ be mapped to some $\lambda(a_1V_1 + a_2V_2) + \mu(V_9)$. In terms of invariant A, $0 = A(V_4 + a_5V_5 + a_8V_8) = A(\lambda(a_1V_1 + a_2V_2) + \mu(V_9)) = \lambda a_1$; therefore $\lambda = 0$. This further implies that $V_4 + a_5V_5 + a_8V_8$ is equivalent to V_9, which is impossible. Therefore, M_1 and M_8 are inequivalent. Similarly, it is straightforward to see from table 3 that all $\{M_i, \ i = 1, \ldots, 12\}$ are inequivalent and inequivalence from M_{13} is justified from table 2.

3. Conservation laws

So many methods are available for the construction of conservation laws [31]. Among all such methods, the technique of Ibragimov [32] is quite popular, but the recent comments from Anco [33] confirm the incompleteness of the theorem. In particular, the formulation proposed by Ibragimov can generate trivial conservation laws and does not always yield all non-trivial conservation laws.

So in this section, we follow a more reliable method for constructing conservation laws. The method is called the direct method, which was first proposed by Anco and Bluman [34] using multipliers given by Olver [4]. The method is well established and the details can be seen in [34–36].

Before proceeding towards the construction of non-trivial local conservation laws, we need to familiarise with some of the basic terminologies used in the direct method. In the following, we state some of the definitions and theorems.

DEFINITION 1

The Euler operator with respect to dependent variable u^j is the operator defined by

$$
E_{u^j} = \frac{\delta}{\delta u^j} = \frac{\partial}{\partial u^j} + \sum_{s=1}^{\infty}(-1)^s D_{i_1}\ldots D_{i_s}\frac{\partial}{\partial u^j_{i_1\ldots i_s}},
\quad \text{for each } j = 1, \ldots, m, \tag{23}
$$

where D_i is the total differentiation with respect to independent variable $\mathbf{x} = (x^1, \ldots, x^n)$. The Euler operator defined by (23) has an interesting property that it can annihilate any divergence expression $D_i\Psi^i(u)$, and this property establishes the following theorem.

Theorem 1. *The equation* $E_{u^j}F(x, u, \partial u, \ldots, \partial^p u) \equiv 0$ *holds if and only if*

$$F(x, u, \partial u, \ldots, \partial^p u) \equiv D_i \Psi^i(x, u, \partial u, \ldots, \partial^{p-1} u),\quad i = 1, \ldots, n, \tag{24}$$

holds for some functions $\Psi^i(x, u, \partial u, \ldots, \partial^{p-1} u)$.

Proof. The proof of the theorem is given in [37]. □

Theorem 1 aids in the construction of local conservation laws through multipliers, and so the second theorem is stated as follows.

Theorem 2. *A set of non-singular multipliers* $\{\Lambda_\sigma(x, u, \partial u, \ldots, \partial^l u)\}_{\sigma=1}^N$ *yields a local conservation law for PDE system* $R^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0$ *if and only if*

$$E_{u^j}(\Lambda_\sigma(x, u, \partial u, \ldots, \partial^l u) R^\sigma(x, u, \partial u, \ldots, \partial^k u)) \equiv 0, \quad j = 1, \ldots, m. \tag{25}$$

Proof. The proof of the theorem is given in [37]. □

The set of eqs (25) are linear determining equations for all local conservation law multipliers up to order l. Solving the determining equations (25) is the same as solving the determining equations for infinitesimals in the Lie symmetry analysis. When all the multipliers are determined, the divergence expression (24) may be written as

$$\Lambda_\sigma(u) R^\sigma[u] = D_i \Psi^i[u], \tag{26}$$

which gives local conservation laws

$$\mathrm{Div}\, \Psi[u] = 0, \tag{27}$$

holding on the solution space of system $R^\sigma[u] = R^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0$, $\sigma = 1, \ldots, N$. For a given set of multipliers, the divergence expression (26) may be solved for fluxes $\Psi^i[u]$ in several ways. In one of the ways, when the multipliers are simple, the fluxes can be obtained by equating derivatives (26) and sometimes inversion of the divergence expression (26) for fluxes is too complex to handle. In such cases, the homotopy operator may be used to invert the divergence expression (26). The detailed discussion and implementation of the homotopy operator is given in [38] and references therein.

Now we shall use Theorem 1 to construct conservation law multipliers for (2). We consider zero-order multipliers $\Lambda_1(x, y, z, t, u, v, \psi)$, $\Lambda_2(x, y, z, t, u, v, \psi)$, $\Lambda_3(x, y, z, t, u, v, \psi)$, and such multipliers are given by determining equations (25). So we have

$$\begin{aligned} E_{u^j}[&\Lambda_1(x, y, z, t, u, v, \psi)\Delta_1 \\ &+ \Lambda_2(x, y, z, t, u, v, \psi)\Delta_2 \\ &+ \Lambda_3(x, y, z, t, u, v, \psi)\Delta_3] \equiv 0, \\ &j = 1, 2, 3, \end{aligned} \tag{28}$$

where E_{u^j} is the Euler operator (23) for $u^1 = u, u^2 = v$ and $u^3 = \psi$. The determining equations (28) split with respect to partial derivatives of u, v and ψ. This yields a linear determining system for Λ's, which can be solved by the same algorithmic method used for solving the determining equations for infinitesimal symmetries. The solution of such determining equations gives

$$\Lambda_1 = -v,\ \Lambda_2 = u,\ \Lambda_3 = 0, \tag{29a}$$

$$\Lambda_1 = 0,\ \Lambda_2 = 0,\quad \Lambda_3 = t\exp(ax + by)\sin(\sqrt{a^2 + b^2}\, z), \tag{29b}$$

$$\Lambda_1 = 0,\ \Lambda_2 = 0,\quad \Lambda_3 = t\exp(ax + by)\cos(\sqrt{a^2 + b^2}\, z), \tag{29c}$$

where $a = \pm 1$ and $b = \pm 1$.

Each multiplier from set (29a)–(29c) determines a local conservation law (27) in the format $D_x\Psi_1(x, y, z, t, u, v, \psi) + D_y\Psi_2(x, y, z, t, u, v, \psi) + D_y\Psi_3(x, y, z, t, u, v, \psi) + D_t\Psi_4(x, y, z, t, u, v, \psi) = 0$ with the characteristic form

$$\begin{aligned} &\Lambda_1(x, y, z, t, u)\Delta_1 + \Lambda_2(x, y, z, t, u)\Delta_2 \\ &\quad + \Lambda_3(x, y, z, t, u)\Delta_3 \\ &= D_x\Psi_1(x, y, z, t, u, v, \psi) \\ &\quad + D_y\Psi_2(x, y, z, t, u, v, \psi) \\ &\quad + D_z\Psi_3(x, y, z, t, u, v, \psi) \\ &\quad + D_t\Psi_4(x, y, z, t, u, v, \psi). \end{aligned} \tag{30}$$

The inversion of divergence expression (30) can be carried out by four-dimensional homotopy operator [38], and results read as follows:

- $\Lambda_1 = -v,\ \Lambda_2 = u,\ \Lambda_3 = 0.$

  $$\begin{aligned} \Psi_1 &= uv_x - u_x v, \\ \Psi_2 &= uv_y - u_y v, \\ \Psi_3 &= uv_z - u_z v, \\ \Psi_4 &= u_t v - uv_t. \end{aligned}$$

- $\Lambda_1 = 0,\ \Lambda_2 = 0,\ \Lambda_3 = t\exp(ax + by) \times \sin(\sqrt{a^2 + b^2}\, z)$.

  $$\begin{aligned} \Psi_1 &= t\,\mathrm{e}^{ax+by}\sin(\sqrt{a^2+b^2}z) \\ &\quad \times (ac^2\psi + au^2 + av^2 - c^2\psi_x - 2uu_x \\ &\quad - 2vv_x), \\ \Psi_2 &= t\,\mathrm{e}^{ax+by}\sin(\sqrt{a^2+b^2}z) \\ &\quad \times (bc^2\psi + bu^2 + bv^2 - c^2\psi_y - 2uu_y \\ &\quad - 2vv_y), \\ \Psi_3 &= t\,\mathrm{e}^{ax+by}(\cos(\sqrt{a^2+b^2}z)\sqrt{a^2+b^2}c^2\psi \\ &\quad + \cos(\sqrt{a^2+b^2}z)\sqrt{a^2+b^2}u^2 \end{aligned}$$

$$
\begin{aligned}
&+\cos(\sqrt{a^2+b^2}z)\sqrt{a^2+b^2}v^2\\
&-\sin(\sqrt{a^2+b^2}z)c^2\psi_z\\
&-2\sin(\sqrt{a^2+b^2}z)uu_z\\
&-2\sin(\sqrt{a^2+b^2}z)vv_z),\\
\Psi_4 &= e^{ax+by}\sin(\sqrt{a^2+b^2}z)\,(tw_t-\psi)\,.
\end{aligned}
$$

- $\Lambda_1 = 0,\ \Lambda_2 = 0,\ \Lambda_3 = t\exp(ax+by) \times \cos(\sqrt{a^2+b^2}\,z)$.

$$
\begin{aligned}
\Psi_1 &= t\,e^{ax+by}\cos(\sqrt{a^2+b^2}z)\\
&\quad\times(ac^2\psi+au^2+av^2-c^2\psi_x\\
&\quad-2uu_x-2vv_x),\\
\Psi_2 &= t\,e^{ax+by}\cos(\sqrt{a^2+b^2}z)\\
&\quad\times(bc^2\psi+bu^2+bv^2-c^2\psi_y\\
&\quad-2uu_y-2vv_y),\\
\Psi_3 &= -t\,e^{ax+by}(\sqrt{a^2+b^2}\sin(\sqrt{a^2+b^2}z)c^2\psi\\
&\quad+\sqrt{a^2+b^2}\sin(\sqrt{a^2+b^2}z)u^2\\
&\quad+\sqrt{a^2+b^2}\sin(\sqrt{a^2+b^2}z)v^2\\
&\quad+\cos(\sqrt{a^2+b^2}z)c^2\psi_z\\
&\quad+2\cos(\sqrt{a^2+b^2}z)uu_z\\
&\quad+2\cos(\sqrt{a^2+b^2}z)vv_z),\\
\Psi_4 &= e^{ax+by}\cos(\sqrt{a^2+b^2}z)\,(tw_t-\psi).
\end{aligned}
$$

4. Painlevé analysis of KGZ equation

To investigate the singularity structure of (2), the local Laurent expansion in a neighbourhood of non-characteristic manifold $g(z_1,\ldots,z_n)=0$ can be used [39].

The Laurent expansion can be taken in the form

$$
u = g^\alpha\sum_{j=0}^{\infty}u_j\,g^j,\quad v = g^\alpha\sum_{j=0}^{\infty}v_j\,g^j,
$$

$$
\psi = g^\alpha\sum_{j=0}^{\infty}\psi_j\,g^j, \tag{31}
$$

where $g=g(x,y,z,t)$.

The leading order analysis can be carried out by letting $u=u_0\,g^{\alpha_1}$, $v=v_0\,g^{\alpha_2}$, $\psi=\psi_0\,g^{\alpha_3}$. Inserting into (2) and balancing the most dominant terms, we obtain four branches as follows:

First branch

$$
\begin{aligned}
&\alpha_1=-1,\ \alpha_2=-1,\ \alpha_3=-2,\\
&\psi_0=-2(g_t^2-g_x^2-g_y^2-g_z^2),\\
&u_0=\frac{1}{g_x^2+g_y^2+g_z^2}(-(g_x^2+g_y^2+g_z^2)\\
&\quad\times(-2g_x^2g_t^2c^2-2g_y^2g_t^2c^2\\
&\quad-2g_z^2g_t^2c^2+2g_x^4c^2+4g_x^2g_y^2c^2\\
&\quad+4g_x^2g_z^2c^2+2g_y^4c^2+4g_y^2g_z^2c^2\\
&\quad+2g_z^4c^2+2g_t^4-2g_x^2g_t^2-2g_y^2g_t^2\\
&\quad-2g_z^2g_t^2+g_x^2v_0^2+g_y^2v_0^2\\
&\quad+g_z^2v_0^2))^{1/2},
\end{aligned}
\tag{32}
$$

where v_0 is arbitrary.

Second branch

$$
\begin{aligned}
&\alpha_1=-1,\ \alpha_2=-1,\ \alpha_3=-2\\
&\psi_0=-2(g_t^2-g_x^2-g_y^2-g_z{}^2),\\
&u_0=-\frac{1}{g_x^2+g_y^2+g_z^2}(-(g_x^2+g_y^2+g_z^2)\\
&\quad\times(-2g_x^2g_t^2c^2-2g_y^2g_t^2c^2\\
&\quad-2g_z^2g_t^2c^2+2g_x^4c^2+4g_x^2g_y^2c^2\\
&\quad+4g_x^2g_z^2c^2+2g_y^4c^2+4g_y^2g_z^2c^2\\
&\quad+2g_z^4c^2+2g_t^4-2g_x^2g_t^2-2g_y^2g_t^2\\
&\quad-2g_z^2g_t^2+g_x^2v_0^2+g_y^2v_0^2+g_z^2v_0^2))^{1/2},
\end{aligned}
\tag{33}
$$

where v_0 is arbitrary.

Third branch

$$
\begin{aligned}
&\alpha_1=-2,\ \alpha_2=-2,\ \alpha_3=-2,\\
&\psi_0=-6(g_t^2-g_x^2-g_y^2-g_z^2),\ u_0=\mathrm{i}v_0,
\end{aligned}
\tag{34}
$$

where v_0 is arbitrary.

Fourth branch

$$
\begin{aligned}
&\alpha_1=-2,\ \alpha_2=-2,\ \alpha_3=-2,\\
&\psi_0=-6(g_t^2-g_x^2-g_y^2-g_z^2),\\
&u_0=-\mathrm{i}v_0,
\end{aligned}
\tag{35}
$$

where v_0 is arbitrary.

The next step is to determine the resonant points corresponding to each branch. For the branch given by (32), we substitute

$$
\begin{aligned}
&u=u_0\,g^{-1}+u_j\,g^{j-1},\ v=v_0\,g^{-1}+v_j\,g^{j-1},\\
&\psi=\psi_0\,g^{-2}+\psi_j\,g^{j-2}
\end{aligned}
\tag{36}
$$

into (2) and retaining the most singular part. The resonance points read as

$$r = -1, 0, 2, 3, 4. \tag{37}$$

The first resonance is called Fuch index, and it is always there because of the arbitrary nature of singularity manifold $g(x, y, z, t)$. Same resonant points are found for branch (33) but unfortunately resonant points for branches (34) and (35) cannot be calculated.

Further, we consider the generalised Laurent expansions corresponding to first branch (32) in the form

$$\begin{aligned} u &= \frac{u_0}{g} + u_1 + u_2\, g + u_3\, g^2 + u_4\, g^3, \\ v &= \frac{v_0}{g} + v_1 + v_2\, g + v_3\, g^2 + v_4\, g^3, \\ \psi &= \frac{\psi_0}{g^2} + \frac{\psi_1}{g} + \psi_2 + \psi_3\, g + \psi_4\, g^2, \end{aligned} \tag{38}$$

where u_j, v_j, ψ_j $(j = 0, \ldots, 4)$ are analytic functions of (x, y, z, t) in the neighbourhood of singularity manifold g. Substitution of (38) into (2) leads to complex recursion relations for u_j, v_j, ψ_j. Further, the simplified analysis can be carried out using Kruskal's ansätz [39]

$$\begin{aligned} &g(x, y, z, t) = x + y + z - h(t), \\ &u_j = u_j(t),\ v_j = v_j(t),\ \psi_j = \psi_j(t), \\ &\quad \text{for } j = 0, \ldots, 4. \end{aligned} \tag{39}$$

This simplification is justifiable by the implicit function theorem whenever singular manifold is non-characteristic. Substituting (38) and (39) into (2) and collecting the coefficient of g corresponding to resonance $r = 2$ lead us to the compatibility condition

$$\begin{aligned} &-216 g_t^5 g_{t,t,t} c^4 - 108 g_t^4 g_{t,t}^2 c^4 + 192 g_t^7 g_{t,t,t} c^2 \\ &\quad + 192\, g_t^6 g_{t,t}^2 c^2 - 40 g_t^9 g_{t,t,t} - 20 g_t^8 g_{t,t}^2 \\ &\quad - 144 g_t^3 g_{t,t} c^4 + 972 g_t^3 g_{t,t,t} c^4 + 1566\, g_t^2 g_{t,t}^2 c^4 \\ &\quad + 240 g_t^5 g_{t,t} c^2 - 960 g_t^5 g_{t,t,t} c^2 - 1872 g_t^4 g_{t,t}^2 c^2 \\ &\quad + 212 g_t^7 g_{t,t,t} + 242\, g_t^6 g_{t,t}^2 - 1512\, g_t g_{t,t} c^4 \\ &\quad - 756\, g_t g_{t,t,t} c^4 - 378 g_{t,t}^2 c^4 - 168 g_t^3 g_{t,t} c^2 \\ &\quad + 984 g_t^3 g_{t,t,t} c^2 + 1548 g_t^2 g_{t,t}^2 c^2 - 244\, g_t^5 g_{t,t,t} \\ &\quad - 154 g_t^4 g_{t,t}^2 + 84\, g_t c^2 g_{t,t,t} + 42 g_{t,t}^2 c^2 \\ &\quad - 28 g_t^3 g_{t,t,t} + 42 g_t^2 g_{t,t}^2 = 0, \end{aligned}$$

which is clearly not satisfied for arbitrary g. Therefore, the KGZ equation fails to pass the Painlevé test for the first branch (32), and it is presumably not integrable. Same can also be verified for the second branch (33). Since the Painlevé property fails at both the branches, it is irrelevant to discuss the remaining branches (34) and (35) for the Painlevé property.

Remark 3. In the Painlevé analysis, the leading order and resonant points have been determined using the Maple package 'wkptest' [40]. Although this package quickly gives out the desired results, due to the complexity of system (2) it failed to verify compatibility conditions at resonant points. To encounter this problem, we have used Kruskal's ansätz (39) which considerably simplified the complex algebraic calculations.

5. Conclusion

In summary, the Lie symmetry analysis is performed in a symbolic manner to investigate the symmetries of KGZ equations. The corresponding Lie algebra is classified into one- and two-dimensional optimal subalgebras. Beside the Killing form, some more invariants are introduced in (17) and (18) to verify the optimality of the derived systems. Apart from the usual symmetry analysis and classifications, we have demonstrated the construction of several local non-trivial conservation laws based on the direct method of Anco and Bluman. It is pertinent to mention here that unlike the new conservation theorem of Ibragimov [32], which sometimes give trivial conservation laws, the direct method has given non-trivial conservation laws in a straightforward manner. Further, the package 'wkptest' built for 'Maple' and detailed manual calculations show that the equations under study are not Painlevé integrable. Although eqs (2) are not Painlevé integrable, the other integrable properties such as Bäcklund transformations and Lax pairs can still be explored, which we intend to explore in our future study.

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Appendix A. Maple code for full adjoint action

Let $M_{i,j}$ be (i, j)th entry in table 1.

$> F := (i, j) \to M_{i,j}:$

$> M := \text{Matrix}(9, F):$

This command will assign M the adjoint matrix equivalent to adjoint table 1.

$> \epsilon := E:$

$> J := [0, 9, 2, 8, 1, 6, 4, 7, 3, 5]:$

The order in J should be reversed when writing full adjoint action.

$> G_0 := \text{add}(a_k \cdot V_k, k = 1, \ldots, 9)$

$$G_0 := a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5 + a_6V_6 + a_7V_7 + a_8V_8 + a_9V_9$$

> for j from 2 to nops(J) do
  $k := J_j$
  $F_k := \mathrm{expand}(T_k \cdot G_{J_{j-1}})$
for ii to 9 do for jj to 9 do
  $F_k := \mathrm{expand}(\mathrm{algsubs}(T_{ii} \cdot V_{jj} = M_{ii,jj}, F_k))$
  od od
  $G_k := \mathrm{expand}(\mathrm{algsubs}(E = \epsilon_k, F_k))$
od

This final procedure will produce full adjoint action given by (12). For writing full adjoint in matrix format, the following procedure can be used:

> SM:= proc(ex, V, A)
Matrix(map($t \rightarrow$ [seq(coeff(t, A_i), $i = 1 \ldots$ nops(A))], map2(coeff, ex, V)))
end proc;
> SM(G_5, [seq($V_i, i = 1 \ldots 9$)], [seq($a_i, i = 1 \ldots 9$)])

Appendix B

$$\begin{aligned}
\tilde{a}_1 &= a_1,\\
\tilde{a}_2 &= -\cos(\epsilon_2)\cos(\epsilon_5)\sin(\epsilon_3)a_5\\
&\quad + \sin(\epsilon_2)\cos(\epsilon_5)\sin(\epsilon_3)a_3\\
&\quad + \cos(\epsilon_2)\sin(\epsilon_5)a_3 + \sin(\epsilon_2)\sin(\epsilon_5)a_5\\
&\quad + \cos(\epsilon_5)\cos(\epsilon_3)a_2,\\
\tilde{a}_3 &= \cos(\epsilon_2)\sin(\epsilon_5)\sin(\epsilon_3)a_5\\
&\quad - \sin(\epsilon_2)\sin(\epsilon_5)\sin(\epsilon_3)a_3\\
&\quad + \cos(\epsilon_2)\cos(\epsilon_5)a_3 + \sin(\epsilon_2)\cos(\epsilon_5)a_5\\
&\quad - \sin(\epsilon_5)\cos(\epsilon_3)a_2,\\
\tilde{a}_4 &= a_4\cos(\epsilon_2)e^{\epsilon_1}\cos(\epsilon_3) + a_3\cos(\epsilon_2)\epsilon_4\sin(\epsilon_3)\\
&\quad + a_5\cos(\epsilon_2)\epsilon_6\sin(\epsilon_3) - a_3\cos(\epsilon_2)\epsilon_7\cos(\epsilon_3)\\
&\quad + a_6\sin(\epsilon_2)e^{\epsilon_1}\cos(\epsilon_3) - a_3\sin(\epsilon_2)\epsilon_6\sin(\epsilon_3)\\
&\quad + a_5\sin(\epsilon_2)\epsilon_4\sin(\epsilon_3) - a_5\sin(\epsilon_2)\epsilon_7\cos(\epsilon_3)\\
&\quad + a_7e^{\epsilon_1}\sin(\epsilon_3) - a_1\epsilon_7\sin(\epsilon_3)\\
&\quad - a_1\epsilon_4\cos(\epsilon_3) - a_2\epsilon_6\cos(\epsilon_3),\\
\tilde{a}_5 &= a_5\cos(\epsilon_2)\cos(\epsilon_3) - a_3\sin(\epsilon_2)\cos(\epsilon_3)\\
&\quad + a_2\sin(\epsilon_3),\\
\tilde{a}_6 &= -\cos(\epsilon_2)e^{\epsilon_1}\sin(\epsilon_5)\sin(\epsilon_3)a_4\\
&\quad + \cos(\epsilon_2)\sin(\epsilon_5)\sin(\epsilon_3)a_3\epsilon_7\\
&\quad + \cos(\epsilon_2)\sin(\epsilon_5)\cos(\epsilon_3)a_3\epsilon_4\\
&\quad + \cos(\epsilon_2)\sin(\epsilon_5)\cos(\epsilon_3)a_5\epsilon_6\\
&\quad - \sin(\epsilon_2)e^{\epsilon_1}\sin(\epsilon_5)\sin(\epsilon_3)a_6\\
&\quad + \sin(\epsilon_2)\sin(\epsilon_5)\sin(\epsilon_3)a_5\epsilon_7\\
&\quad - \sin(\epsilon_2)\sin(\epsilon_5)\cos(\epsilon_3)a_3\epsilon_6\\
&\quad + \sin(\epsilon_2)\sin(\epsilon_5)\cos(\epsilon_3)a_5\epsilon_4\\
&\quad + \cos(\epsilon_2)e^{\epsilon_1}\cos(\epsilon_5)a_6 - \cos(\epsilon_2)\cos(\epsilon_5)a_5\epsilon_7\\
&\quad - \sin(\epsilon_2)e^{\epsilon_1}\cos(\epsilon_5)a_4 + \sin(\epsilon_2)\cos(\epsilon_5)a_3\epsilon_7\\
&\quad + e^{\epsilon_1}\sin(\epsilon_5)\cos(\epsilon_3)a_7\\
&\quad + \sin(\epsilon_5)\sin(\epsilon_3)a_1\epsilon_4 + \sin(\epsilon_5)\sin(\epsilon_3)a_2\epsilon_6\\
&\quad - \sin(\epsilon_5)\cos(\epsilon_3)a_1\epsilon_7\\
&\quad - \cos(\epsilon_5)a_1\epsilon_6 + \cos(\epsilon_5)a_2\epsilon_4,\\
\tilde{a}_7 &= -\cos(\epsilon_2)e^{\epsilon_1}\cos(\epsilon_5)\sin(\epsilon_3)a_4\\
&\quad + \cos(\epsilon_2)\cos(\epsilon_5)\sin(\epsilon_3)a_3\epsilon_7\\
&\quad + \cos(\epsilon_2)\cos(\epsilon_5)\cos(\epsilon_3)a_3\epsilon_4\\
&\quad + \cos(\epsilon_2)\cos(\epsilon_5)\cos(\epsilon_3)a_5\epsilon_6\\
&\quad - \sin(\epsilon_2)e^{\epsilon_1}\cos(\epsilon_5)\sin(\epsilon_3)a_6\\
&\quad + \sin(\epsilon_2)\cos(\epsilon_5)\sin(\epsilon_3)a_5\epsilon_7\\
&\quad - \sin(\epsilon_2)\cos(\epsilon_5)\cos(\epsilon_3)a_3\epsilon_6\\
&\quad + \sin(\epsilon_2)\cos(\epsilon_5)\cos(\epsilon_3)a_5\epsilon_4\\
&\quad - \cos(\epsilon_2)e^{\epsilon_1}\sin(\epsilon_5)a_6 + \cos(\epsilon_2)\sin(\epsilon_5)a_5\epsilon_7\\
&\quad + \sin(\epsilon_2)e^{\epsilon_1}\sin(\epsilon_5)a_4\\
&\quad - \sin(\epsilon_2)\sin(\epsilon_5)a_3\epsilon_7 + e^{\epsilon_1}\cos(\epsilon_5)\cos(\epsilon_3)a_7\\
&\quad + \cos(\epsilon_5)\sin(\epsilon_3)a_1\epsilon_4\\
&\quad + \cos(\epsilon_5)\sin(\epsilon_3)a_2\epsilon_6 - \cos(\epsilon_5)\cos(\epsilon_3)a_1\epsilon_7\\
&\quad + \sin(\epsilon_5)a_1\epsilon_6 - \sin(\epsilon_5)a_2\epsilon_4,\\
\tilde{a}_8 &= -e^{\epsilon_1}a_1\epsilon_8 + e^{\epsilon_1}a_8,\\
\tilde{a}_9 &= a_9.
\end{aligned}$$

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