Proc. Indian Acad. Sci. (Math. Sci.) (2018) 128:58
https://doi.org/10.1007/s12044-018-0435-y

Recognition of PSL(2, 2^a) by the orders of vanishing elements

JINSHAN ZHANG

Department of Mathematics, Shantou University, Shantou 515063,
People's Republic of China
E-mail: zjscdut@163.com

MS received 28 March 2017; revised 18 September 2017; accepted 16 October 2017;
published online 25 October 2018

Abstract. Here, we show that the simple groups PSL(2, 2^a), $a \geq 2$, are characterized by the orders of vanishing elements.

Keywords. Finite groups; characters; vanishing elements.

2000 Mathematics Subject Classification. 20C15.

1. Introduction

Let G be a finite group and Irr(G) be the set of irreducible characters of G. Denote by cd(G) the set of irreducible character degrees of G. The *character degree graph* of G is defined as follows: the vertices of this graph are the prime divisors of the irreducible character degrees of the group G and two distinct vertices p and q are joined by an edge if there exists an irreducible character degree of G which is divisible by pq. This graph was introduced in [12]. Recently, there has been much interest in the influence of arithmetical conditions on degrees of irreducible characters of a group G on the structure of G. For instance, Khosravi *et al.* [10] have proved that PSL(2, p^2) is uniquely determined by its order and its character degree graph. Jiang *et al.* [9] proved that simple groups PSL(2, 2^a) can be uniquely determined by its order and its character degree graph.

The goal of this paper is to introduce a new characterization for the finite group PSL(2, 2^a). Given a finite group G, a vanishing element of G is an element $g \in G$ such that $\chi(g) = 0$ for some irreducible complex character χ of G. Denote by $\pi_e(G)$ the set of orders of elements of G. We will denote the set of vanishing elements of G by Van(G). Our aim in this paper is to analyse a particular subset of $\pi_e(G)$, the set Vo(G) of the orders of elements in Van(G). We know that Vo(G) encodes some information about the structure of G (see [3,5,17]).

For a set Ω of positive integers, let $h(\Omega)$ be the number of isomorphism classes of finite group G such that $\pi_e(G) = \Omega$. For a given group G, we have $h(\pi_e(G)) \geq 1$. A group G is called characterizable (or recognizable) if $h(\pi_e(G)) = 1$. We define the following.

DEFINITION 1.1

For a set Ω of positive integers, let $v(\Omega)$ be the number of isomorphism classes of finite group G such that Vo$(G) = \Omega$. For a given non-abelian group G, we have $v(\text{Vo}(G)) \geq$

1. A non-abelian group G is called V-characterizable (or V-recognizable) if $\upsilon(\mathrm{Vo}(G)) = 1$.

In [16,17], it is shown that the following simple groups are V-recognizable: $L_2(q)$, where $q \in \{5, 7, 8, 17\}$, $L_3(4)$, A_7, $Sz(2^{2m+1})$. In this paper, we continue this work and obtain the following result:

Main Theorem. *Let G be a finite group. Then $G \cong \mathrm{PSL}(2, 2^a)$ $(a \geq 2)$ if and only if $\mathrm{Vo}(G) = \mathrm{Vo}(\mathrm{PSL}(2, 2^a))$.*

In this paper, G always denotes a finite group. Notation is standard and is taken from [6] and [8].

2. Preliminary results

Given a finite set of positive integers X, the *prime graph* $\Pi(X)$ is defined as the simple undirected graph whose vertices are the primes p such that there exists an element of X divisible by p, and two distinct vertices p and q are adjacent if and only if there exists an element of X divisible by p and q. For a finite group G, the graph $\Pi(\pi_e(G))$, which we denote by $GK(G)$, is also known as the Gruenberg—Kegel graph of G. Denote the vertex set of $GK(G)$ by $\pi(G)$. We denote the prime graph $\Pi(\mathrm{Vo}(G))$ by $\Gamma(G)$, which is called the vanishing prime graph of G. The vanishing prime graph was introduced in [3,4]. The following lemma provides some properties of the vanishing prime graph of a finite group and its relationship with the Gruenberg–Kegel graph. In what follows, we shall denote by $V(\mathcal{G})$ the vertex set of a graph $\mathcal{G}$, and by $n(\mathcal{G})$ the number of connected components of $\mathcal{G}$.

Lemma 2.1 [3,4]. *Let G be a finite group. Then the following hold*:

(1) *If G is solvable, then $\Gamma(G)$ has at most two connected components.*
(2) *If G is nonsolvable and $\Gamma(G)$ is disconnected, then G has a unique non-abelian composition factor S, and $n(\Gamma(G)) \leq n(GK(S))$ unless G is isomorphic to A_7.*

Lemma 2.2 [2, *Proposition* 2.1]. *Let G be a non-abelian simple group and p a prime number. If G is of Lie type, or if $p \geq 5$, then there exists $\chi \in \mathrm{Irr}(G)$ of p-defect zero.*

In the following lemma, we collect some basic remarks relating to the vanishing elements of a group G and the vanishing elements of the quotients of G. We shall freely use these results.

Lemma 2.3 [3,5]. *Let N be a normal subgroup of G.*

(1) *Any character of G/N can be viewed, by inflation, as a character of G. In particular, if $xN \in \mathrm{Van}(G/N)$, then $xN \subseteq \mathrm{Van}(G)$.*
(2) *If $p \in \pi(N)$ and N has an irreducible character of p-defect zero, then every element of N of order divisible by p is a vanishing element of G.*
(3) *If N is a normal subgroup of G and $m \in \mathrm{Vo}(G/N)$, then there exists an integer n such that $mn \in \mathrm{Vo}(G)$.*

Lemma 2.4 [8, *Theorem* 8.17]. *Let $\chi \in \mathrm{Irr}(G)$ and suppose that $p \nmid |G|/\chi(1)$ for some prime p. Then $\chi(g) = 0$ whenever $p \mid o(g)$.*

Remark 2.5. Let G be a simple group of Lie type. By Lemma 2.2, G has characters of p-defect zero for every prime p, and hence by Lemma 2.4, every non-identity element of G is a vanishing element. Hence $\mathrm{Vo}(G) = \pi_e(G) - \{1\}$.

Remark 2.6. By Remark 2.5, $\mathrm{Vo}(\mathrm{PSL}(2, 2^{\mathrm{a}})) = \pi_{\mathrm{e}}(\mathrm{PSL}(2, 2^{\mathrm{a}})) - \{1\} = \{2$, all factors of $(2^a - 1)$ and $(2^a + 1)\} - \{1\}$. Therefore the graph $\Gamma(\mathrm{PSL}(2, 2^a))$ has three connected components and all connected components are complete graphs.

3. Proof of the main theorem

Lemma 3.1 [3, *Proposition* 2.10]. *Let S be a sporadic simple group, or an alternating group on n letters with $n \geq 8$. Then S has an irreducible character ϕ which extends to* $\mathrm{Aut}(S)$ *and an element g of order* 6 *such that $\phi(g) = 0$.*

Lemma 3.2 [9, *Lemma* 2.6]. *Let a, b be two negative integers with $a > b \geq 2$. If $b \mid a$, then $\frac{2^{2a}-1}{2^{2b}-1} > b$.*

The proof of the main theorem uses the classification of non-solvable CIT-groups. A group G is called a CIT-group if G is a group of even order containing no element of order $2p$, with p an odd prime (see the introduction part of [14]). In the following, we give the proof of the main theorem.

Proof of the main theorem. We assume that $\mathrm{Vo}(G) = \mathrm{Vo}(\mathrm{PSL}(2, 2^{\mathrm{a}}))$. Then by Remark 2.6, $\Gamma(G)$ has three connected components. By part (1) of Lemma 2.1, G is non-solvable. Let N be the solvable radical of G. Then by part (2) of Lemma 2.1, G has a normal series

$$1 \leq N < M \leq G,$$

where G/M is a solvable group, and M/N is a non-cyclic simple group. Now we consider the group $\bar{G} := G/N$. Denote by $\bar{M}$ the group M/N. As N is the solvable radical of G, $\bar{G} \leq \mathrm{Aut}(\bar{M})$ and $G/M \leq \mathrm{Out}(\bar{M})$.

Step 1. *$\bar{M}$ is isomorphic to* $\mathrm{PSL}(2, 2^b)$, *for some positive integer b*. Let $\bar{M}$ be a sporadic simple group, or an alternating group on n letters with $n \geq 8$. Then, by Lemma 3.1, $\bar{M}$ has an irreducible character ϕ which extends to $\mathrm{Aut}(\bar{M})$ and an element g of order 6 such that $\phi(g) = 0$. So $g \in \mathrm{Van}(\bar{G})$, and thus $\mathrm{Van}(G)$ contains an element of order divisible by 6, a contradiction.

Now, we assume that $\bar{M} \cong A_7$. Then $M = G$, as otherwise $\bar{G} \cong S_7$, and hence G, have vanishing elements of order divisible by 6, a contradiction. As $\bar{M} \cong A_7$ and $M = G$, $4 \in \mathrm{Vo}(\bar{G})$, and thus $\mathrm{Van}(G)$ contains an element of order divisible by 4, a contradiction. By the classification theorem of finite simple groups, we can now suppose that $\bar{M}$ is a simple group of Lie type (note that $A_5 \cong L_2(5)$ and $A_6 \cong L_2(9)$). Then by Lemma 2.2, for any prime divisor p of $|\bar{M}|$, there exists $\chi_p \in \mathrm{Irr}(\bar{M})$ such that χ_p is of p-defect zero, and so every element of $\bar{M}$ of order divisible by p is a vanishing element of $\bar{G}$. Therefore, every non-identity element of $\bar{M}$ is a vanishing element of $\bar{G}$.

On the other hand, by Lemma 2.1, $n(GK(\bar{M})) \geq 3$. Considering $\pi_1(\bar{M}) = \{2\}$, we now inspect the groups with ≥ 3 prime graph components listed in [15, Tables Id and Ie] and [11, Table 3]. We collect the connected components of $\Gamma(\bar{M})$ in Table 1.

Notice that ${}^2B_2(q)$ contains elements of order 4. As the elements of even order in $\bar{M}$ are of order 2, it follows by [1] that $\bar{M}$ is isomorphic to $\mathrm{PSL}(2, 2^b)$, for some positive integer b.

Table 1. Connected components of $\Gamma(\bar{M})$.

$\bar{M}$	π_1	π_2	π_3	π_4
$L_2(9)$	2	5	3	
$A_1(q)$	2	$\pi(q-1)$	$\pi(q+1)$	
$A_2(2)$	2	7	3	
$A_2(4)$	2	7	5	3
${}^2B_2(q)$	2	$\pi(q-1)$	$\pi(q-\sqrt{2q}+1)$	$\pi(q+\sqrt{2q}+1)$

Step 2. $N = 1$. Assume that $N > 1$. Let $1 \leq V < N$ such that N/V is a chief factor of G. Then N/V is an elementary abelian p-group, for some prime p. Now, we consider the group $\tilde{G} := G/V$. As $\tilde{M}/\tilde{N} \cong M/N \cong \mathrm{PSL}(2, 2^b)$, it follows by Remark 2.5 that for any prime p in $\pi(\tilde{M}/\tilde{N})$, $\tilde{M}/\tilde{N}$ has an irreducible character of p-defect zero, and so every element of $\tilde{M}/\tilde{N}$ of order divisible by p is a vanishing element of $\tilde{G}$. Hence every non-identity element of $\tilde{M}/\tilde{N}$ is a vanishing element of $\tilde{G}$, and thus $\tilde{M}\backslash\tilde{N} \subseteq \mathrm{Van}(\tilde{G})$. On the other hand, we have that $\mathrm{Vo}(G) = \{2, \text{all factors of } (2^a-1) \text{ and } (2^a+1)\} - \{1\}$. Therefore, since every element of $\mathrm{Vo}(\tilde{G})$ is a factor of some element in $\mathrm{Vo}(G)$, we get

$$\pi_e(\tilde{M}\backslash\tilde{N}) \subseteq \left\{2, \text{all factors of}\,(2^a-1) \text{ and } (2^a+1)\right\}.$$

Notice that $\tilde{N}$ is an elementary abelian p-group. Thus we infer

$$\pi_e(\tilde{M}) = \pi_e(\tilde{M}\backslash\tilde{N}) \cup \pi_e(\tilde{N}) = \pi_e(\tilde{M}\backslash\tilde{N}) \cup \{1, p\}.$$

Hence, $\tilde{M}$ is a CIT-group, then by [14, III, Theorem 5], we obtain that $p = 2$. For any element x in $\tilde{M}\backslash\tilde{N}$, we get that $o(x) = o(x\tilde{N})$, or $2 \cdot o(x\tilde{N})$. Note that $\tilde{M}/\tilde{N} \cong \mathrm{PSL}(2, 2^b)$, therefore, since $\tilde{M}/\tilde{N}$ does not contain elements of order 4, we get

$$\pi_e(\tilde{M}) = \{2, \text{all factors of}\,(2^b-1) \text{ and } (2^b+1)\}.$$

Then by [13], $\tilde{M} \cong \mathrm{PSL}(2, 2^b)$, a contradiction. Hence $N = 1$.

Step 3. *M isomorphic to* $\mathrm{PSL}(2, 2^a)$. As $N = 1$, $M \cong \mathrm{PSL}(2, 2^b)$. It is well-known that $\mathrm{Out}(M) \cong C_b$. Hence $G/M \leq C_b$. Now, we show that $b = a$. Assume that $b < a$. Since any non-identity element of M is a vanishing element of G and $\mathrm{Vo}(G) = \{2, \text{all factors of } (2^a-1) \text{ and } (2^a+1)\} - \{1\}$, it is easy to see that $2^b - 1 \mid 2^a - 1$. Let $a = bq + r$, where $0 \leq r < b$. Then we have

$$\begin{aligned} 2^a - 1 &= 2^r(2^b)^q - 2^r + 2^r - 1 \\ &= 2^r[(2^b)^q - 1] + (2^r - 1) \\ &= 2^r(2^b-1)[(2^b)^{q-1} + \cdots + 2^b + 1] + (2^r - 1). \end{aligned}$$

Hence, by $2^b - 1 \mid 2^a - 1$, we get that $2^b - 1 \mid 2^r - 1$, and so $r = 0$, namely, $b \mid a$. On the other hand, we easily get

$$
\begin{aligned}
|G/M| &\geq \frac{(2^a - 1)(2^a + 1)}{(2^b - 1)(2^b + 1)} \\
&= \frac{2^{2a-1}}{2^{2b-1}} > b \text{ (see Lemma 3.2).}
\end{aligned}
$$

Hence we obtain a contradiction (note that $G/M \leq C_b$).

Step 4. *G isomorphic to* PSL$(2, 2^a)$. Assume that $G > M$. First, we suppose that there exists an odd prime r such that $r \mid |G/M|$. By [7, Chap. XI. Theorem 5.10], M has an unique irreducible character ω such that $\omega(1) = |M|_2$, and so ω is invariant in G.

Let R be a normal subgroup of G with $M < R \leq G$ and $|R/M| = r$ (note that the outer automorphism group of M is cyclic). As ω is invariant in G, it is also invariant in R. So it follows by [8, Corollary 11.22] that ω is extensible to R, namely, there exists $\nu \in \mathrm{Irr}(R)$ such that $\nu_R = \omega$. Hence ν is of 2-defect zero, and every element of R of order divisible by 2 is a vanishing element of G. So it follows by hypothesis that R is a CIT-group. Therefore, R has a normal 2-group U such that R/U is isomorphic to one of the following groups (see [14, III, Theorem 5]): $L_2(q)$, $q = 2^k$, $k \geq 2$ or $q = p$ is a Fermat prime or Mersenne prime, or $q = 9$; $Sz(2^{2n_1+1})$, $n_1 \geq 1$; $L_3(4)$; M_9.

Then by Jordan–Hölder theorem, we obtain that $r = 2$, a contradiction, which implies that $|G/M|$ is a power of 2. For any odd prime factor s of $|G|$, there exists an element α in $\mathrm{Irr}(M)$ such that α is of s-defect zero. Let β be an irreducible constituent of α^G. Then we get that β is of s-defect zero. So it follows by the hypothesis that G has no element of order $2s$. In consideration of the arbitrariness of s, we conclude that G is a CIT-group. Note that N is the solvable radical of G and $N = 1$; it follows by [14, III, Theorem 5]) that $M = G$, and so G isomorphic to PSL$(2, 2^a)$. The proof is complete. □

Acknowledgements

The author is grateful to the referee for pointing out some inaccuracies in an earlier version of the paper as well as for helpful comments that greatly improved the exposition of the paper. This work is supported by the NNSF of China (11671245) and also supported by the Opening Project of Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing (2017QZJ01).

References

[1] Conway J H, Curtis R T, Norton S P, Park R A and Wilson R A, Atlas of finite groups (1985) (Oxford: Clarendon Press)

[2] Dolfi S, Pacifici E, Sanus L and Spiga P, On the orders of zeros of irreducible characters, *J. Algebra*, **321** (2009) 345–352

[3] Dolfi S, Pacifici E, Sanus L and Spiga P, On the vanishing prime graph of finite groups, *J. London Math. Soc.*, **82** (2010) 167–183

[4] Dolfi S, Pacifici E, Sanus L and Spiga P, On the vanishing prime graph of solvable groups, *J. Group Theory*, **13** (2010) 189–206

[5] Ghasemabadi M F, Iranmanesh A and Mavadatpour F, A new characterization of some finite simple groups, *Sib. Math. J.*, **56** (2015) 78–82

[6] Huppert B, Character Theory of Finite Groups (1998) (Berlin: de Gruyter)
[7] Huppert B and Blackburn N, Finite groups III (1982) (Berlin: Springer)
[8] Isaacs I M, Character theory of finite groups (1976) (Cambridge: Academic Press)
[9] Jiang Q H, Shao C G and Zhang J S, New characterization of $\mathrm{PSL}(2, 2^a)$ by character degree graph and order, submitted
[10] Khosravi B, Khosravi B, Khosravi B and Momen Z, Recognition of the simple group $\mathrm{PSL}(2, p^2)$ by character degree graph and order, *Monatsh. Math.*, **178** (2015) 251–257
[11] Kondratev A S, On prime graph components of finite simple groups, *Mat. Sb.*, **180** (1989) 787–797
[12] Manz O, Staszewski R and Willems W, On the number of components of a graph related to character degrees, *Proc. Am. Math. Soc.*, **103** (1988) 31–37
[13] Shi W J, A characteristic property of J_1 and $\mathrm{PSL}(2, 2^n)$, *Adv. Math. (China)*, **16** (1987) 397–401
[14] Suzuki M, Finite groups with nilpotent centralizers, *Trans. Am. Math. Soc.*, **99** (1961) 425–470
[15] Williama J S, Prime graph components of finite groups, *J. Algebra*, **69** (1981) 487–513
[16] Zhang J S, Shen Z C and Shao C G, Recognition of some finite simple groups by the orders of vanishing elements, submitted
[17] Zhang J S, Shao C G and Shen Z C, A new characterization of Suzuki's simple groups, *J. Algebra Appl.*, **16** (2017) 6 pages

COMMUNICATING EDITOR: B Sury