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Iteration of certain exponential-like meromorphic functions

TARUN KUMAR CHAKRA[1,*], TARAKANTA NAYAK[1]
and KEDARNATH SENAPATI[2]

[1]School of Basic Sciences, Indian Institute of Technology Bhubaneswar, Bhubaneswar 752 050, India
[2]Department of Mathematical and Computational Sciences, National Institute of Technology, Surathkal, Mangalore 575 025, India
*Corresponding author.
E-mail: tkc10@iitbbs.ac.in; tnayak@iitbbs.ac.in; kedar@nitk.edu.in

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Abstract. The dynamics of functions $f_\lambda(z) = \lambda \frac{e^z}{z+1}$ for $z \in \mathbb{C}, \lambda > 0$ is studied showing that there exists $\lambda^* > 0$ such that the Julia set of f_λ is disconnected for $0 < \lambda < \lambda^*$ whereas it is the whole Riemann sphere for $\lambda > \lambda^*$. Further, for $0 < \lambda < \lambda^*$, the Julia set is a disjoint union of two topologically and dynamically distinct completely invariant subsets, one of which is totally disconnected. The union of the escaping set and the backward orbit of ∞ is shown to be disconnected for $0 < \lambda < \lambda^*$ whereas it is connected for $\lambda > \lambda^*$. For complex λ, it is proved that either all multiply connected Fatou components ultimately land on an attracting or parabolic domain containing the omitted value of the function or the Julia set is connected. In the latter case, the Fatou set can be empty or consists of Siegel disks. All these possibilities are shown to occur for suitable parameters. Meromorphic functions $E_n(z) = e^z(1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!})^{-1}$, which we call exponential-like, are studied as a generalization of $f(z) = \frac{e^z}{z+1}$ which is nothing but $E_1(z)$. This name is justified by showing that E_n has an omitted value 0 and there are no other finite singular value. In fact, it is shown that there is only one singularity over 0 as well as over ∞ and both are direct. Non-existence of Herman rings are proved for λE_n.

Keywords. Chaotic burst; meromorphic function; Fatou set; Julia set; escaping set.

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1. Introduction

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a transcendental meromorphic function. The set of points $z \in \mathbb{C}$ in a neighborhood of which the sequence of iterates $\{f^n\}_{n>0}$ is defined and forms a normal family is called the Fatou set of f and is denoted by $\mathcal{F}(f)$. The Julia set, denoted by $\mathcal{J}(f)$, is the complement of $\mathcal{F}(f)$ in $\hat{\mathbb{C}}$. Complex dynamics studies the Fatou sets and the Julia sets of meromorphic functions. The Fatou set is open by definition and each of its maximally connected subset is known as a Fatou component. A Fatou component U is called p-periodic if p is the smallest natural number satisfying $f^p(U) \subseteq U$. If for all the

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points z in U, $\lim_{n\to\infty} f^{pn}(z) = z_0$ where z_0 is an attracting p-periodic point lying in U, the component U is called an attracting domain. If the above holds for all $z \in U$, where z_0 is a parabolic p-periodic point, then U is called a parabolic domain and in this case, z_0 lies on the boundary of U. The Fatou component U is said to be a Siegel disk if there exists an analytic homeomorphism $\phi : U \to \{z : |z| < 1\}$ such that $\phi(f^p(\phi^{-1}(z))) = e^{i2\pi\alpha}z$ for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. In this case, there is a p-periodic point z_0 in U and $(f^p)'(z_0) = e^{2\pi i\alpha}$. This is called the multiplier of z_0 and α is called the rotation number of the Siegel disk. It is known that if the argument of the multiplier of a p-periodic point is a Brjuno number, then there is a Siegel disk containing z_0 [5]. If there exists an analytic homeomorphism $\phi : U \to \{z : 1 < |z| < r\}$ such that $\phi(f^p(\phi^{-1}(z))) = e^{i2\pi\alpha}z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then U is called a Herman ring. If for $z \in U$, $\lim_{n\to\infty} f^{pn}(z) = z_0$ and $f^p(z_0)$ is not defined, then U is called a Baker domain. It is also possible for a Fatou component U that $f^m(U) \cap f^n(U) = \emptyset$ for all $m \neq n$. Such a component is called wandering.

The Julia sets of f_λ changes from a connected nowhere dense subset to the whole sphere as the parameter λ crosses the value $1/e$ from the left-hand side [6]. This phenomena is known as *chaotic burst* in the literature. Initiated by this, the exponential family (λe^z for $\lambda \in \mathbb{C}$) has been investigated later by many researchers and it is probably the most comprehensively studied one-parameter family of transcendental functions. Several aspects of exponential dynamics including the escaping set, Siegel disks and the parameter space have been investigated. All these investigations have been mostly influenced by some key facts. There is only one singular value, namely 0 and that is an omitted value. Further, there is only one singularity over it. There is also a single singularity over the point at infinity. In less technical term, the pre-image of each ball around the origin, not necessarily with small radius, is a left half plane. Lastly, a number of dynamical aspects are tractable because the exponential is periodic. Attempts are made to investigate dynamics of entire functions that are in one way or the other similar to, or generalizations of the exponential maps [8,14,15].

Meromorphic functions that can be said to be similar to the exponential are also studied recently. In [12], it has been observed that chaotic burst occurs for the one parameter family of Joukowski-exponential maps $\{g_\lambda = \lambda(e^z + 1 + \frac{1}{e^z+1}) : \lambda > 0\}$ at the parameter value $\lambda^* \approx 0.266$. In spite of this similarity, the Julia set of g_λ is found to be disconnected when $\lambda < \lambda^*$. In fact, it is a disjoint union of two completely invariant subsets, one of which is totally disconnected. This is a natural consequence of the presence of poles. Each g_λ has two asymptotic values, ∞ and 2λ like exponential maps and has an additional critical value -2λ. Like exponential maps, Joukowski-exponential maps are periodic.

This paper is an attempt to study meromorphic functions which are not periodic but seems to be a more natural candidate for the meromorphic generalization of the exponential. We consider $E_n(z) = \frac{e^z}{P_n(z)}$, where $P_n(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$. The function E_n has n poles giving rise to some amount of complexity into the study. However, it remains simple like exponential in terms of singular values. The function E_n has an omitted value. In fact, the omitted value is also the critical point and there are no other finite singular value. Thus it has a single orbit of finite singular values. Further, there is only one singularity over 0 as well as over ∞ and both are direct (shown in the last section) which is exactly the situation with the exponential maps. In addition to this, the singular values are real and the real line is preserved which simplifies the dynamical study of λE_n for real λ giving insight for the case of complex parameters. This is why we suggest to call these maps exponential-like meromorphic functions. However, this article is mostly concerned with the simplest case, i.e., $n = 1$.

Let $f_\lambda = \lambda E_1$ and consider $\mathcal{K} = \{f_\lambda(z) = \lambda f(z) : f(z) = \frac{e^z}{z+1} \text{ for } z \in \mathbb{C},\ \lambda > 0\}$. The function $f_\lambda(z)$ has a single pole, namely -1 and that is not an omitted value. It is proved that, for $f_\lambda \in \mathcal{K}$, there exists $\lambda^* > 0$ such that

(1) when $0 < \lambda < \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ is the attracting basin of a real positive fixed point,
(2) when $\lambda = \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ is the parabolic basin corresponding to a real rationally indifferent fixed point and
(3) for $\lambda > \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ is empty.

Further, it is shown that $\mathcal{F}(f_\lambda)$ is infinitely connected for $0 < \lambda < \lambda^*$. In other words, the Julia set is disconnected and a detailed investigation is made in this case. More precisely, the Julia set is shown to consist of an unbounded forward invariant component, infinitely many non-singleton bounded components and singleton components. The singleton Julia components form a totally disconnected and completely invariant subset of $\mathcal{J}(f_\lambda)$ (Theorem 12). The Julia sets are plotted for different values of λ.

The escaping set $I(f)$ of a meromorphic function f is defined as $\{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}$. The change in topology of $I(f_\lambda)$ is investigated showing that the union of the backward orbit of ∞ and $I(f_\lambda)$ is disconnected for $0 < \lambda < \lambda^*$ and is connected for $\lambda > \lambda^*$ (Theorem 13). A comparison between f_λ, the exponential and the Joukowski exponential maps is made in table 1.

Dynamics of f_λ for complex λ is discussed. It is shown that either all multiply connected Fatou components ultimately land on an attracting or parabolic domain containing the omitted value or the Julia set is connected (Theorem 14). In the latter case, the Julia set can be the whole sphere and Siegel disks can occur. All these possibilities are shown to be true for suitable parameters (Theorem 16). Certain dynamical similarities are found between λE_n and f_λ. In particular, Theorems 14 and 16 remain true for λE_n. A more general proof for non-existence of Herman rings is provided.

In section 2, all the singular values are found and the nature of all the singularities lying over them are determined. Section 3 discusses the dynamics of f_λ. The Julia set and the escaping set of f_λ for $\lambda > 0$, $\lambda \neq \lambda^*$ are studied in detail in section 4. This section also discusses the algorithm used to generate pictures of Julia sets of f_λ. Some detailed remarks on dynamics of f_λ, $\lambda \in \mathbb{C}$ are made in section 5 and the section 6 discusses some dynamical aspects of functions λE_n.

We use the following notations throughout this paper. Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\}$. Let $A \subset \hat{\mathbb{C}}$ be any domain. Then the boundary and the closure of A are denoted by ∂A and $\bar{A}$ respectively. Also, the pre-image $\{z \in \mathbb{C} : f(z) \in A\}$ of A is denoted by $f^{-1}(A)$. Let the backward orbit $O^-(\infty)$ of ∞ be defined as the set $\{z \in \mathbb{C} : f^n(z) = \infty \text{ for some } n \in \mathbb{N}\}$.

2. Singular values of f_λ

All the critical values and the asymptotic values of a function f and all the limit points of these values are known as the singular values of f. The set of singular values of f_λ is denoted by $\text{sing}(f_\lambda^{-1})$. These are also known as the singularities of the inverse function f_λ^{-1}. The definition and classification of singularities of inverse function of transcendental meromorphic functions can be found in [3]. A complex number a is called an omitted value of f if $f(z) \neq a$ for any z. Omitted values are known to be asymptotic values.

Lemma 1. *The function f has only one critical value, namely* 1 *and two asymptotic values namely*, 0 *and* ∞.

Proof. Since $f'(z) = \frac{ze^z}{(z+1)^2} = 0$ only for $z = 0$, $f(0) = 1$ is the critical value of f. If $\gamma_1(t) = -t$ for $t \in [0, \infty)$ then $\lim_{t\to\infty} f(\gamma_1(t)) = 0$. If $\gamma_2(t) = t$ for $t \in [0, \infty)$ then $\lim_{t\to\infty} f(\gamma_2(t)) = \infty$. This shows that both 0 and ∞ are asymptotic values of f. If a meromorphic function of finite order ρ has only finitely many critical values, then it has at most 2ρ asymptotic values (see [3], Corollary 3). Since the order of f is one and 1 is the only critical value of f, it has at most two asymptotic values. This proves that 0 and ∞ are the only asymptotic values of f. □

From Lemma 1, it follows that, $\mathrm{sing}(f_\lambda^{-1}) = \{\lambda, 0, \infty\}$. The asymptotic values of f_λ are 0 and ∞, and the critical value is λ. Note that the asymptotic value 0 is the only omitted value of the function and is mapped to λ. In order to determine the nature of singularities of f_λ^{-1} lying over the asymptotic values, a careful look at the pre-images of disks around 0 is necessary. The pre-image of every disk containing the omitted value 0 is clearly unbounded (see [10], Lemma 2.1). Further details are found in the following lemma. Let $D_r = \{z \in \mathbb{C} : |z| < r\}$ and $f^{-1}(D_r)$ be denoted by A_r.

Lemma 2. *For each $r > 0$, the set A_r is connected. Further, the following are true*:

(1) *For $r < 1$, A_r is simply connected and its boundary ∂A_r is an unbounded Jordan curve in $\hat{\mathbb{C}}$.*
(2) *For $r = 1$, A_r is simply connected and $\partial A_r = \gamma_1 \cup \gamma_2$, $\gamma_1 \cap \gamma_2 = \{0\}$, where γ_1 is a bounded Jordan curve and γ_2 is an unbounded Jordan curve in $\hat{\mathbb{C}}$.*
(3) *For $r > 1$, A_r is doubly connected and $\partial A_r = \tau_1 \cup \tau_2$, $\tau_1 \cap \tau_2 = \emptyset$, where τ_1 is a bounded Jordan curve and τ_2 is an unbounded Jordan curve in $\hat{\mathbb{C}}$.*

Proof. For $r > 0$, let $f^{-1}(\partial D_r) = \{z \in \mathbb{C} : |f(z)| = r\}$. Note that $|f(z)| = r$ if and only if $|e^z| = r|z+1|$. This is same as $y = \pm\sqrt{\frac{e^{2x}}{r^2} - (x+1)^2}$. Let $\varphi_r(x) = \frac{e^{2x}}{r^2} - (x+1)^2 = \frac{(x+1)^2}{r^2}\psi_r(x)$, where $\psi_r(x) = \frac{e^{2x}}{(x+1)^2} - r^2$. It is important to note that φ_r and ψ_r have same sign for all values of x and r. We have $\psi_r'(x) = \frac{2xe^{2x}}{(x+1)^3} > 0$ for $x < -1$ and $x > 0$, and $\psi_r'(x) < 0$ for $-1 < x < 0$. Also, $\lim_{x\to-\infty} \psi_r(x) = -r^2$ and $\lim_{x\to-1} \psi_r(x) = \infty$. By the continuity of $\psi_r(x)$ there exists a unique $x_0 < -1$ such that $\psi_r(x_0) = 0$. But $\lim_{x\to-1} \psi_r(x) = \infty$ when x approaches -1 from the right-hand side, $\psi_r(x)$ decreases to the minimum value $1 - r^2$ at $x = 0$ and then increases to ∞ as $x \to \infty$. For different values of r, we have the following cases:

(1) For $r < 1$, it follows that $\psi_r(x) = 0$ has no solution in $(-1, \infty)$. So, $\psi_r(x) < 0$ for $x < x_0$, $\psi_r(x_0) = 0$ and $\psi_r(x) > 0$ for $x > x_0$. Consequently, $\varphi_r(x) < 0$ for $x < x_0$ and $\varphi_r(x_0) = 0$ and $\varphi_r(x) > 0$ for $x > x_0$ (see figure 1(a)).

The function $\sqrt{\varphi_r(x)}$ is defined as a real valued function only when $\varphi_r(x) \geq 0$, that is, when $x \geq x_0$. So ∂A_r is the union of the graphs of $y_1 = \sqrt{\varphi_r(x)}$ and $y_2 = -\sqrt{\varphi_r(x)}$. Note that $|y_i| \to \infty$ when $x \to \infty$ for $i = 1, 2$. Hence ∂A_r is an unbounded Jordan curve and both of its ends tend to ∞ in the right half plane. Since the negative real axis is an asymptotic path corresponding to 0, we have that $A_r = \{x + iy : |y| > \sqrt{\varphi_r(x)},\ x > x_0\} \cup \{x + iy : x \leq x_0\} \setminus \{x_0\}$ which is clearly simply connected (see figure 1(d)).

(2) For $r = 1$, $\psi_r(x) = 0$ has a unique solution 0 in $(-1, \infty)$. This implies that $\psi_r(x) < 0$ for $x < x_0$, $\psi_r(x_0) = 0$, $\psi_r(x) > 0$ for $x \in (x_0, 0)$, $\psi_r(0) = 0$ and $\psi_r(x) > 0$ for $x \in (0, \infty)$. Consequently, $\varphi_r(x) < 0$ for $x < x_0$ and $\varphi_r(x_0) = 0 = \varphi_r(0)$ and $\varphi_r(x) > 0$ for $x \in (x_0, 0) \cup (0, \infty)$ (see figure 1(b)).

In this case, $\sqrt{\varphi_r(x)}$ is defined as a real valued function only when $x > x_0$. Let $\gamma_1^+ = \{x + iy : y = \sqrt{\varphi_r(x)},\ x_0 < x < 0\}$ and $\gamma_2^+ = \{x + iy : y = \sqrt{\varphi_r(x)},\ x > 0\}$. Let $\gamma_1^- = \{z : \bar{z} \in \gamma_1^+\}$, $\gamma_2^- = \{z : \bar{z} \in \gamma_2^+\}$ and $\gamma_i = \overline{\gamma_i^+ \cup \gamma_i^-}$ for $i = 1, 2$. Since each of the branches $\pm\sqrt{\varphi_r(x)}$ in $(x_0, 0)$ are bounded, this implies that γ_1 is bounded. Similarly, as each of the branches $\pm\sqrt{\varphi_r(x)}$ in $(0, \infty)$ are unbounded, we have that γ_2 is unbounded. Note that $\gamma_1 \cap \gamma_2 = \{0\}$. Now, $\partial A_r = \gamma_1 \cup \gamma_2$ and $A_r = \{x + iy : |y| > \sqrt{\varphi_r(x)},\ x \in (x_0, \infty)\} \cup \{x + iy : x \leq x_0\} \setminus \{x_0\}$ which is simply connected (see figure 1(e)).

(3) For $r > 1$, we see that there are two solutions x_1, x_2 in $(-1, \infty)$ of $\psi_r(x) = 0$ with $x_1 < x_2$. So $\psi_r(x) = 0$ has three real roots. Finally, as in the above cases, $\varphi_r(x) < 0$ for $x \in (-\infty, x_0) \cup (x_1, x_2)$ and $\varphi_r(x) = 0$ for $x = x_0,\ x_1,\ x_2$ and $\varphi_r(x) > 0$ for $x \in (x_0, x_1) \cup (x_2, \infty)$ (see figure 1(c)).

In this case, $\sqrt{\varphi_r(x)}$ is defined only when $x \in (x_0, x_1) \cup (x_2, \infty)$. Let $\tau_1^+ = \{x + iy : y = \sqrt{\varphi_r(x)},\ x_0 < x < x_1\}$ and $\tau_2^+ = \{x + iy : y = \sqrt{\varphi_r(x)},\ x > x_2\}$. Let $\tau_1^- = \{z : \bar{z} \in \tau_1^+\}$, $\tau_2^- = \{z : \bar{z} \in \tau_2^+\}$ and $\tau_i = \overline{\tau_i^+ \cup \tau_i^-}$ for $i = 1, 2$.

As in Case (2), we see that τ_1 is bounded and τ_2 is unbounded. Also, $\tau_1 \cap \tau_2 = \emptyset$ and $\partial A_r = \tau_1 \cup \tau_2$ and $A_r = \{x + iy : |y| > \sqrt{\varphi_r(x)},\ x \in (x_0, x_1) \cup (x_2, \infty)\} \cup \{x + iy : x \in (-\infty, x_0] \cup [x_1, x_2]\} \setminus \{x_0, x_1, x_2\}$ which is doubly connected (see figure 1(f)). □

The regions D_r are shown red in figures 1(g), (h) and (i). The pre-images A_r are dotted red in figures 1(d), (e) and (f).

As a consequence of the above lemma, we have the following.

Lemma 3. *There is exactly one singularity of* f^{-1} *over the point* 0. *Also, there is only one singularity of* f^{-1} *lying over* ∞.

Proof. For all $r < 1$, $f^{-1}(D_r)$ is simply connected. This means that there is exactly one singularity of f^{-1} over 0.

If $r > 1$, then $f^{-1}(\hat{\mathbb{C}} \setminus D_r)$ has two components; a bounded component V_1 containing the pole, and an unbounded component V_2 contained in the right half plane $\{z \in \mathbb{C} : \mathfrak{Re}(z) > 0\}$. Note that $f : V_1 \to \hat{\mathbb{C}} \setminus D_r$ is bijective not corresponding to any singularity. There is only one singularity of f^{-1} lying over ∞. □

It may be noted that the singularity over 0 (as well as that over ∞) is logarithmic.

3. Dynamics of f_λ

All the singular values of f_λ are on the real line and the real line is preserved by f_λ. Hence, it is important to know the iterative behavior of the function f_λ on $\mathbb{R}$.

Consider the function $f(x) = \frac{e^x}{1+x}$. It is clear that $f(x) > 0$ when $x > -1$, $f(x) < 0$ when $x < -1$ and $f(x)$ is continuous everywhere except at the point $x = -1$ which is the pole of f. Since $f'(x) = \frac{xe^x}{(1+x)^2}$, the function f is strictly decreasing

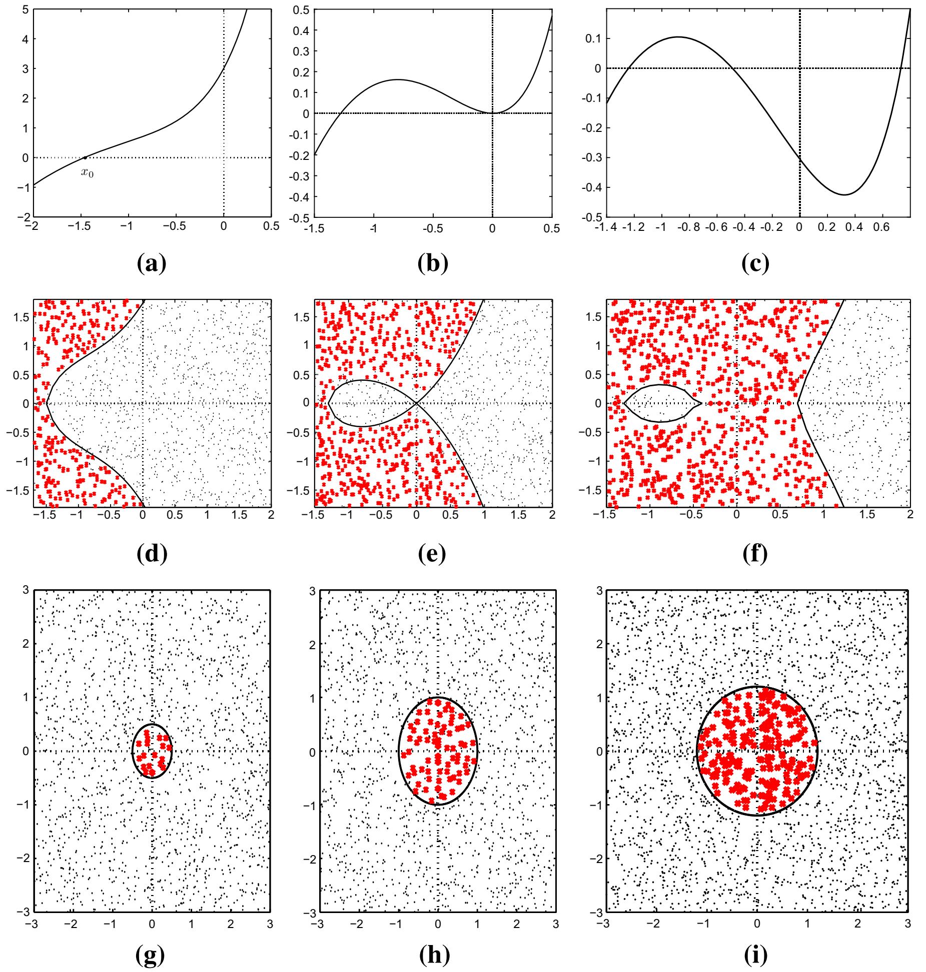
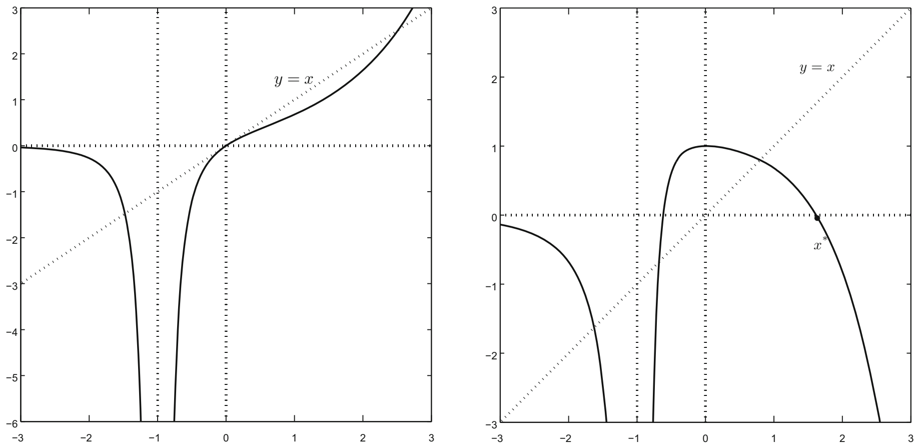


Figure 1. Graphs of $\varphi_r(x)$ for (**a**) $r = 0.5$, (**b**) $r = 1$ and (**c**) $r = 1.2$; A_r for (**d**) $r = 0.5$, (**e**) $r = 1$ and (**f**) $r = 1.2$; D_r for (**g**) $r = 0.5$, (**h**) $r = 1$ and (**i**) $r = 1.2$.

in $(-\infty, 0)$ and strictly increasing in $(0, \infty)$. As $f''(x) = \frac{(1+x^2)\mathrm{e}^x}{(1+x)^3}$, the function f' is decreasing in $(-\infty, -1)$ and is increasing in $(-1, \infty)$. Note that $\lim_{x\to -1} f'(x) = -\infty$, $\lim_{x\to\infty} f'(x) = \infty$ and $\lim_{x\to -\infty} f'(x) = 0$ (see figure 2(a)). As f is continuous in $(-1, \infty)$ and is decreasing in $(-1, 0)$ and increasing in $(0, \infty)$, it attains its local minimum at $x = 0$ and the minimum value is $f(0) = 1$. Moreover, $f(x) \to 0$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.

Consider the function $\phi(x) = f(x) - xf'(x)$ for $x \geq 0$. As $\phi'(x) = -xf''(x) < 0$ for $x > 0$, $\phi(x)$ is decreasing in $\mathbb{R}^+$. We have $\phi(0) = 1$ and $\phi(x) \to -\infty$ as $x \to \infty$. By the continuity of ϕ in $\mathbb{R}^+$, there exists a unique $x^* > 0$ such that (see figure 2(b))

$$\phi(x) \begin{cases} > 0 & \text{for } 0 < x < x^* \\ = 0 & \text{for } x = x^* \\ < 0 & \text{for } x > x^*. \end{cases} \tag{1}$$



(a) **(b)**

Figure 2. Graphs of (**a**) $f'(x)$ and (**b**) $\phi(x)$.

Define $\lambda^* = \frac{1}{f'(x^*)}$. It is numerically found that, $x^* = (\frac{\sqrt{5}}{2} + \frac{1}{2}) \approx 1.618033988749895$ which is incidentally the golden ratio and $\lambda^* \approx 0.839962094657175$. This λ^* is going to be the parameter value at which a sudden change in the dynamics of f_λ occurs.

Define $g_\lambda(x) = f_\lambda(x) - x$ for $x \in \mathbb{R}$ and note that $g'_\lambda(x) = \lambda \frac{x e^x}{(1+x)^2} - 1$ and $g''_\lambda(x) = \lambda \frac{(1+x^2)e^x}{(1+x)^3} > 0$ for $x > 0$. Therefore, $g'_\lambda(x)$ is increasing in $\mathbb{R}^+$ and $g'_\lambda(x) \to \infty$ as $x \to \infty$. Since $g'_\lambda(0) = -1$ and $g'_\lambda(x)$ is continuous and strictly increasing in $(0, \infty)$, there exists a point $x_\lambda > 0$ such that

$$g'_\lambda(x) \begin{cases} < 0 & \text{for } x \in (0, x_\lambda) \\ = 0 & \text{for } x = x_\lambda \\ > 0 & \text{for } x \in (x_\lambda, \infty). \end{cases} \tag{2}$$

Thus $g_\lambda(x)$ decreases in $(0, x_\lambda)$ and attains its minimum value at $x = x_\lambda$, and then increases to ∞ in (x_λ, ∞). This gives that $\lambda = \frac{1}{f'(x_\lambda)}$. Now we present the complete iterative behavior of f_λ on the real line.

Recall that $\mathcal{K} = \{f_\lambda(z) = \lambda f(z) : f(z) = \frac{e^z}{z+1} \text{ for } z \in \mathbb{C}, \ \lambda > 0\}$.

Theorem 4. *Let $f_\lambda \in \mathcal{K}$. Then the following are true* (*see figure* 3).

(1) *For $0 < \lambda < \lambda^*$, f_λ has two positive real fixed points a_λ and r_λ with $a_\lambda < r_\lambda$, where a_λ is attracting and r_λ is repelling. Further*, $\lim_{n\to\infty} f_\lambda^n(x) = a_\lambda$ *for* $0 \le x < r_\lambda$ *and* $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ *for* $x > r_\lambda$.
(2) *For $\lambda = \lambda^*$, f_λ has only one positive real fixed point $x = x^*$ and x^* is rationally indifferent. Further*, $\lim_{n\to\infty} f_\lambda^n(x) = x^*$ *for* $0 \le x \le x^*$ *and* $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ *for* $x > x^*$.
(3) *For $\lambda > \lambda^*$, f_λ has no positive real fixed point. Further*, $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ *for all* $x > 0$.
(4) *For every value of λ, f_λ has a negative fixed point $\tilde{r}_\lambda$ and that is repelling.*

Proof.

(1) If $0 < \lambda < \lambda^*$, then $\frac{1}{f'(x_\lambda)} < \frac{1}{f'(x^*)}$. Since f' is increasing in $\mathbb{R}^+$, we have $x_\lambda > x^*$. So $\phi(x_\lambda) < 0$ by equation (1). This implies that $g_\lambda(x_\lambda) = f_\lambda(x_\lambda) - x_\lambda < 0$. But $g_\lambda(0) = \lambda > 0$ and by equation (2), there exists two real numbers a_λ and r_λ with $0 < a_\lambda < x_\lambda < r_\lambda$ such that $g_\lambda(a_\lambda) = g_\lambda(r_\lambda) = 0$. Thus f_λ has exactly two fixed points a_λ and r_λ in $\mathbb{R}^+$. Since f' is increasing in $\mathbb{R}^+$, we get $0 < f'_\lambda(a_\lambda) < f'_\lambda(x_\lambda) = 1 < f'_\lambda(r_\lambda)$. So a_λ is the attracting fixed point and r_λ is the repelling fixed point of f_λ. Note that $f_\lambda(x) > x$ for $0 \leq x < a_\lambda$ and $f_\lambda(x) < x$ for $a_\lambda < x < r_\lambda$. Since $f_\lambda(x)$ is increasing in $\mathbb{R}^+$, the sequence $\{f_\lambda^n(x)\}_{n\geq 0}$ is increasing and bounded above by a_λ for $0 \leq x < a_\lambda$. Also $\{f_\lambda^n(x)\}_{n\geq 0}$ is decreasing and bounded below by a_λ for $a_\lambda < x < r_\lambda$. Hence by the monotone convergence theorem, $\lim_{n\to\infty} f_\lambda^n(x) = a_\lambda$ for $0 \leq x < r_\lambda$. Now $f_\lambda(x) > x$ for $x > r_\lambda$ gives that $\{f_\lambda^n(x)\}_{n\geq 0}$ is increasing and not bounded above and hence $\lim_{n\to\infty} f_\lambda^n(x) = \infty$.
(2) If $\lambda = \lambda^*$, then $f'(x_\lambda) = f'(x^*)$ and the injectivity of f' on the positive real line implies that $x_\lambda = x^*$. Note that x^* is the only root of $g_{\lambda^*}(x) = 0$. Hence $f_{\lambda^*}(x)$ has only one fixed point in $\mathbb{R}^+$ and it is rationally indifferent. The sequence $\{f_{\lambda^*}^n(x)\}_{n\geq 0}$ is increasing and bounded above by x^* for $0 \leq x < x^*$. By the monotone convergence theorem, we have $\lim_{n\to\infty} f_{\lambda^*}^n(x) = x^*$ for $0 \leq x < x^*$. For $x > x^*$, the sequence $\{f_{\lambda^*}^n(x)\}_{n\geq 0}$ is increasing and not bounded above and hence $\lim_{n\to\infty} f_{\lambda^*}^n(x) = \infty$.
(3) If $\lambda > \lambda^*$, then $\frac{1}{f'(x_\lambda)} > \frac{1}{f'(x^*)}$ and it follows that $x^* > x_\lambda$. So, we have $\phi(x_\lambda) > 0$ and it follows that $g_\lambda(x_\lambda) > 0$. Since the minimum value of g_λ in $\mathbb{R}^+$ is positive, we conclude that $g_\lambda(x) > 0$ for all $x > 0$. Thus for $x > 0$, we have that $f_\lambda(x)$ is increasing and $f_\lambda(x) > x$. This implies that $\{f_\lambda^n(x)\}_{n\geq 0}$ is an increasing sequence of positive real numbers, which is not bounded above and hence $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ for all $x > 0$.
(4) Note that $g_\lambda(x) \to \infty$ as $x \to -\infty$ and $g_\lambda(x) \to -\infty$ as $x \to -1$ from the left-hand side. We see that $g'_\lambda < 0$ in $\mathbb{R}^-$. So g_λ is strictly decreasing in $\mathbb{R}^-$. Therefore, there exists a unique $\tilde{r}_\lambda \in \mathbb{R}^-$ such that $g_\lambda(\tilde{r}_\lambda) = 0$. That is $f_\lambda(\tilde{r}_\lambda) = \tilde{r}_\lambda$. By Lemma 1, the singular values of f_λ are 0, λ and ∞. Since $f_\lambda(0) = \lambda$, we are essentially having a single forward orbit of finite singular values. Let $\tilde{r}_\lambda$ be either attracting or a parabolic fixed point. Then there exists an immediate attracting or parabolic basin U_0 such that $U_0 \cap \text{sing}(f_\lambda^{-1}) \neq \emptyset$. But it is proved in (1), (2) and (3) of this theorem that $\{f^n(0)\}_{n\geq 0}$ converges to either a_λ or x^* or ∞. Therefore, $\tilde{r}_\lambda$ cannot be attracting or parabolic. Now, $f'_\lambda(\tilde{r}_\lambda)$ is a real number and hence $\tilde{r}_\lambda$ cannot be irrationally indifferent. So $\tilde{r}_\lambda$ is a repelling fixed point of f_λ. □

Remark 5.

(1) For $0 < \lambda < \lambda^*$, the map $f_\lambda : (-1, 0) \to (\lambda, \infty)$ is bijective and decreasing. Since $r_\lambda > \lambda$, there exists a unique point $\xi \in (-1, 0)$ such that $f_\lambda(\xi) = r_\lambda$ and $f_\lambda(x) > r_\lambda$ for $x \in (-1, \xi)$ and $\lambda < f_\lambda(x) < r_\lambda$ for $x \in (\xi, 0)$. Then by Theorem 4, $\lim_{n\to\infty} f_\lambda^n(x) = a_\lambda$ for $x \in (\xi, 0)$ and $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ for $x \in (-1, \xi)$.
(2) For $\lambda = \lambda^*$, $f_\lambda : (-1, 0) \to (\lambda, \infty)$ is bijective. Since $x^* > \lambda$, there exists $\eta \in (-1, 0)$ such that $f_\lambda(\eta) = x^*$ and $f_\lambda(x) > x^*$ when $x \in (-1, \eta)$ and $0 < f_\lambda(x) < x^*$ when $x \in (\eta, 0)$. Thus we have $\lim_{n\to\infty} f_\lambda^n(x) = x^*$ for $x \in (\eta, 0)$ and $\lim_{n\to\infty} f_\lambda^n(x) = \infty$ for $x \in (-1, \eta)$ by Theorem 4.

The following theorem gives a complete description of the Fatou set of f_λ.

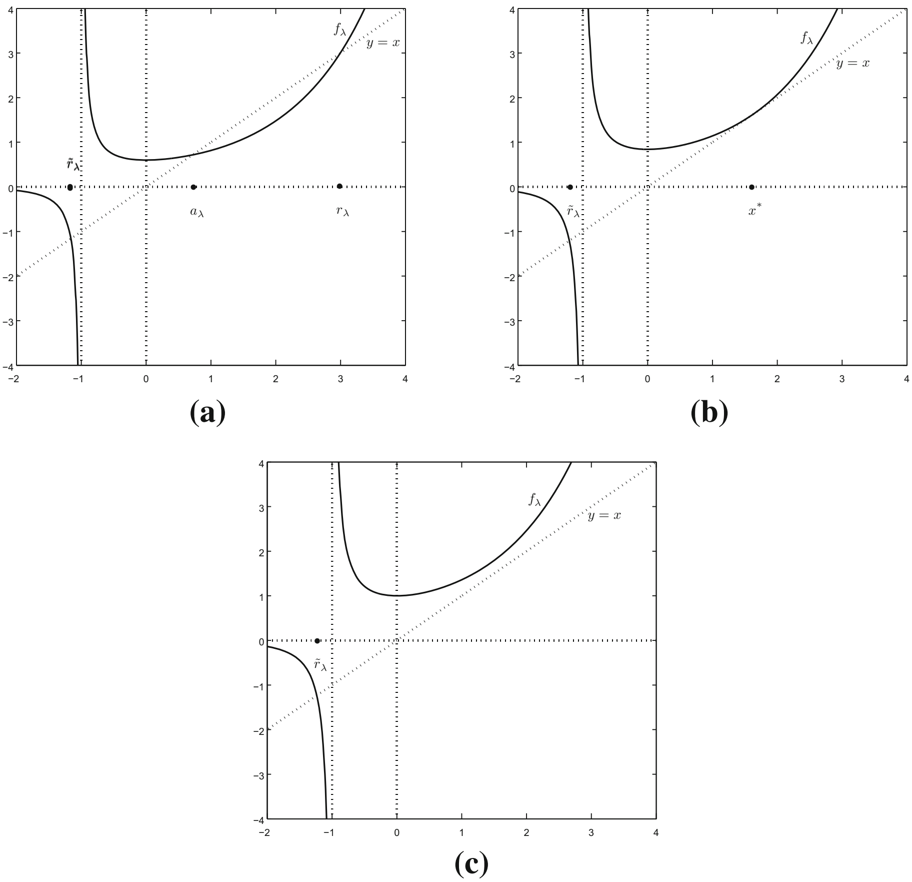


Figure 3. Graphs of $f_\lambda(x) = \lambda \frac{e^x}{1+x}$ for (**a**) $0 < \lambda < \lambda^*$, (**b**) $\lambda = \lambda^*$ and (**c**) $\lambda > \lambda^*$.

Theorem 6. *Let $f_\lambda \in \mathcal{K}$. Then the dynamics of f_λ is as follows*:

(1) *If $0 < \lambda < \lambda^*$, then the Fatou set $\mathcal{F}(f_\lambda)$ is the attracting basin of the real positive fixed point.*
(2) *If $\lambda = \lambda^*$, then the Fatou set $\mathcal{F}(f_\lambda)$ is the parabolic basin corresponding to the real rationally indifferent fixed point.*
(3) *If $\lambda > \lambda^*$, then the Fatou set $\mathcal{F}(f_\lambda)$ is empty.*

Proof. Since f_λ has finitely many singular values by Lemma 1, the function f_λ has neither wandering domain nor a Baker domain [2]. Recall that the omitted value 0 is also the critical point of f_λ. Thus the forward orbit of 0, $O^+(0)$ is the post singular set. The forward orbit converges either to an attracting or a parabolic real fixed point or it escapes to ∞. So $\overline{O^+(0)}$ cannot contain the boundary of any Fatou component. Therefore, f_λ has no Herman ring or Siegel disk.

Since all finite singular values of f_λ are in $\mathbb{R}$ and $f_\lambda(\mathbb{R}) \subset \mathbb{R}$, any periodic Fatou component corresponds to a real non-repelling periodic point. Further, there is at most one cycle of Fatou components as $O^+(0)$ has a single limit point in $\hat{\mathbb{C}}$.

(1) The function f_λ has only one attracting fixed point a_λ for $0 < \lambda < \lambda^*$. Therefore, it follows that $\mathcal{F}(f_\lambda) = A(a_\lambda)$ for $0 < \lambda < \lambda^*$, where $A(a_\lambda) = \{z \in \mathbb{C} : \lim_{n\to\infty} f_\lambda^n(z) = a_\lambda\}$ is the attracting basin of a_λ.
(2) For $\lambda = \lambda^*$, f_λ has a rationally indifferent fixed point namely x^*. This implies that $\mathcal{F}(f_\lambda) = P(x^*)$, where $P(x^*) = \{z \in \mathbb{C} : \lim_{n\to\infty} f_\lambda^n(z) = x^*\}$ denotes the parabolic basin corresponding to x^*.
(3) If $\lambda > \lambda^*$, then $O^+(0)$ converges to ∞. Therefore, $\mathcal{F}(f_\lambda) = \emptyset$ for $\lambda > \lambda^*$. □

Theorem 6 gives the following characterization of the Julia set, in terms of λ of f_λ which is computationally useful to generate the pictures of the Julia sets.

COROLLARY 7

Let $f_\lambda \in \mathcal{K}$. If

(1) $0 < \lambda < \lambda^*$, *then the Julia set $\mathcal{J}(f_\lambda)$ is the complement of the basin of attraction of the real fixed point a_λ,*
(2) $\lambda = \lambda^*$, *then the Julia set $\mathcal{J}(f_\lambda)$ is the complement of the parabolic basin corresponding to the real rationally indifferent fixed point x^*,*
(3) $\lambda > \lambda^*$, *then the Julia set $\mathcal{J}(f_\lambda)$ is $\hat{\mathbb{C}}$.*

4. Topology of Julia sets

Now we prove the following lemmas which are used in the forthcoming theorems.

Lemma 8. *Let $f_\lambda \in \mathcal{K}$. Then $z \in \mathcal{J}(f_\lambda)$ if and only if $\bar{z} \in \mathcal{J}(f_\lambda)$.*

Proof. Let $z \in \mathcal{J}(f_\lambda)$. Observe that $f_\lambda(\bar{z}) = \overline{f_\lambda(z)}$ and consequently, $f_\lambda^n(\bar{z}) = \overline{f_\lambda^n(z)}$ for all $z \in \mathbb{C}$ and $n \in \mathbb{N}$. For $z \in \mathcal{J}(f_\lambda)$, the sequence $\{f_\lambda^n\}_{n>0}$ is not normal at z. This gives that $\{\overline{f_\lambda^n}\}_{n>0}$ is also not normal at z. Therefore, $\{f_\lambda^n\}_{n>0}$ is not normal at $\bar{z}$ and $\bar{z} \in \mathcal{J}(f_\lambda)$. The converse follows similarly. □

For $\lambda > \lambda^*$, the Julia set is $\hat{\mathbb{C}}$ and hence connected. Now we make a detailed study of $\mathcal{J}(f_\lambda)$ for $0 < \lambda < \lambda^*$.

Lemma 9. *For $0 < \lambda < \lambda^*$, $A(a_\lambda) \cap \mathbb{R}$ is disconnected. In fact, $A(a_\lambda) \cap \mathbb{R}$ consists of infinitely many disjoint bounded intervals.*

Proof. For $0 < \lambda < \lambda^*$, by Remark 5(1), there exists $\xi \in (-1, 0)$ such that $(\xi, 0) \subset A(a_\lambda)$ and $(-1, \xi) \subset \mathcal{J}(f_\lambda)$. This proof uses the backward invariance of the Fatou set. Let $A_0 = [\xi, 0]$ and $E_0 = [-1, \xi]$. The map $f_\lambda : (-\infty, -1) \to (-\infty, 0)$ is bijective and strictly decreasing. Therefore there is ξ_{-1}, the pre-image of ξ, and p_{-1}, the pre-image of -1 such that $f_\lambda(-\infty, \xi_{-1}) = A_0$ and $f_\lambda(\xi_{-1}, p_{-1}) = E_0$. Letting $A_{-1} = (-\infty, \xi_{-1})$ and $E_{-1} = (\xi_{-1}, p_{-1})$, it is seen that A_{-1} lies left to E_{-1} whereas A_0 lies right to E_0. The interval E_{-1} is between A_0 and A_{-1}. Note that $f_\lambda : (p_{-1}, -1) \to (-\infty, -1)$ is bijective and strictly decreasing. Repeating the argument again, two intervals A_{-2} and E_{-2} can be found such that $f_\lambda(A_{-2}) = A_{-1}$ and $f_\lambda(E_{-2}) = E_{-1}$. Further, the right-hand side of A_{-2}

is -1 and the left-hand side of E_{-2} is the pre-image p_{-2} of p_{-1}. It is important to note that $p_{-2} > p_{-1}$. Further E_{-2} is between A_{-1} and A_{-2}, and E_0 is between A_{-2} and A_0. One can go on arguing this way to show that $A_{-n} \cap A_{-m} = \emptyset$ for $n \neq m$. This shows that $A(a_\lambda) \cap \mathbb{R}$ consists of infinitely many disjoint bounded intervals. □

Remark 10. It follows that the negative repelling fixed point is the accumulation point of A_{-n}'s as well as of E_{-n}'s.

Theorem 11. *For $0 < \lambda < \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ is connected. Further, it is infinitely connected.*

Proof. We know that $\lim_{n\to\infty} f_\lambda^n(x) = a_\lambda$ for $\xi \leq x < r_\lambda$ (by Remark 5). The Fatou set of f_λ is the attracting basin $A(a_\lambda)$ for $0 < \lambda < \lambda^*$. Let $\mathrm{Im}(a_\lambda)$ be the immediate basin of attraction of a_λ. By definition, $\mathrm{Im}(a_\lambda)$ is the forward invariant Fatou component containing a_λ. Note that $A(a_\lambda) = \mathrm{Im}(a_\lambda)$ if $\mathrm{Im}(a_\lambda)$ is backward invariant. To prove that $\mathcal{F}(f_\lambda)$ is connected, it is sufficient to show that $\mathrm{Im}(a_\lambda)$ is the backward invariant. Note that a_λ is a pre-image of itself giving that $a_\lambda \in f_\lambda^{-1}(\mathrm{Im}(a_\lambda)) \cap \mathrm{Im}(a_\lambda)$. It now becomes sufficient to show that $f_\lambda^{-1}(\mathrm{Im}(a_\lambda))$ is connected. On the contrary, let U_1 and U_2 be two components of $f_\lambda^{-1}(\mathrm{Im}(a_\lambda))$. Let D_r be a ball around 0 of radius r such that $D_r \subset \mathrm{Im}(a_\lambda)$. Since $f_\lambda^{-1}(D_r)$ is connected (by Lemma 3), it is contained in any one of the two components. Suppose it is contained in U_1, then $f_\lambda(U_2) = \mathrm{Im}(a_\lambda) \setminus D_r$. But $\mathrm{Im}(a_\lambda) \setminus f_\lambda(U_2)$ contains at most two points ([9], Theorem 1), which is a contradiction proving that $f_\lambda^{-1}(\mathrm{Im}(a_\lambda))$ is connected. Thus, $\mathrm{Im}(a_\lambda)$ is backward invariant and $\mathcal{F}(f_\lambda)$ is connected for $0 < \lambda < \lambda^*$.

Since $\mathcal{F}(f_\lambda)$ is connected and contains an attracting fixed point, it is invariant. The connectivity of any invariant Fatou component is $1, 2$ or ∞. It is 2 when the component is a Herman ring ([1], Theorem 3.1). Since the Fatou set $\mathcal{F}(f_\lambda)$ is an attracting domain for $0 < \lambda < \lambda^*$, its connectivity is either 1 or ∞. If possible, let $\mathcal{F}(f_\lambda)$ be simply connected. Then the Julia set $\mathcal{J}(f_\lambda)$ is connected. As the point at ∞ and the pole -1 are in $\mathcal{J}(f_\lambda)$, there is an unbounded connected subset J_{-1} of the Julia set containing -1. Now, $\overline{J_{-1}} = \{z \in \mathbb{C} : \overline{z} \in J_{-1}\}$ is also in the Julia set (by Lemma 8). Thus $J = J_{-1} \cup \overline{J_{-1}}$ is in the Julia set. Note that J_{-1} cannot be completely contained in the real line by Lemma 9. Therefore $\hat{\mathbb{C}} \setminus J$ has at least two components each intersecting the Fatou set of f_λ. It contradicts the fact that $\mathcal{F}(f_\lambda)$ is connected. Therefore $\mathcal{F}(f_\lambda)$ is infinitely connected for $0 < \lambda < \lambda^*$. □

Since the Fatou set is infinitely connected, the Julia set $\mathcal{J}(f_\lambda)$ is disconnected for $0 < \lambda < \lambda^*$. In the following result, a deeper study of Julia components is made.

Let $\mathcal{J}_s$ denote the set of all singleton Julia components and ${\mathcal{J}_s}^c = \mathcal{J}(f_\lambda) \setminus \mathcal{J}_s$ be the set of all non-singleton Julia components. Then we have the following result.

Theorem 12. *For $0 < \lambda < \lambda^*$, the Julia set can be written as $\mathcal{J}(f_\lambda) = \mathcal{J}_s \cup {\mathcal{J}_s}^c$ with $\mathcal{J}_s \neq \emptyset$, ${\mathcal{J}_s}^c \neq \emptyset$ and $\mathcal{J}_s \cap {\mathcal{J}_s}^c = \emptyset$. The set ${\mathcal{J}_s}^c$ consists of an unbounded forward invariant component and infinitely many non-singleton bounded components. Further, both the sets $\mathcal{J}_s$ and ${\mathcal{J}_s}^c$ are completely invariant.*

Proof. Since the Fatou set $\mathcal{F}(f_\lambda)$ is connected with connectivity greater than three for $0 < \lambda < \lambda^*$ (Theorem 11), singleton components are dense in the Julia set (see [7],

Table 1. Comparison of dynamics of f_λ with other classes of functions.

Dynamics of λe^z	Dynamics of $\lambda(e^z+1+\frac{1}{e^z+1})$	Dynamics of $\lambda\frac{e^z}{z+1}$
Singular values and periodicity		
Entire	Meromorphic	Meromorphic
No pole	Poles at $z_k = i\pi(2k+1),\ k\in\mathbb{Z}$	Pole at -1
No critical value	-2λ is the critical value	λ is the critical value
The asymptotic values are 0 and ∞	The asymptotic values are 2λ and ∞	The asymptotic values are 0 and ∞
Number of singularities each over 0 and ∞ is one and those are logarithmic	One singularity over 2λ which is logarithmic, at least one direct singularity over ∞	Number of singularities each over 0 and ∞ is one and those are logarithmic
Periodic	Periodic	Not periodic
The Julia set		
Is connected but nowhere dense for $0<\lambda<1/e$	Consists of an unbounded component, non-singleton bounded components and singleton components for $0<\lambda<\lambda^*\approx 0.2666$	Consists of an unboundedcomponent, non-singleton bounded components and singleton components for $0<\lambda<\lambda^*\approx 0.84$
$\hat{\mathbb{C}}$ for $\lambda>1/e$	$\hat{\mathbb{C}}$ for $\lambda>\lambda^*$	$\hat{\mathbb{C}}$ for $\lambda>\lambda^*$

Theorem A). So, $\mathcal{J}_s \neq \emptyset$. It follows from Theorem 4 that the Julia set $\mathcal{J}(f_\lambda)$ contains the interval $[r_\lambda,\infty)$, which is contained in an unbounded forward invariant component of $\mathcal{J}(f_\lambda)\cap\mathbb{C}$. This implies that ${\mathcal{J}_s}^c \neq \emptyset$. There are components of $\mathcal{J}(f_\lambda)$ containing intervals in $\mathbb{R}$ by Lemma 9. Any such component cannot be contained in an unbounded, connected subset of the Julia set. Otherwise, using Lemma 8 and following the argument used in the last part of Theorem 11, it can be shown that the Fatou set is disconnected. Thus any such component of $\mathcal{J}(f_\lambda)$ is bounded. It follows from Lemma 9 that the number of bounded Julia components is infinite. So, the set of all non-singleton Julia components ${\mathcal{J}_s}^c$ consists of bounded as well as unbounded components. Clearly, $\mathcal{J}_s\cap{\mathcal{J}_s}^c=\emptyset$.

To prove that both of $\mathcal{J}_s$ and ${\mathcal{J}_s}^c$ are completely invariant, note that the image of a singleton Julia component is singleton. If possible, let a singleton Julia component be mapped into a non-singleton Julia component, say γ. Now, since γ is disjoint from the forward orbit of the singular values of f_λ (because the forward orbit of singular values are in $A(a_\lambda)$), we can define a branch of f_λ^{-1} on γ. Since each branch of f_λ^{-1} on γ is bijective and γ contains infinitely many points, $f_\lambda^{-1}(\gamma)$ cannot be a single point. This implies that γ is singleton. Similarly, it follows that the inverse image of a singleton component of $\mathcal{J}(f_\lambda)$ is also a singleton. Thus the set of singleton Julia components $\mathcal{J}_s$ is a completely invariant subset of $\mathcal{J}(f_\lambda)$. Therefore, the set of non-singleton components ${\mathcal{J}_s}^c$ is also completely invariant. □

A comparison of dynamics of f_λ with other classes of functions is given in table 1.

The escaping set of a transcendental meromorphic function f is defined as $I(f) = \{z : f^n(z)$ is defined for all n and $f^n(z) \to \infty$ as $n \to \infty\}$. Since, f_λ is a meromorphic function with a direct singularity over ∞ (by Lemma 3), $I(f_\lambda)$ has an unbounded component and $I(f_\lambda) \cap \mathcal{J}(f_\lambda)$ contains a continua by Theorems 1.1 and 1.2 of [4]. The following result provides a more accurate description.

Theorem 13. *Let $f_\lambda \in \mathcal{K}$ and $O^-(\infty)$ denote the backward orbit of ∞. Then $I(f_\lambda) \cup O^-(\infty)$ is disconnected for $0 < \lambda < \lambda^*$ whereas this set is connected for $\lambda > \lambda^*$.*

Proof. The claim follows for $0 < \lambda < \lambda^*$ since the Julia set is disconnected. For $\lambda > \lambda^*$, the proof is as follows. Note that -1 is the only pole of f_λ and $(-1, \infty) \subset I(f_\lambda)$. Let $L_0 = [-1, \infty) \cup \{\infty\}$ and $L_{-(n+1)} = f_\lambda^{-1}(L_{-n})$ for $n = 0, 1, 2, \ldots$. Clearly L_0 is connected. Let l be a maximally connected subset of L_{-1}. If l is unbounded, then $l \cup L_0$ is connected. If l is bounded then it must contain -1, the only pole of f_λ. This is because $f_\lambda : l \to L_0 \setminus \{0\}$ is onto. Therefore, the union of all components of L_{-1} and L_0 is connected which gives that L_{-1} is connected. Note that $L_0 \subset L_{-1}$. Arguing inductively, it follows that L_{-n} is connected and $L_{-n} \subset L_{-(n+1)}$ for all n.

Let $K = \cup_{n=0}^{\infty} L_{-n}$. Then for every $w_1, w_2 \in K$, there are n_1, n_2 such that $w_i \in L_{-n_i}, i = 1, 2$. Consequently, a path can be found in L_{-m} where $m = \max\{n_1, n_2\}$ which joins w_1 and w_2. Therefore, K is connected. Clearly $K \subset I(f_\lambda) \cup O^-(\infty)$. Since K is dense in the Julia set and in particular in $I(f_\lambda) \cup O^-(\infty)$, it follows that $I(f_\lambda) \cup O^-(\infty)$ is connected. □

For $0 < \lambda < \lambda^*$, the backward invariance of the escaping set gives that $I(f_\lambda)$ is disconnected. But for $\lambda > \lambda^*$, the above theorem does not conclude anything about the connectedness of the escaping set of f_λ.

The Julia sets of f_λ for different positive values of λ are generated in the rectangular domain $R = \{z \in \mathbb{C} : -3 \le \mathfrak{Re}(z) \le 8,\ -5 \le \mathfrak{Im}(z) \le 5\}$, where 500 iterations of the functions are considered (see figure 4). The following algorithm is used for this purpose:

(1) Select a rectangle in the complex plane and construct a $K_1 \times K_2$ grid therein.
(2) Corresponding to the attracting fixed point a_λ, choose a real number $B > 0$ such that all the images under f_λ of the points in $D = \{z \in \mathbb{C} : |z - a_\lambda| < B\}$ remain inside the ball D. Such a choice can actually be made.
(3) For each grid point z, compute $f_\lambda^N(z)$, i.e., up to a maximum of N iteration.
(4) For each grid point, find out the least natural number i, $1 \le i \le N$ for which $|f_\lambda^i(z) - a_\lambda| < B$. Then color the original grid point (pixel) according to the iteration number $k = N - i$ with the color scale 'jet', in which the color gradient starts from blue via green and yellow to red. Note that, the colors produced by the gradient vary continuously with position, producing smooth color transitions.

In order to construct a grid in the rectangular domain R, we have chosen $K_1 = 11{,}000$ and $K_2 = 10{,}000$ in Step (1). The points with largest k-value, receive the color red which indicates the earliest entry into the ball D. Similarly the points whose k-value is zero are colored blue and it indicates that the concerned points have never entered into the ball D. The Julia set is contained in the blue region and the red region is the approximation of the Fatou set. Figure 4(a) shows the basin of attraction of the attracting fixed point a_λ when $\lambda < \lambda^*$. It is clear from the figure that the Julia set of the map f_λ is disconnected and a nowhere dense subset of $\hat{\mathbb{C}}$ in this case. A zoomed-in version of a significant portion is

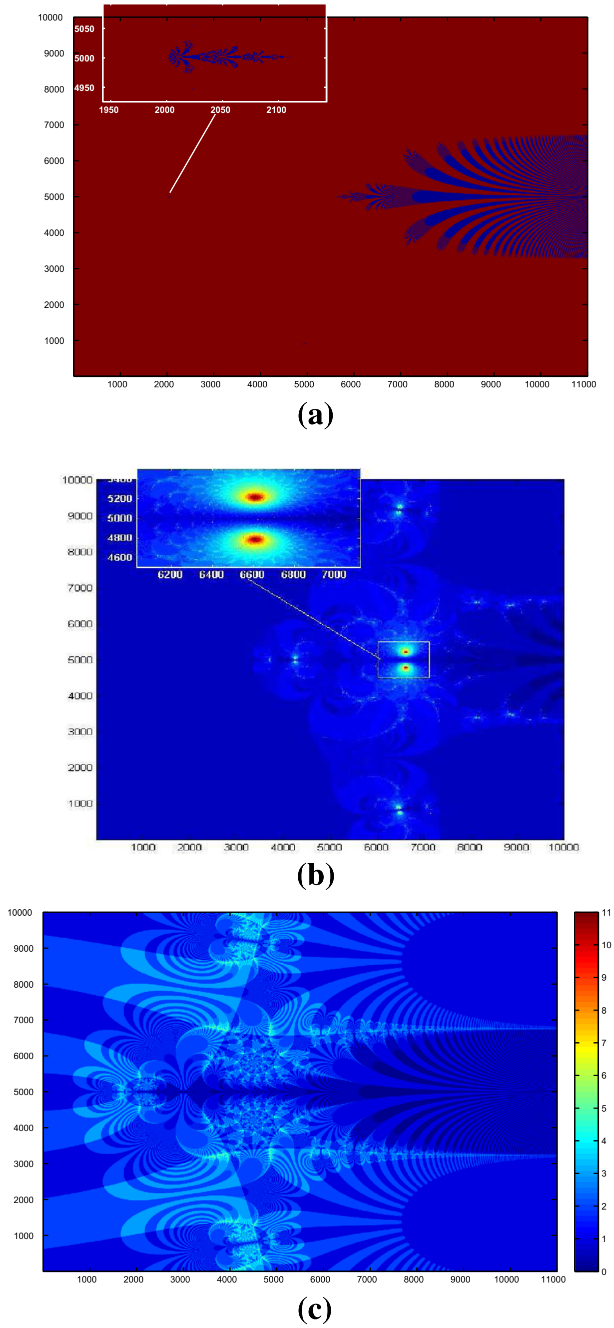


Figure 4. Julia sets of $f_\lambda(z)$ for (**a**) $\lambda = 0.7$, (**b**) $\lambda = \lambda^* \approx 0.84$ and (**c**) $\lambda = 1$.

produced inside figure 4(a), which demonstrates the disconnectedness of the Julia set. The parabolic fixed point x^* is replaced in the place of the attracting fixed point a_λ in Step (4) of the above algorithm to produce figure 4(b), which displays the dynamics of f_λ for $\lambda = \lambda^*$. Note that, in this case, the Fatou set is the parabolic basin of the parabolic fixed point x^*.

In order to produce figure 4(c), a_λ is replaced by 0 in Step (4) of the above algorithm and it makes sense since most of the points escape to ∞ in this case. In figure 4(c), it is seen that the Julia set is $\hat{\mathbb{C}}$ when $\lambda > \lambda^*$. Note the dramatic change in the size of the Julia set for this value of λ. This phenomenon as described earlier is known as chaotic burst and it occurs in the Julia set when λ increases through $\lambda^* \approx 0.84$.

5. On dynamics of f_λ for $\lambda \in \mathbf{C}$

For $\lambda \in \mathbb{C}$, the function $f_\lambda(z) = \lambda \frac{e^z}{z+1}$ omits 0 and -1 is its only pole. The finite number of singular values of f_λ implies that it has no Baker domain or wandering domain. It is proved in Theorem 1.4 of [11] that, if a meromorphic function with exactly one pole omits at least one value, then the Fatou set can not have a Herman ring. Thus f_λ has no Herman ring. It is proved in Theorem 3.8 of [10] that if a meromorphic function omits one value and takes another value in $\hat{\mathbb{C}}$ only finitely many times and its Fatou set is non-empty, then each multiply connected Fatou component lands on a Fatou component (under forward iteration of the function) containing the omitted value. We summarize these.

Theorem 14. *Exactly one of the following is true about the dynamics of f_λ, $\lambda \in \mathbb{C}$.*

(1) *The Fatou set of f_λ is empty.*
(2) *All the Fatou components of f_λ are simply connected.*
(3) *There is a Fatou component U of f_λ containing* 0 *such that for every multiply connected Fatou component V, there is a k such that $f_\lambda^k(V) \subseteq U$.*

Remark 15.

(1) The first and third possibility of the above theorem are already shown to be true for some real λ in Theorem 6.
(2) The last possibility shows that the omitted value is in the Fatou set whenever there is a multiply connected Fatou component. That the converse is not true can be seen by considering $\lambda < 0$. In this case, f_λ maps $(-1, \infty)$ onto $(-\infty, \lambda)$ which is then mapped onto $(0, f_\lambda(\lambda))$ by f_λ. Thus f_λ^2 has a fixed point in $(0, f_\lambda(\lambda))$. Since there is no real fixed point, this must be a two periodic point, say z_0. It also follows that each point in $(-1, \infty)$ tends to z_0 under iteration of f_λ^2 and each point in $(-\infty, -1)$ tends to $f_\lambda(z_0)$ under iteration of f_λ^2. The pole -1 is on the common boundary of the two Fatou components U_1, U_2 (these are either attracting domains or parabolic domains) containing $(f_\lambda(z_0), -1)$ and $(-1, z_0)$ respectively. This means that the component of the Julia set containing -1 is unbounded. Now, it follows from Remark 5(i) of [13] that all Fatou components are simply connected.

If the omitted value is in the Julia set of f_λ and the Fatou set is non-empty, then all the Fatou components of f_λ are simply connected by (3) of the above result. Since the singular values have only one forward orbit, it follows that the Fatou set does not contain any attracting basin or parabolic basin. In this case, the Fatou set, if non-empty, can contain Siegel disks only. In fact, this is true for many suitably chosen values of λ. The following result actually shows that f_λ has fixed points with all possible multipliers.

Theorem 16. *For every complex number β, there is a λ such that f_λ has a fixed point whose multiplier is β. In particular, the following are true.*

(1) *For every Brjuno number* θ, *there is a* λ *such that* f_λ *has an invariant Siegel disk whose rotation number is* θ.
(2) *There is a bounded domain* $\mathcal{A}$ *in the plane such that if* $\lambda \in \mathcal{A}$, *then the Fatou set of* f_λ *is an invariant attracting domain. Further, it is connected.*

Proof. Let $g(z) = \frac{z^2}{z+1}$ and $\bar{\mathbb{D}}$ be the closure of the unit disk. Then $D = g^{-1}(\bar{\mathbb{D}})$ is a connected closed domain containing the origin. This is because 0 is the only pre-image of itself. The domain D does not contain -1. The image of D under $h(z) = \mathrm{e}^{-z} z(z+1)$ is a closed bounded domain. Let it be denoted by $\mathcal{A}$. Choose z such that $g(z) = \beta$. Then for $\lambda = h(z)$, $f_\lambda(z) = h(z) f(z) = z$ and $f_\lambda'(z) = \beta$ as desired.

(1) Let z_0 be such that $g(z_0) = \mathrm{e}^{2\pi i \theta}$, where θ is a Brjuno number. Then, for $\lambda_0 = h(z_0)$, z_0 is a fixed point of f_{λ_0} with multiplier equal to $\mathrm{e}^{2\pi i \theta}$. In other words, the function f_{λ_0} is linearizable around the irrationally indifferent fixed point z_0 and there is a Siegel disk containing z_0.
(2) Choosing z such that $g(z)$ is in the interior of $\mathbb{D}$, it is seen that f_λ has an attracting fixed point at z. That the Fatou set is the attracting basin and is connected follows from the proof of Theorem 11. □

It is important to note that 0 is the only critical point of f_λ and is also the omitted value. So, it cannot be a periodic point and consequently, f_λ cannot have a super-attracting domain of any period.

6. More general exponential-like functions

Consider the polynomial $P_n(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$. Then $P_n(z) - P_n'(z) = \frac{1}{n!} z^n$. The function $E_n(z) = \frac{\mathrm{e}^z}{P_n(z)}$ has only one critical point, namely 0. It is also the omitted value and the only finite asymptotic value of the function. The point at ∞ is an asymptotic value corresponding to the positive real axis as its asymptotic path. In fact, it can be seen that $E_n(z) \to \infty$ whenever $z \to \infty$ along any curve $\alpha(t)$ whose real part tends to $+\infty$ as $t \to \infty$. If there are at least two transcendental singularities over ∞, then for all sufficiently large $R > 0$, there are two disjoint unbounded components of $f^{-1}(D_R)$ where $D_R = \{z : |z| > R\}$. These components are known as tracts in the literature. Now, these tracts must be unbounded towards right and an unbounded curve $\alpha : [0, \infty) \to \mathbb{C}$ can be found in their complement such that $\mathfrak{Re}(\alpha(t)) \to +\infty$ as $t \to \infty$. But by definition, $|E_n(\alpha(t))| < R$ for all t which is not true. Therefore, there is only one transcendental singularity over ∞. Note that 0 is not a root of P_n. If $P_n(z_0) = 0$, then $P_n'(z_0) = -\frac{1}{n!} z_0^n$ which is nonzero. This shows that all roots of P_n are simple and that there are no algebraic singularity over ∞. The situation, at least as long as singularities are concerned, is same as that of f_λ which is nothing but λE_1.

It can be shown that Theorem 16 holds true for λE_n for all n. The proof follows by taking $g(z) = \frac{z^{n+1}}{n! P_n(z)}$ and $h(z) = \frac{z P_n(z)}{\mathrm{e}^z}$. To prove that Theorem 14 remains true for λE_n, it must be shown that λE_n has no Herman ring. Note that λE_n has more than one pole for $n \geq 2$ for which Theorem 1.4 of [11] does not apply. However, there is a simple proof that applies to a more general class of functions which includes λE_n for all n.

Theorem 17. *If f is a meromorphic function and it has an omitted value which is a critical point then f has no Herman ring.*

Proof. If there is a Herman ring H of period p, then consider an f^p-invariant curve in H. As argued in Lemma 2.1 of [11], it follows that $f^k(\gamma)$ surrounds the omitted value of f for some k. If $f^k(\gamma)$ surrounds no pole, then $f^{k+1}(\gamma)$ will have winding number bigger than one with respect to each point it surrounds. This is not possible since $f^{k+1}(\gamma)$ is an f^p-invariant curve in a Herman ring. Therefore, there is at least a pole surrounded by $f^k(\gamma)$. However, this gives that $f^{k+n}(\gamma)$ surrounds the omitted value for all n, again by Lemma 2.1 of [11], giving that for i, j, either $f^i(H) \subset f^j(H)$ or $f^j(H) \subset f^i(H)$. This arrangement of Herman rings is known as nested. But nested Herman rings cannot exist for functions with omitted value by Theorem 1.3 of [11]. □

Further investigation on dynamics of E_n is to be reported later.

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