

INTEGRATION OVER NONRECTIFIABLE PATHS WITH APPLICATIONS

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Abstract: We study a generalized contour integral along a nonrectifiable path and its applications.

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1. Introduction

Many boundary value problems in complex analysis have solutions in terms of contour integrals. An example is the Riemann boundary value problem stated as follows. Given an oriented curve Γ in the complex plane \mathbb{C} , find an analytic function $\Phi(z)$ on $\overline{\mathbb{C}} \setminus \Gamma$ with the left limit $\Phi^+(t)$ and right limit $\Phi^-(t)$ at each interior point t of Γ satisfying the boundary condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad (1)$$

where $G(t)$ and $g(t)$ are known functions on Γ . Usually this boundary condition is supplemented with some restrictions on the growth of the required function at the endpoints of Γ . It is a classical complex analysis problem with many applications; see [1–5] and others.

In the literature, the solutions to this problem are constructed as integrals along Γ . For instance, for $G(t) \equiv 1$ (the jump problem) the solution is the Cauchy-type integral

$$\Phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(t) dt}{t - z}, \quad z \notin \Gamma. \quad (2)$$

This leads to the two problems:

Firstly, contour integration $\int_{\Gamma} \cdot dz$ is defined only for rectifiable paths Γ , while (1) is meaningful for every Jordan curve. This entails the problem of developing techniques for solving these boundary value problems for nonrectifiable paths which amounts in fact to extending the concept of contour integral to nonrectifiable paths.

For nonsmooth rectifiable contour conditions for the existence of the left and right limits $\Phi^{\pm}(t)$ of the Cauchy-type integral (2) are established in [6, 7]. Clearly, upon the definition of this integral along nonrectifiable paths, the problem of existence of the limit arises immediately.

The first of these problems was solved for the first time in [8–10]; also see the survey [11]. The original solution of the Riemann boundary value problem avoided the generalization of the contour integral to nonrectifiable paths. It was obtained using regular quasisolutions (see the references above). The concept of contour integral was extended to nonrectifiable paths later; see [12, 13], as well as the references in [12]. These articles also include a series of conditions for the existence of boundary values of generalized Cauchy-type integral along nonrectifiable paths.

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The definitions of integral along nonrectifiable paths proposed in [12, 13] cover both closed curves and open paths, but the results of those articles apply mainly to closed curves. This article deals with nonrectifiable Jordan paths and demonstrates the salient features of the case of open paths.

Let us start with defining integration along nonrectifiable paths. Then we proceed to the Cauchy-type integral and some analogs along these paths, and finish the article with applications to boundary value problems.

2. Integration Along a Nonrectifiable Path

Let us begin with heuristic arguments. Suppose that Γ is a simple Jordan path in the complex plane which begins at a_1 and ends at a_2 . If it is rectifiable then the integral $\int_{\Gamma} u(z) dz$ exists for every continuous function u on Γ and can be identified with the continuous functional (distribution)

$$C_0^{\infty}(\mathbb{C}) \ni \phi \mapsto \int_{\Gamma} \phi(z) u(z) dz.$$

Suppose further that there is a compactly supported function $U(z)$ continuously differentiable on $\mathbb{C} \setminus \Gamma$ such that the limit values $U^+(t)$ and $U^-(t)$ at each point $t \in \Gamma' := \Gamma \setminus \{a_1, a_2\}$ are related as

$$U^+(t) - U^-(t) = u(t), \quad t \in \Gamma'. \quad (3)$$

Moreover, if the first partial derivatives of U are integrable then the Green's formula yields

$$\int_{\Gamma} \phi(z) u(z) dz = - \iint_{\mathbb{C}} \frac{\partial U \phi}{\partial \bar{z}} dz d\bar{z},$$

where, as usual,

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial z} := \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right),$$

$$dz = dx + i dy, \quad d\bar{z} = dx - i dy.$$

The existence of the double integral in the last equality clearly has nothing to do with the rectifiability of Γ . This leads to the following two-stage definition.

DEFINITION 1. Refer as an *integrator* of a function $u(t)$ defined on Γ to a continuously differentiable function $U(z)$ in $\mathbb{C} \setminus \Gamma$ with compact support such that at each $t \in \Gamma' := \Gamma \setminus \{a_1, a_2\}$ the limit values $U^+(t)$ and $U^-(t)$ are related by (3).

An integrator is called *complete* whenever its first partial derivatives are integrable.

If u admits a complete integrator U then

$$C^1(\mathbb{C}) \ni \phi \mapsto - \iint_{\mathbb{C}} \frac{\partial U \phi}{\partial \bar{z}} dz d\bar{z} \quad (4)$$

is called the *integration* of $u dz$ along Γ and denoted by $\int_{\Gamma} \phi(z) u(z) dz$.

The analogous functional

$$C^1(\mathbb{C}) \ni \phi \mapsto \iint_{\mathbb{C}} \frac{\partial U \phi}{\partial z} dz d\bar{z} \quad (5)$$

is called the *integration* of $u d\bar{z}$ along Γ and denoted by $\int_{\Gamma} \phi(z) u(z) d\bar{z}$. Certainly, this enables us to define integration along a nonrectifiable path Γ for the differential forms $udz + vd\bar{z}$ too, provided that both u and v admit complete integrators.

Since u can admit several integrators, when it is necessary to emphasize the dependence of integration in (4) on the integrator, we denote the latter by $(U) \int_{\Gamma} \phi(z)u(z) dz$. Under certain conditions, different integrators can produce the same integration. Let us present a result of this kind.

As usual, say that a function $f(t)$ on a set $A \subset \mathbb{C}$ satisfies there *Hölder's condition* with exponent $\nu \in (0, 1]$ and write $f \in H_{\nu}(A)$, whenever

$$h_{\nu}(f; A) := \sup \left\{ \frac{|f(t_1) - f(t_2)|}{|t_1 - t_2|^{\nu}} : t_1, t_2 \in A, t_1 \neq t_2 \right\} < \infty.$$

Denote the Hausdorff dimension of A by $\text{dmh } A$; see [14] for instance.

Lemma 1. *Consider a domain δ of the complex plane such that $a_1, a_2 \notin \delta$ and the intersection $\gamma := \delta \cap \Gamma$ is a path partitioning δ into two components δ^{\pm} . If a function u defined on Γ has two complete integrators U_1 and U_2 satisfying Hölder's condition with exponent $\nu > \text{dmh } \gamma - 1$ on the closures of δ^{\pm} then*

$$(U_1) \int_{\Gamma} \phi(z)u(z) dz = (U_2) \int_{\Gamma} \phi(z)u(z) dz$$

for every $\phi \in C_0^1(\delta)$.

PROOF. Fix $\alpha \in (\text{dmh } \gamma, 1 + \nu)$. By assumption, for every $\varepsilon > 0$ we can cover γ by a family of disks $B_j = \{z : |z - z_j| \leq r_j\}$, where $j = 1, 2, \dots$, such that $r_j < \varepsilon$ and $\sum_{j>0} r_j^{\alpha} < \varepsilon$. Denote the boundary of their union $\mathbf{B} = \bigcup_{j>0} B_j$ by Λ .

Put $D := U_1 - U_2$. This difference is obviously continuous and satisfies Hölder's condition with exponent ν on γ .

Given $\phi \in C_0^1(\gamma)$, we have

$$\left| (U_1) \int_{\Gamma} u \phi dz - (U_2) \int_{\Gamma} u \phi dz \right| \leq \left| \iint_{\mathbf{B}} \frac{\partial(D\phi)}{\partial \bar{z}} dz d\bar{z} \right| + \left| \int_{\Lambda} D\phi dz \right|.$$

The first term on the right-hand side vanishes as $\varepsilon \rightarrow 0$ because Γ has zero planar measure. It remains to show that the second term vanishes too.

Without loss of generality we may assume that the covering of γ by disks is finite and no disk is completely covered by the union of the remaining disks. Following the proof of Dolzhenko's Theorem [15], enumerate these disks in the decreasing order of radii and put $\Delta_1 = B_1$, $\Delta_2 = B_2 \setminus \Delta_1$, $\Delta_3 = B_3 \setminus \bigcup_{k=1}^2 \Delta_k$, $\Delta_4 = B_4 \setminus \bigcup_{k=1}^3 \Delta_k$, and so on. This expresses \mathbf{B} as the union of finitely many nonoverlapping simply-connected domains $\Delta_k \subset B_k$, whose boundaries $\lambda_k = \partial \Delta_k$ consist of circular paths of radii $\geq r_k$ lying inside B_k . Therefore, the length of λ_k is at most $2\pi r_k$. It is obvious that

$$\begin{aligned} \int_{\Lambda} D\phi dz &= \sum_k \int_{\lambda_k} D\phi dz, \\ \int_{\lambda_k} D\phi dz &= \int_{\lambda_k} D(z_k)\phi(z) dz + \int_{\lambda_k} (D(z) - D(z_k))\phi(z) dz. \end{aligned}$$

We have

$$\int_{\lambda_k} D(z_k)\phi(z) dz = - \iint_{\Delta_k} D(z_k) \frac{\partial \phi}{\partial \bar{z}} dz d\bar{z}.$$

Since $D \frac{\partial \phi}{\partial \bar{z}}$ is bounded, it follows that

$$\left| \sum_k \int_{\lambda_k} D(z_k)\phi(z) dz \right| \leq c \|\mathbf{B}\|,$$

where $\|\cdot\|$ stands for the planar measure. Thus, this sum vanishes as $\varepsilon \rightarrow 0$. Finally,

$$\left| \sum_k \int_{\lambda_k} (D(z) - D(z_k)) \phi(z) dz \right| \leq c \sum_k r_k^{\nu+1} \leq c \sum_k r_k^\alpha$$

for $\nu > \alpha - 1$, and this sum vanishes as $\varepsilon \rightarrow 0$ too. The proof of Lemma 1 is complete. \square

Suppose that a function $\nu(t)$ with $0 < \nu(t) \leq 1$ is defined on Γ' . Put a compactly supported continuously differentiable function $U(z)$ on $\mathbb{C} \setminus \Gamma$ into the class $\mathbb{H}_\nu(\Gamma')$ provided that each $t \in \Gamma'$ has in \mathbb{C} a neighborhood δ such that $a_1, a_2 \notin \delta$ and the intersection $\gamma := \delta \cap \Gamma$ is a path partitioning δ into two components δ^\pm ; furthermore, $U(z)$ satisfies Hölder's condition with exponent $\nu(t)$ on the closures of δ^\pm .

Refer to a function $\mathfrak{h}(t)$ on Γ' as the *local Hausdorff dimension* whenever for each $t \in \Gamma'$ there is a path $\gamma \subset \Gamma'$ of dimension $\text{dmh } \gamma = \mathfrak{h}(t)$ containing t .

Lemma 1 implies the following theorem.

Theorem 1. *Suppose that a path Γ' has local Hausdorff dimension related to the function $\nu(t)$ by the inequality $\nu(t) > \mathfrak{h}(t) - 1$ for $t \in \Gamma'$. If a function u on Γ' admits two complete integrators U_1 and U_2 of class $\mathbb{H}_\nu(\Gamma')$, then the integrations $(U_1) \int_\Gamma \cdot u(z) dz$ and $(U_2) \int_\Gamma \cdot u(z) dz$ coincide.*

In some cases it is expedient to require the integrability of $\frac{\partial U}{\partial \bar{z}}$ (to define integration of the form $u dz$) or $\frac{\partial U}{\partial z}$ (accordingly for $v d\bar{z}$) rather than that of the partial derivatives of the integrator with respect to x and y . We always mark transition to this version of the definition as integration *in the sense of* Definition 1'. Theorem 1 obviously remains valid.

3. The Existence of Integrators

Let us discuss conditions under which the function u on Γ admits an integrator. We will express them in terms of Marcinkiewicz exponents, the metric characteristics of a nonrectifiable curve introduced for the first time in [16, 17].

Suppose that Γ is still of planar measure zero. For $t \in \Gamma$ and $r > 0$ put $B(t; r) := \{z : |z - t| < r\}$. If $t \in \Gamma'$ and r is sufficiently small then Γ partitions $B(t; r)$ into the two sets $B^\pm(t; r)$ lying on the left and on the right of Γ respectively. Given $p > 0$, put

$$I_p^\pm(t; r) = \iint_{B^\pm(t; r)} \frac{dx dy}{\text{dist}^p(x + iy, \Gamma)}, \quad I_p(a_j; r) = \iint_{B(a_j; r)} \frac{dx dy}{\text{dist}^p(x + iy, \Gamma)}.$$

DEFINITION 2. The *inner* and *outer Marcinkiewicz exponents* of Γ at $t \in \Gamma'$ are defined as

$$\mathfrak{m}^\pm(\Gamma; t) := \sup\{p : \lim_{r \rightarrow 0} I_p^\pm(t; r) < \infty\}.$$

The quantity

$$\mathfrak{m}(\Gamma; t) := \max\{\mathfrak{m}^+(\Gamma; t), \mathfrak{m}^-(\Gamma; t)\}$$

is the *Marcinkiewicz exponent* at this point, while at the endpoints of Γ the Marcinkiewicz exponent is equal to

$$\mathfrak{m}(\Gamma; a_j) := \sup\{p : \lim_{r \rightarrow 0} I_p(a_j; r) < \infty\}, \quad j = 1, 2.$$

At the endpoints of Γ the Marcinkiewicz exponent has a special definition because Γ does not partition small neighborhoods of these endpoints into components.

Let us describe some properties of Marcinkiewicz exponents now available; see [18] for instance. First of all, if d is the upper Minkowski dimension (see [14] for instance) of an arbitrary path $\gamma \subset \Gamma$ containing a point t then all Marcinkiewicz exponents defined above lie in the segment $[2 - d, 1]$. Since the Minkowski dimension of every rectifiable path equals 1, at a point t contained in an arbitrarily small

rectifiable path $\gamma \subset \Gamma$ the Marcinkiewicz exponent equals 1 as well. Some curves constructed in [17, 18] have $\mathbf{m}(\Gamma; t) > 2 - d$ at interior points. Let us give an example of calculation of the Marcinkiewicz exponent at the endpoint of a path.

EXAMPLE 1. For a decreasing and vanishing sequence r_1, r_2, r_3, \dots of positive numbers, consider the circles $C_j := \{z : |z| = r_j\}$ and rays $R_\pm := \{xe^{\pm i\alpha} : 0 \leq x < +\infty\}$, where α is a small positive number. These rays cross C_j at the points $z_j^\pm := r_j e^{\pm i\alpha}$. Denote by A_j the larger of the two circular paths on C_j connecting these points; namely, $A_j = \{r_j e^{i\theta} : \alpha \leq \theta \leq 2\pi - \alpha\}$, and by L_j the rectilinear segment connecting z_j^- and z_{j+1}^+ . The union of all A_j and all L_j , closed by the point 0, constitutes a simple Jordan path Γ with endpoints 0 and z_1^+ . This path is rectifiable iff the series $\sum r_j$ converges. Assume that this series diverges. Simple calculations show that the integral $I_p(a_j; r)$ for $p \leq 1$ converges or diverges simultaneously with the series $\sum r_j(r_j - r_{j+1})^{1-p}$, i.e.

$$\mathbf{m}(\Gamma; 0) = \sup \left\{ p : \sum_{j=1}^{\infty} r_j(r_j - r_{j+1})^{1-p} < \infty \right\}.$$

In particular, for $r_j = j^{-m}$ with $0 < m \leq 1$ we obtain $\mathbf{m}(\Gamma; 0) = 2m/(1+m)$. Near the other endpoint z_1^+ the path Γ is rectifiable and $\mathbf{m}(\Gamma; z_1^+) = 1$.

To construct an integrator $U(z)$ for a function $u(t)$ on Γ , assume that near each $t \in \Gamma$ the function satisfies Hölder's condition with exponent $\nu(t)$, where $\nu : \Gamma \mapsto (0, 1]$ is a prescribed varying exponent, and denote the class of these functions by $H_\nu^{\text{loc}}(\Gamma)$.

Fix a small $\varepsilon > 0$ and refine from the covering of Γ by open disks $B(t; \varepsilon)$ with $t \in \Gamma$ a finite subcovering $B(t_j; \varepsilon)$ for $j = 1, 2, \dots, n$. Without loss of generality we may assume that the subcovering contains disks centered at the endpoints of the path. Take sufficiently small ε , so that u inside $B(t_j; \varepsilon)$ satisfies Hölder's condition with exponent $\nu(t_j)$ and, moreover,

- if t_j is an endpoint of Γ then the integral $I_p(t_j; r)$ is finite for $p = \mathbf{m}(\Gamma; t_j) - \varepsilon$;
- if $t_j \in \Gamma'$ and $\mathbf{m}(\Gamma; t) = \mathbf{m}^+(\Gamma; t)$ then the integral $I_p^+(t_j; r)$ is finite for $p = \mathbf{m}^+(\Gamma; t_j) - \varepsilon$;
- if $t_j \in \Gamma'$ and $\mathbf{m}(\Gamma; t) = \mathbf{m}^-(\Gamma; t)$ then the integral $I_p^-(t_j; r)$ is finite for $p = \mathbf{m}^-(\Gamma; t_j) - \varepsilon$.

Take a partition $\{\psi_j(z)\}_{j=1}^n$ of unity subordinate to the covering $\{B(t_j; \varepsilon)\}_{j=1}^n$; i.e., a collection of smooth functions $0 \leq \psi_j(z) \leq 1$ such that $\text{supp } \psi_j \subset B(t_j; \varepsilon)$ for $j = 1, 2, \dots, n$ and $\sum_{j=1}^n \psi_j(t) = 1$ for $t \in \Gamma$. Construct an integrator U as the sum of integrators U_j of $u_j := u\psi_j$.

If t_j is an interior point of Γ and $\mathbf{m}^+(\Gamma; t) \geq \mathbf{m}^-(\Gamma; t)$ then let U_j equal to $\chi_j^+ \mathcal{E}_0(u\psi_j)$, where χ_j^+ is the characteristic function of $B^+(t_j; \varepsilon)$, while $\mathcal{E}_0(u\psi_j)$ is the Whitney continuation of the function $u\psi_j$ supported on Γ to the whole complex plane. The following properties of this continuation are available; see [19] for instance. If A is a compact set on the complex plane and f is a function on A satisfying Hölder's condition with exponent $\mu \in (0, 1]$ then the Whitney continuation $F = \mathcal{E}_0(f)$ is defined on the whole complex plane so that $F|_A = f$ and $F \in H_\mu(\mathbb{C})$, while in $\mathbb{C} \setminus A$ it has partial derivatives satisfying the estimate $|\nabla F(z)| \leq C \text{dist}^{\mu-1}(z, A)$. Hence, we conclude that U_j on Γ has jump $u\psi_j$, while $U_j \in H_{\nu(t_j)}(B^+(t_j; \varepsilon))$ with $\text{supp } U_j \subset B^+(t_j; \varepsilon)$, and the derivative $\frac{\partial U_j}{\partial \bar{z}}$ is integrable to every power less than $\frac{\mathbf{m}(\Gamma; t_j) - \varepsilon}{1 - \nu(t_j)}$.

If t_j is an interior point of Γ at which $\mathbf{m}^-(\Gamma; t) \geq \mathbf{m}^+(\Gamma; t)$ then we put $U_j = -\chi_j^- \mathcal{E}_0(u\psi_j)$, where χ_j^- is the characteristic function of $B^-(t_j; \varepsilon)$. This integrator enjoys similar properties.

Thus, under the condition

$$\nu(t) > 1 - \mathbf{m}(\Gamma; t), \quad t \in \Gamma', \quad (6)$$

we can choose ε so small that the derivatives $\frac{\partial U_j}{\partial \bar{z}}$, with $t_j \in \Gamma'$, turn out integrable.

Finally, suppose that t_j is one of the endpoints a_1 and a_2 . In the previous cases we used the characteristic functions χ^\pm with unit jump on the path containing t_j , which is impossible now because Γ does not partition $B(t_j; \varepsilon)$. Instead of those characteristic functions we use the logarithm.

Refer as the *logarithmic kernel* of Γ to the single-valued branch of the logarithm

$$K_\Gamma(z) := \frac{1}{2\pi i} \log \frac{z - a_2}{z - a_1}, \quad (7)$$

chosen by cutting along Γ and requiring that $K_\Gamma(\infty) = 0$. These conditions determine K_Γ uniquely; this function has unit jump on Γ' , while its behavior at the endpoints a_1 and a_2 is determined by the geometric properties of Γ : its spiralling rate near the endpoints. Without spiralling, the real part of the logarithmic kernel is obviously bounded, while the imaginary part has logarithmic singularities at the endpoints of Γ . It is known that if Γ is rectifiable then $K_\Gamma(z) = o(|z - a_j|)$ for $j = 1, 2$, see [10], while if in addition the total length of the parts of Γ lying inside an arbitrary disk is commensurable with its radius, then $K_\Gamma(z) = O(\log |z - a_j|^{-1})$ for $j = 1, 2$; see [20]. If two paths Γ^1 and Γ^2 share the initial point and no other points near it, then $K_{\Gamma^1}(z) - K_{\Gamma^2}(z)$ is bounded near the common initial point; the same applies to the terminal points.

If t_j is the initial or terminal point of Γ then take the function U_j equal to $K_\Gamma \mathcal{E}_0(u\psi_j)$. This product has jump $u\psi_j$ on Γ' , but the integrability power of its derivatives depends not only on the Marcinkiewicz exponent of Γ at this point, but also on the singularity order of its logarithmic kernel, which can be arbitrarily large. For instance, in the example above if $r_j = j^{-m}$ with $0 < m \leq 1$ then near the origin $K_\Gamma(z)$ behaves like $|z|^{-1/m}$. Put Γ into the class $\mathfrak{S}(p_1, p_2)$ if near the point a_j its logarithmic kernel is integrable to every power less than p_j , for $j = 1, 2$. Say that the in Example 1 is of class $\mathfrak{S}(2m, \infty)$. Of certain interest are the paths Γ whose endpoints can be connected by a smooth path Λ disjoint from the interior of Γ . We call them *closable*. It is obvious that every closable path satisfies $K_\Gamma(z) = K_\Lambda(z) + O(1)$ and is of class $\mathfrak{S}(\infty, \infty)$.

If near an endpoint $U_j = K_\Gamma \mathcal{E}_0(u\psi_j)$ and $\Gamma \in \mathfrak{S}(p_1, p_2)$, for $p_1, p_2 > 1$, then the first partial derivatives of U_j are integrable near this point to every power less than $\frac{p_j \mathfrak{m}(\Gamma; a_j)}{p_j(1 - \nu(a_j)) + \mathfrak{m}(\Gamma; a_j)}$, while this fraction is greater than 1 provided that

$$\nu(a_j) > 1 - \frac{p_j - 1}{p_j} \mathfrak{m}(\Gamma; a_j). \quad (8)$$

Therefore, on assuming (6) at all interior points of the path and (8) at its endpoints, we see that the sum $U = \sum U_j$ is a complete integrator. This yields the following

Theorem 2. *Suppose that a function $u(t)$ with local Hölder exponent $\nu(t)$ is defined on a path $\Gamma \in \mathfrak{S}(p_1, p_2)$ with $p_1, p_2 > 1$. This function is integrable along Γ in the sense of Definition 1 provided that (6) holds at all interior points of the path and (8) holds at its endpoints.*

The condition $p_1, p_2 > 1$ is rather restrictive. For instance, the path in Example 1 meets it only for $m > 1/2$. When the condition is violated, the class of integrable functions is narrower.

Put a path Γ into the class $\mathcal{S}(m_1, m_2)$ whenever $K_\Gamma(z) = O(|z - a_j|^{-m_j})$ near a_j , with $m_j > 0$ for $j = 1, 2$. It is obvious that $\mathcal{S}(m_1, m_2) \subset \mathfrak{S}(2/m_1, 2/m_2)$. If at least one m_j is greater than or equal to 2 then the class $\mathcal{S}(m_1, m_2)$ fails to embed into $\mathfrak{S}(p_1, p_2)$ with $p_j > 1$ for $j = 1, 2$.

Say that a function $u(t) \in H_\nu^{\text{loc}}(\Gamma)$ on a path Γ has *Taylor derivatives* at a_j of orders through $m \in \mathbb{N}$ whenever we can express it as $u(t) = P(t, \bar{t}) + |t - a_j|^{m+1} u_1(t)$, where $P(z, \bar{z})$ is an algebraic polynomial in the variables z and \bar{z} of degree at most m , called a *Taylor polynomial*, and $u_1(t) \in H_\nu^{\text{loc}}(\Gamma)$. We can define the continuation of the function to the whole complex plane as $P(z, \bar{z}) + |z - a_j|^{m+1} \mathcal{E}_0 u_1(z)$. Replacing in the proof of Theorem 2 the Whitney continuation in a neighborhood of an endpoint by this continuation, we arrive at the following result.

Theorem 3. *Suppose that a function $u(t) \in H_\nu^{\text{loc}}(\Gamma)$ is defined on a path $\Gamma \in \mathcal{S}(m_1, m_2)$. If at the endpoints this function has Taylor derivatives of orders above m_j for $j = 1, 2$, while its Taylor polynomials are holomorphic (independent of \bar{z}), and moreover (6) holds at the interior points of the path, while (8) with $p_j = 2/m_j$ holds at its endpoints for $j = 1, 2$; then the integral $\int_\Gamma u(z) dz$ exists in the sense of Definition 1'.*

Observe that the conjugate logarithmic kernel $\overline{K_\Gamma(z)}$ also has unit jump on Γ' , which enables us to obtain an analog of Theorem 3 for the integral $\int_\Gamma v(z) d\bar{z}$.

4. Some Integral Representations

Consider the Cauchy-type integral (2) over a nonrectifiable path. Fixing $z \in \mathbb{C} \setminus \Gamma$, denote by $\mathcal{CA}(w, z)$ a function of class $C_0^\infty(\mathbb{C})$ with respect to the variable w coinciding with $(2\pi i(w - z))^{-1}$ near Γ and vanishing identically in a neighborhood of z . Then the Cauchy-type integral with density u along the nonrectifiable path Γ is the function

$$\Phi(z) = \int_\Gamma \mathcal{CA}(w, z) u(w) dw, \quad (9)$$

i.e., the result of applying the functional $\int_\Gamma \cdot u dw$ of Section 3 to $\mathcal{CA}(w, z)$.

Under the assumptions of Theorem 2 this function is holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ and vanishes at infinity. Simple rearrangements yield

$$\Phi(z) = U(z) - \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial U}{\partial \bar{w}} \frac{dw d\bar{w}}{w - z}. \quad (10)$$

Recall that $U = \sum U_j$ and the support of U_j is concentrated in a small neighborhood of the point $t_j \in \Gamma$. If it is an interior point of Γ then the derivative $\frac{\partial U_j}{\partial \bar{z}}$ is integrable to an arbitrary power less than $\frac{\mathfrak{m}(\Gamma; t_j) - \varepsilon}{1 - \nu(t_j)}$. This fraction is greater than 2 under the condition

$$\nu(t) > 1 - \frac{1}{2} \mathfrak{m}(\Gamma; t), \quad t \in \Gamma', \quad (11)$$

i.e., see [21] for instance, the function $\iint_{\mathbb{C}} \frac{\partial U_j}{\partial \bar{w}} \frac{dw d\bar{w}}{w - z}$ is continuous in a neighborhood of t_j and satisfies there Hölder's condition with an arbitrary exponent at most $\mu(t_j)$, where

$$\mu(t) := 1 - \frac{2(1 - \nu(t))}{\mathfrak{m}(\Gamma; t)}. \quad (12)$$

If t_j is the initial or terminal point of Γ then, as we have seen, under condition (8) the derivative $\frac{\partial U_j}{\partial \bar{z}}$ is integrable near this point to a power greater than 1, and so $\iint_{\mathbb{C}} \frac{\partial U_j}{\partial \bar{w}} \frac{dw d\bar{w}}{w - z}$ is integrable there to a power greater than 2. Summarizing, we obtain the following

Theorem 4. *Suppose that $u(t)$ is defined on a path $\Gamma \in \mathfrak{S}(p_1, p_2)$, where $p_1, p_2 > 1$, with local Hölder exponent $\nu(t)$ and, moreover, conditions (11) and (8) are met. Then the Cauchy-type integral (9) enjoys the following properties:*

- (i) $\Phi(z)$ is holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ and vanishes at infinity;
- (ii) $\Phi(z)$ has continuous right and left limit at each point $t \in \Gamma'$ whose difference equals $u(t)$;
- (iii) $\Phi \in \mathbb{H}_\nu(\Gamma')$, where $0 < \nu(t) < \mu(t)$ for $t \in \Gamma'$;
- (iv) near the endpoints of the path the function $\Phi(z)$ is integrable to a power greater than 2.

Applications demand a finer description of the behavior of Cauchy-type integral near the endpoints of the path. Consider firstly the situation of $\Gamma \in \mathcal{S}(m_1, m_2)$ with $m_1, m_2 < 1$. Near the endpoints the derivatives of the integrator constructed above are integrable to arbitrary power less than

$$\frac{2\mathfrak{m}(\Gamma; a_j)}{2(1 - \nu(a_j)) + m_j \mathfrak{m}(\Gamma; a_j)}, \quad j = 1, 2,$$

and under the condition

$$\nu(t_j) > 1 - \frac{1 - m_j}{2} \mathfrak{m}(\Gamma; a_j), \quad j = 1, 2, \quad (13)$$

these quantities are greater than 2. Hence, under this condition the integral term in (10) is continuous at these points; i.e.,

$$\Phi(z) = K_\Gamma(z)\mathcal{E}_0(u)(z) + C_j + o(1) \quad \text{as } z \rightarrow a_j,$$

where C_j are constants. In turn,

$$K_\Gamma(z)\mathcal{E}_0(u)(z) = u(a_j)K_\Gamma(z) + O(|z - a_j|^{\nu(t_j)-m_j}).$$

Finally, we obtain the following corollary.

Corollary 1. *Suppose that a function $u(t)$ is defined on a path $\Gamma \in \mathcal{S}(m_1, m_2)$, where $m_1, m_2 < 1$, with the local Hölder exponent $\nu(t)$, conditions (11) and (13) are met, and $\nu(t_j) \geq m_j$ for $j = 1, 2$. Then claims (i)–(iii) of Theorem 4 remain valid, while claim (iv) can be strengthened as follows:*

(v) $\Phi(z) = u(a_j)K_\Gamma(z) + C_j + o(1)$ near the endpoints.

If $\Gamma \in \mathcal{S}(m_1, m_2)$ with large exponents m_1 and m_2 then the Cauchy integral along this path has different asymptotics at the endpoints. Arguing as in Section 3, we obtain the following corollary.

Corollary 2. *Suppose that a function $u(t)$ on a path $\Gamma \in \mathcal{S}(m_1, m_2)$, where $m_1, m_2 \geq 2$, can be expressed near the endpoints as $u(t) = P_j(t) + (t - a_j)^{n_j}u_j(t)$, where P_j are polynomials of degree below n_j , the integers n_j are greater than m_j , the functions u_j are of local Hölder exponent $\nu(t)$, and conditions (11) and*

$$\nu(t_j) > 1 - \frac{n_j - m_j}{2} \mathfrak{m}(\Gamma; a_j), \quad j = 1, 2, \quad (14)$$

are satisfied. Then claims (i)–(iii) of Theorem 4 remain valid, while claim (iv) is replaced by the following condition:

(vi) $\Phi(z) = P_j(z)K_\Gamma(z) + O(1)$ near the endpoints.

Let us present another representation with an integral along a nonrectifiable path.

Consider the Beltrami equation

$$\frac{\partial \Phi}{\partial \bar{z}} = \mu(z) \frac{\partial \Phi}{\partial z}. \quad (15)$$

This generalization of the Cauchy–Riemann equations has numerous applications; see [21] for instance.

Consider a nonconstant function $f(z)$ holomorphic in the domain Δ of the complex plane, a positive number α , and put

$$\mu = \beta \frac{f}{\bar{f}} \frac{\bar{f}'}{f'}, \quad (16)$$

where $\beta := \frac{\alpha}{1+\alpha}$. As [22] shows, every solution of the Beltrami equation (15) with coefficient μ in an arbitrary domain D with $\bar{D} \subset \Delta$ and rectifiable boundary Γ can be expressed as

$$\Phi(z) = \frac{1}{2\pi i(1-\beta)} \int_{\Gamma} \Phi(t)(P(t, z) dt + Q(t, z) d\bar{t}), \quad (17)$$

where

$$P(t, z) := \frac{f'(t)}{f(t) - f(z) \left| \frac{f(z)}{f(t)} \right|^{2\alpha}}, \quad (18)$$

$$Q(t, z) := \frac{\beta f(t) \overline{f'(t)}}{f(t)(f(t) - f(z) \left| \frac{f(z)}{f(t)} \right|^{2\alpha})}. \quad (19)$$

The arguments of [22] and Section 1 here lead to the following

Theorem 5. Suppose that a nonrectifiable path $\Gamma \in \mathfrak{S}(p_1, p_2)$, where $p_1, p_2 > 1$, lies in Δ , a function $f(z)$ is holomorphic in Δ , and $\alpha > 0$. If $u(t) \in H_\nu^{\text{loc}}(\Gamma)$ is defined on this path and conditions (11) and (8) are met then the generalized integral

$$\Phi(z) = \frac{1}{2\pi i(1-\beta)} \int_{\Gamma} u(t)(P(t, z) dt + Q(t, z) d\bar{t}) \quad (20)$$

exists, where P and Q are the kernels (18) and (19) respectively, and $\Phi(z)$ enjoys the properties:

- (i) $\Phi(z)$ satisfies the Beltrami equation (15) with coefficient (16) in $\Delta \setminus \Gamma$;
- (ii) $\Phi(z)$ has continuous right and left limits at each point $t \in \Gamma'$ whose difference equals $u(t)$;
- (iii) $\Phi \in \mathbb{H}_\nu(\Gamma')$, where $0 < \nu(t) < \mu(t)$ and $t \in \Gamma'$;
- (iv) $\Phi(z)$ is integrable to a power greater than 2 near the endpoints.

5. Applications to Boundary Value Problems

Let us start with the jump problem for holomorphic functions in the following formulation.

Problem 1. Find a holomorphic function $\Phi(z)$ in $\overline{\mathbb{C}} \setminus \Gamma$ vanishing at infinity, possessing right and left limits at $t \in \Gamma'$ related by the boundary condition

$$\Phi^+(t) - \Phi^-(t) = u(t), \quad (21)$$

where $u(t)$ is a prescribed function, and integrable near the endpoints of the path to a power greater than 2.

It is obvious that under the hypotheses of Theorem 4 the generalized Cauchy-type integral (9) is a solution to this problem. Let us discuss the question of its uniqueness. In general it is not unique. Indeed, as [15] established, if $\text{dmh } \Gamma > 1$ then there exists a function $F(z) \not\equiv 0$ continuous on the whole complex plane, holomorphic on $\overline{\mathbb{C}} \setminus \Gamma$, and vanishing at infinity. Consequently, the sum of this function and the Cauchy-type integral (9) is another solution to Problem 1. At the same time [15] this function cannot satisfy Hölder's condition on Γ' with exponent greater than or equal to $\text{dmh } \Gamma - 1$. Therefore, the difference of two solutions to Problem 1 of class $\mathbb{H}_\nu(\Gamma')$ with $\nu(t) > \mathfrak{h}(t) - 1$ for $t \in \Gamma'$ can be singular only at the endpoints a_1 and a_2 ; furthermore, the integrability conditions for the solutions to power $p > 2$ implies that these singularities are removable. Thus, we have the following statement.

Theorem 6. Suppose that the path Γ and the jump $u(t)$ satisfy the hypotheses of Theorem 4. Then Problem 1 is solvable and one of its solutions is the Cauchy-type integral (9). Furthermore, if

$$\mu(t) := 1 - \frac{2(1 - \nu(t))}{\mathfrak{m}(\Gamma; t)} > \mathfrak{h}(t) - 1, \quad t \in \Gamma', \quad (22)$$

where \mathfrak{h} is the local Hausdorff dimension of Γ , then the function (9) is the unique solution to Problem 1 in the class $\mathbb{H}_\nu(\Gamma')$ with $\nu(t) > \mathfrak{h}(t) - 1$ for $t \in \Gamma'$.

In the jump problem, the condition

$$\Phi(z) = O(|z - a_j|^{-\gamma_j}), \quad 0 \leq \gamma_j < 1, \quad j = 1, 2, \quad (23)$$

at the endpoints, see [1–5], is often imposed on the required function instead of integrability condition to a power greater than 2. For this formulation of the problem Theorem 6 holds if we replace in it the reference to Theorem 4 with that to Corollary 1.

Under the conditions of Corollary 2, formula (v) of Corollary 1 describes the behavior of the Cauchy-type integral near the endpoints. Thus, the integral (9) is integrable near the endpoints to a power greater than 2 or satisfies (23) only in the case that the Taylor polynomials of the jump $u(t)$ at the endpoints have zeroes of sufficiently high order at these points. If this is not so then the problem may lack solutions.

Consider the homogeneous Riemann problem.

Problem 2. Find a holomorphic function $\Phi(z) \in \mathbb{H}_\nu(\Gamma')$ on $\overline{\mathbb{C}} \setminus \Gamma$ vanishing at infinity and possessing right and left limits at $t \in \Gamma'$ related by the boundary condition

$$\Phi^+(t) = G(t)\Phi^-(t), \quad (24)$$

where $G(t)$ is a prescribed function, while $\nu(t) > \mathfrak{h}(t) - 1$ for $t \in \Gamma'$, and bounded near the endpoints of the path.

Assume that $\Gamma \in \mathcal{S}(m_1, m_2)$ with $m_1, m_2 < 1$, while the coefficient $G \in H_\nu^{\text{loc}}(\Gamma)$ is nonvanishing. Then $G(t) = \exp u(t)$ with $u \in H_\nu^{\text{loc}}(\Gamma)$. Under the hypotheses of Corollary 1 the Cauchy-type integral yields the function

$$\Phi(z) = \int_{\Gamma} \mathcal{C}\mathcal{A}(w, z)u(w) dw$$

with jump u on Γ' and the estimates $\Phi(z) = u(a_j)(z)K_\Gamma(z) + O(1)$ at the endpoints of Γ . Put $X(z) = \exp \Phi(z)$. It is obvious that this function satisfies the boundary condition (24), while

$$|X(z)| = \exp \operatorname{Re}(u(a_j)(z)K_\Gamma(z) + O(1))$$

at the endpoints. Put $u(a_j) = \sigma_j + i\varsigma_j$, where σ_j and ς_j are real numbers. Then

$$\operatorname{Re}(u(a_j)(z)K_\Gamma(z)) = \sigma_j A(z) + \frac{\varsigma_j}{2\pi} \log \left| \frac{z - a_2}{z - a_1} \right|,$$

where $A(z)$ is a single-valued branch of the argument $\frac{1}{2\pi} \arg \frac{z - a_2}{z - a_1}$ chosen by cutting along the path Γ and requiring that $A(\infty) = 0$. Put

$$b_j := \liminf_{z \rightarrow a_j} \frac{\sigma_j A(z)}{\log |z - a_j|}, \quad B_j := \limsup_{r \rightarrow 0} \max \left\{ \frac{\sigma_j A(z)}{\log |z - a_j|} : |z - a_j| = r \right\}.$$

Following the arguments of [10], we obtain the next result.

Theorem 7. Suppose that $\Gamma \in \mathcal{S}(m_1, m_2)$, where $m_1, m_2 < 1$, the coefficient $G \in H_\nu^{\text{loc}}(\Gamma)$ is nonvanishing, conditions (11), (13), and (22) hold, and $\nu(t_j) \geq m_j$ for $j = 1, 2$. Then

if one of the B_j for $j = 1, 2$ is $-\infty$ then Problem 2 lacks nontrivial solutions;

if one of the b_j is $+\infty$ and the other is distinct from $-\infty$ then the number of linearly independent solutions is infinite;

if B_1 and B_2 are finite then the number of linearly independent solutions equals $\max\{0, \varkappa\}$, where $\varkappa = \varkappa_1 + \varkappa_2$ with $\varkappa_j = [b_j + (-1)^j(2\pi)^{-1}\varsigma_j]$; for $\varkappa > 0$ the general solution is of the form

$$\Phi(z) = P(z)X(z)(z - a_1)^{-\varkappa_1}(z - a_2)^{-\varkappa_2},$$

where $P(z)$ is an arbitrary polynomial of degree at most \varkappa .

Here $[x]$ stands for the largest integer not exceeding x .

Let us present one result concerning the paths that spiral fast near the endpoints. Suppose that $\Gamma \in \mathcal{S}(m_1, m_2)$, where $m_1, m_2 \geq 2$, and that the coefficient $G(t) \in H_\nu^{\text{loc}}(\Gamma)$ is nonvanishing, has Taylor derivatives of sufficiently high order at the endpoints, and furthermore its Taylor polynomials are holomorphic. As in the proof of Theorem 6, take $G(t) = \exp u(t)$, where $u(t)$ enjoys the same properties. Putting

$$X(z) = \exp \int_{\Gamma} \mathcal{C}\mathcal{A}(w, z)u(w) dw,$$

under the hypotheses of Corollary 2 we obtain a function with the asymptotics

$$|X(z)| = \exp \operatorname{Re}(P_j(z)(z)K_\Gamma(z) + O(1))$$

at the endpoints, where P_j are Taylor polynomials for $u(t)$. Suppose that P_j has a zero of order $k_j \geq 1$ at a_j . If $\Phi(z)$ is an arbitrary solution to Problem 2 then the ratio $\Psi = \Phi/X$ can be singular only at a_1 and a_2 . Moreover, this ratio is bounded on the set $\Upsilon_j := \{z : \operatorname{Re} P_j(z)\}$, while in some neighborhoods of the endpoints it admits the estimates $|\Psi(z)| = O(\exp |z - a_j|^{k_j - m_j})$. The set Υ_j consists of the curves partitioning a neighborhood of a_j into $2k_j$ curvilinear sectors of opening π/k_j with vertices at this point. According to the Phragmén–Lindelöf principle, see [23] for instance, for $k_j > m_j - k_j$ the singularity of this function at a_j is removable. Thus, we have justified the next statement.

Theorem 8. Suppose that $\Gamma \in \mathcal{S}(m_1, m_2)$ with $m_1, m_2 \geq 2$, the coefficient $G \in H_\nu^{\text{loc}}(\Gamma)$ is nonvanishing and has Taylor derivatives of orders $n_j > m_j$ at the endpoints of the path, and furthermore, its Taylor polynomials are holomorphic and have zeroes of orders $k_j > m_j/2$ at a_j for $j = 1, 2$. Then under conditions (11), (14), and (22) the unique solution to Problem 2 vanishes identically.

Consider the jump problem for the Beltrami equation on a nonrectifiable path. Suppose that the path Γ lies in Δ where (15) makes sense.

Problem 3. Find a function $\Phi(z)$ satisfying in $\Delta \setminus \Gamma$ equation (15) with coefficients (16), with right and left limits at each $t \in \Gamma'$ related by the boundary condition (21), where $u(t)$ is a prescribed function, and integrable near the endpoints of the path to a power greater than 2.

By analogy with Theorem 6 we obtain the next statement.

Theorem 9. Suppose that the path Γ and jump $u(t)$ satisfy the conditions of Theorem 4. Then Problem 3 is solvable, and one of its solutions is given by (17). Furthermore, if (16) is satisfied, where \mathfrak{h} is the local Hausdorff dimension of Γ , then the general solution to Problem 3 is the sum of (17) with an arbitrary function satisfying the Beltrami equation with the coefficients of (16) in the domain Δ .

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