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Augmentation quotients for real representation rings of cyclic groups

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Abstract. Denote by C_m the cyclic group of order m. Let $\mathcal{R}(C_m)$ be its real representation ring, and $\Delta(C_m)$ its augmentation ideal. In this paper, we give an explicit $\mathbb{Z}$-basis for the n-th power $\Delta^n(C_m)$ and determine the isomorphism class of the n-th augmentation quotient $\Delta^n(C_m)/\Delta^{n+1}(C_m)$ for each positive integer n.

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1. Introduction

Let G be a finite group. A real matrix representation (for convenience, we say 'representation' in the sequel) of G is a group homomorphism

$$\rho : G \longrightarrow \mathrm{GL}_d(\mathbb{R}), \tag{1}$$

where $\mathrm{GL}_d(\mathbb{R})$ is the general linear group of rank d ($d \in \mathbb{N}$). We also say that d is the degree of ρ. (Here we set $\mathrm{GL}_0(\mathbb{R})$ to be the trivial group consisting of the empty matrix.) Two representations ρ and η are said to be *similar* (denoted by $\rho \sim \eta$), if there exists an invertible square matrix P such that

$$\eta(g) = P^{-1}\rho(g)P, \quad \forall\, g \in G. \tag{2}$$

It is easy to see that similarity of representations is an equivalence relation. The equivalence classes are called similarity classes. The similarity class of ρ is denoted by $\bar{\rho}$. The direct sum $\bar{\rho} \oplus \bar{\eta}$ of two similarity classes $\bar{\rho}$ and $\bar{\eta}$ is defined by $\bar{\rho} \oplus \bar{\eta} = \overline{\rho \oplus \eta}$, where

$$\rho \oplus \eta : G \longrightarrow \mathrm{GL}_d(\mathbb{R}) \times \mathrm{GL}_{d'}(\mathbb{R}) \rightarrowtail \mathrm{GL}_{d+d'}(\mathbb{R}). \tag{3}$$

The *real representation ring* $\mathcal{R}(G)$ is the group completion of the monoid (under direct sum $\oplus$) of similarity classes of representations of G. Its addition and multiplication are induced by direct sum and tensor product of matrices, respectively. By [5], $\mathcal{R}(G)$ is a commutative ring with an identity element. Its underlying group is a finitely generated free abelian group with basis on the similarity classes of irreducible representations.

The notion of degree of a representation induces a ring homomorphism

$$\phi : \mathcal{R}(G) \longrightarrow \mathbb{Z}. \tag{4}$$

This homomorphism is called the *augmentation map*. Its kernel $\Delta(G)$ is called the *augmentation ideal* of $\mathcal{R}(G)$. Let $\Delta^n(G)$ and $Q_n(G)$ denote the n-th power of $\Delta(G)$ and the n-th consecutive quotient group $\Delta^n(G)/\Delta^{n+1}(G)$, respectively.

It is an interesting problem to determine the structures of $\Delta^n(G)$ and $Q_n(G)$ since they have many connections with other algebraic branches. A related problem of recent interest has been to settle the same problem for the *complex representation ring* $\mathcal{R}(G, \mathbb{C})$. Chang and collaborators [1–3] solved it for dihedral groups, point groups and generalized quaternion groups respectively. In fact, they proved in [1] that, for any finite abelian group G,

$$\mathcal{R}(G, \mathbb{C}) \cong \mathbb{Z}G, \quad Q_n(G, \mathbb{C}) \cong I^n/I^{n+1}, \tag{5}$$

where $Q_n(G, \mathbb{C})$ and I denote the n-th augmentation quotient for $\mathcal{R}(G, \mathbb{C})$ and the augmentation ideal of $\mathbb{Z}G$, respectively. Karpilovsky raised the problem of determining the isomorphism type of the groups I^n/I^{n+1} in [6]. Chang and Tang [4] solved it, thereby solving the problem for the groups $Q_n(G, \mathbb{C})$.

The goal of this article is to give an explicit $\mathbb{Z}$-basis for each $\Delta^n(C_m)$ and determine the isomorphism class of each $Q_n(C_m)$, where C_m is the cyclic group of order m.

The result also computes $\mathrm{Tor}_1^{\mathcal{R}(C_m)}(\mathcal{R}(C_m)/\Delta^n(C_m), \mathcal{R}(C_m)/\Delta(C_m))$ because for any finite group G, $Q_n(G) \cong \mathrm{Tor}_1^{\mathcal{R}(G)}(\mathcal{R}(G)/\Delta^n(G), \mathcal{R}(G)/\Delta(G))$.

2. Preliminaries

In this section, we provide some useful results about $Q_n(G)$ and finite generated free abelian groups. Chang *et al.* [1] proved similar properties for complex representation rings. Here we omit their proofs since they are almost identical to the proofs in [1].

Theorem 2.1. *For any natural number n, $Q_n(G)$ is a finite abelian $|G|$-torsion group.*

COROLLARY 2.2

For each positive integer n, $\Delta^n(G)$ has free rank $r(G) - 1$, where $r(G)$ is the free rank of $\mathcal{R}(G)$.

Theorem 2.3. *If $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{R}(G)$ has trivial Jacobson radical, then there exist positive integers n_0 and π such that*

$$Q_n(G) \cong Q_{n+\pi}(G) \tag{6}$$

for any $n \geqslant n_0$.

It is well known that any two representations with the same character are similar. Moreover, there is an injective ring homomorphism

$$\chi : \mathcal{R}(G) \longrightarrow \mathbb{R}^G \tag{7}$$

which sends $\bar{\rho}$ to its character χ_ρ for each representation ρ of G.

At last, we recall a classical result about finite generated free abelian groups.

Lemma 2.4. *Let H be a finite generated free abelian group of rank N. If the N elements $g_1, \ldots, g_N$ generate H, then they form a basis of H.*

3. Necessary tools

In this section, we construct a basis of $\Delta(C_m)$ and show some of its basic properties. Since the real and complex representation rings of C_1 (and C_2) are isomorphic, we shall assume $m \geqslant 3$ in the sequel. Denote by g the generator of C_m. Then each representation ρ of C_m depends only on its value at g. Therefore, we use $\rho(g)$ denote ρ.

The following theorem found in [5] classifies the similarity classes of all irreducible representations of C_m.

Theorem 3.1. *Let $\rho_1 = (1)$, $\rho_2 = (-1)$ and*

$$\eta_k = \begin{pmatrix} \cos(2k\pi/m) & \sin(2k\pi/m) \\ -\sin(2k\pi/m) & \cos(2k\pi/m) \end{pmatrix}, \quad k \in \mathbb{Z}. \tag{8}$$

Then all distinct similarity classes of irreducible representations of C_m are

- $\bar{\rho}_1$, $\bar{\eta}_k$, $1 \leqslant k \leqslant (m-1)/2$, *when m odd,*
- $\bar{\rho}_1$, $\bar{\rho}_2$, $\bar{\eta}_k$, $1 \leqslant k \leqslant m/2 - 1$, *when m even.*

For later use, we remind that, by Corollary 2.2 and Theorem 3.1, for each natural number n, $\Delta^n(C_m)$ has free rank $(m-1)/2$ or $m/2$ according to m is odd or even, respectively.

Now we construct a basis of $\Delta(C_m)$. For convenience, we fix the following notation:

- $F = \bar{\rho}_1 - \bar{\rho}_2$, $Y_k = \bar{\eta}_k - \bar{\eta}_{k-1}$, $k \in \mathbb{Z}$.
- For any subset $\mathcal{S} \subset \mathcal{R}(C_m)$, denote by $\mathbb{Z}\mathcal{S}$ the set of all $\mathbb{Z}$-linear combinations of elements of $\mathcal{S}$.

Lemma 3.2. *$\Delta(C_m)$ is the free abelian group based on $\mathcal{B}_m$, where*

$$\mathcal{B}_m = \begin{cases} \{Y_1, \ldots, Y_{(m-1)/2}\}, & \text{if } m \text{ odd}, \\ \{F,\ Y_1, \ldots, Y_{m/2-1}\}, & \text{if } m \text{ even}. \end{cases} \tag{9}$$

Proof. Note that $\mathcal{B}_m$ is contained in $\Delta(C_m)$ and it has a free rank $|\mathcal{B}_m|$. So by Lemma 2.4, we only need to show $\mathcal{B}_m$ generates $\Delta(C_m)$. When m is an odd number, let

$$\omega = a_1\bar{\rho}_1 + \sum_{k=1}^{(m-1)/2} c_k\bar{\eta}_k \in \Delta(C_m). \tag{10}$$

Then $a_1 + 2\sum_{k=1}^{(m-1)/2} c_k = 0$. The short calculations show that $\bar{\eta}_0 = 2\bar{\rho}_1$. So

$$\omega = \sum_{k=1}^{(m-1)/2} c_k(\bar{\eta}_k - \bar{\eta}_0) = \sum_{k=1}^{(m-1)/2} c_k \sum_{j=1}^{k} Y_j \in \mathbb{Z}\mathcal{B}_m. \tag{11}$$

When m is an even number, let

$$\omega = a_1\bar{\rho}_1 + a_2\bar{\rho}_2 + \sum_{k=1}^{m/2-1} c_k\bar{\eta}_k \in \Delta(C_m). \tag{12}$$

Then $a_1 + a_2 + 2\sum_{k=1}^{m/2-1} c_k = 0$. Hence,

$$\omega = a_2(\bar{\rho}_2 - \bar{\rho}_1) + \sum_{k=1}^{m/2-1} c_k(\bar{\eta}_k - \bar{\eta}_0) = -a_2F + \sum_{k=1}^{m/2-1} c_k \sum_{j=1}^{k} Y_j \in \mathbb{Z}\mathcal{B}_m. \tag{13}$$

Together (11) and (13) finish the proof. □

PROPOSITION 3.3

Regarding elements of $\Delta(C_m)$, *we have*

(1) Y_k *depends only on the residue class of k modulo m, and* $Y_k = -Y_{m+1-k}$,
(2) $Y_kY_l = (Y_{k+l} - Y_{k+l-1}) - (Y_{k-l+1} - Y_{k-l}) = \sum_{j=-l+1}^{l-1} Y_{k+j}Y_1$.

In addition, the following identities hold when m even.

(3) $F^n = 2^{n-1}F$, $FY_k = Y_k + Y_{m/2+1-k}$.

Proof. One can easily verify (1) and (3) by calculating the characters of relative representations. For (2), a short calculation shows that $\chi_{\eta_k}\chi_{\eta_l} = \chi_{\eta_{k+l}} + \chi_{\eta_{k-l}}$. So

$$\bar{\eta}_k\bar{\eta}_l = \bar{\eta}_{k+l} + \bar{\eta}_{k-l}. \tag{14}$$

Hence,

$$\begin{aligned} Y_kY_l &= (\bar{\eta}_k - \bar{\eta}_{k-1})(\bar{\eta}_l - \bar{\eta}_{l-1}) \\ &= \bar{\eta}_{k+l} + 2\bar{\eta}_{k-l} - 2\bar{\eta}_{k+l-1} - \bar{\eta}_{k-l+1} - \bar{\eta}_{k-l-1} + \bar{\eta}_{k+l-2} \\ &= (Y_{k+l} - Y_{k+l-1}) - (Y_{k-l+1} - Y_{k-l}) \\ &= \sum_{j=-l+1}^{l-1} \left[(Y_{k+j+1} - Y_{k+j}) - (Y_{k+j} - Y_{k+j-1})\right] \\ &= \sum_{j=-l+1}^{l-1} Y_{k+j}Y_1, \end{aligned} \tag{15}$$

as required. □

Recall that the first goal of this article is to provide an explicit $\mathbb{Z}$-basis for each $\Delta^n(C_m)$. By Lemma 2.4, we just need to find a generating set of $\Delta^n(C_m)$ whose cardinality equals $(m-1)/2$ or $m/2$ according to m is odd or even, respectively. Due to Lemma 3.2, $\Delta^n(C_m)$ is generated by

$$\begin{cases} \left\{ \prod_{i=1}^{n} Y_{k_i} \middle| 1 \leqslant k_i \leqslant (m-1)/2 \right\}, & \text{if } m \text{ odd,} \\ \left\{ F^j \prod_{i=1}^{n-j} Y_{k_i} \middle| 0 \leqslant j \leqslant n, 1 \leqslant k_i \leqslant m/2 \right\}, & \text{if } m \text{ even.} \end{cases}$$

Hence, due to Proposition 3.3, $\Delta^n(C_m)$ is generated by

$$\begin{cases} \{Y_k Y_1^{n-1} | 1 \leqslant k \leqslant m\}, & \text{if } m \text{ odd,} \\ \{F^j Y_k Y_1^{n-j-1} | 0 \leqslant j \leqslant n-1, 1 \leqslant k \leqslant m\} \cup \{F^n\}, & \text{if } m \text{ even.} \end{cases}$$

For later use, we fix the following notation and prove two useful identities about them. Throughout, n and N are natural numbers.

- $\mathcal{S}_{n,0}(N) = \{Y_k Y_1^{n-1} | 1 \leqslant k \leqslant N\}$.
- $\mathcal{S}_{n,j}(N) = \{F^j Y_k Y_1^{n-j-1} | 1 \leqslant k \leqslant N\}$, m is even, $n \geqslant 2$, $1 \leqslant j \leqslant n-1$.

PROPOSITION 3.4

For any $n \geqslant 2$ and each positive integer N, we have

$$\mathbb{Z}\mathcal{S}_{n,0}(N) = \mathbb{Z}\{(Y_k - (2k-1)Y_1)Y_1^{n-2} | 2 \leqslant k \leqslant N+1\}. \tag{16}$$

Moreover, the following identity holds for $0 \leqslant j \leqslant n-2$ when m even.

$$\mathbb{Z}\mathcal{S}_{n,j}(N) = \mathbb{Z}\{F^j (Y_k - (2k-1)Y_1)Y_1^{n-j-2} | 2 \leqslant k \leqslant N+1\}. \tag{17}$$

Proof. Due to Proposition 3.2, we get

$$Y_k Y_1 = (Y_{k+1} - Y_k) - (Y_k - Y_{k-1}). \tag{18}$$

Recall that $Y_0 = -Y_1$. So for each natural number $N \in \mathbb{N}$,

$$O_N^2 \begin{pmatrix} Y_1 Y_1 \\ Y_2 Y_1 \\ \vdots \\ Y_N Y_1 \end{pmatrix} = \begin{pmatrix} Y_2 - 3Y_1 \\ Y_3 - 5Y_1 \\ \vdots \\ Y_{N+1} - (2N+1)Y_1 \end{pmatrix}, \tag{19}$$

where

$$O_N = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{N \times N} \in GL_N(\mathbb{Z}). \tag{20}$$

Therefore,

$$\mathbb{Z}\mathcal{S}_{2,0}(N) = \mathbb{Z}\{Y_k - (2k-1)Y_1 \,|2 \leqslant k \leqslant N+1\}. \tag{21}$$

From this, the proposition follows. □

4. Structure of $Q_n(C_m)$

This section is divided into three subsections according to when m is odd or m is even and $m \equiv 0 (\mathrm{mod}\ 4)$ or $m \equiv 2 (\mathrm{mod}\ 4)$.

4.1 *m is an odd number*

We first give a basis of $\Delta^n(C_m)$ as a free abelian group. Recall that we have assumed $m \geqslant 3$, so $(m-1)/2$ is a positive integer.

Theorem 4.1. *$\Delta^n(C_m)$ is the free abelian group based on $\mathcal{S}_{n,0}((m-1)/2)$.*

Proof. Note that $\mathcal{S}_{n,0}((m-1)/2)$ has cardinality $(m-1)/2$. So by Lemma 2.4, we only need to show that it generates $\Delta^n(C_m)$. Recall that we have already proved that $\Delta^n(C_m)$ is generated by $\mathcal{S}_{n,0}(m)$. Note that $Y_k = -Y_{m+1-k}$, in particular, $Y_{(m+1)/2} = 0$. Then

$$\mathcal{S}_{n,0}(m) \subset \mathbb{Z}\mathcal{S}_{n,0}\Big(\frac{m-1}{2}\Big), \tag{22}$$

as required. □

Now we come to the main result of this subsection.

Theorem 4.2. *When m is an odd number,*

$$Q_n(C_m) \cong C_m \tag{23}$$

for each positive integer n.

Proof. By Theorem 4.1 and its proof, we get, for any natural number n,

$$\begin{aligned} \Delta^{n+1}(C_m) &= \mathbb{Z}\mathcal{S}_{n+1,0}\Big(\frac{m-1}{2}\Big) \\ &= \mathbb{Z}\left\{(Y_k - (2k-1)Y_1)Y_1^{n-1} | 2 \leqslant k \leqslant \frac{m+1}{2}\right\} \\ &= \mathbb{Z}\left\{(Y_k - (2k-1)Y_1)Y_1^{n-1} | 2 \leqslant k \leqslant \frac{m-1}{2}\right\} + m\mathbb{Z}Y_1^n. \end{aligned} \tag{24}$$

Meanwhile, it is easy to see that $\Delta^n(C_m)$ has the basis

$$\left\{(Y_k - (2k-1)Y_1)Y_1^{n-1} | 2 \leqslant k \leqslant \frac{m-1}{2}\right\} \cup \{Y_1^n\}. \tag{25}$$

Therefore,

$$Q_n(C_m) \cong \frac{\mathbb{Z}Y_1^n}{m\mathbb{Z}Y_1^n} \cong C_m. \tag{26}$$

□

4.2 *m is a multiple of* 4

We study this case using the same method as in the above subsection.

Lemma 4.3. *For any $n \geqslant 2$,*

$$F^n \in \mathbb{Z}\mathcal{S}_{n,n-1}(m). \tag{27}$$

Proof. Brief calculations show that $\bar{\eta}_{m/2} = 2\bar{\rho}_2$. Hence,

$$\sum_{k=1}^{m/4} FY_k = \sum_{k=1}^{m/2} Y_k = \bar{\eta}_{m/2} - \bar{\eta}_0 = -2F = -F^2. \tag{28}$$

Then the lemma follows. □

Lemma 4.4. *For any $n \geqslant 3$ and $2 \leqslant j \leqslant n-1$,*

$$\mathcal{S}_{n,j}(m) \subset \mathbb{Z}\mathcal{S}_{n,j-1}(m). \tag{29}$$

Proof. It is easy to see that we only need to show $\mathcal{S}_{3,2}(m) \subset \mathbb{Z}\mathcal{S}_{3,1}(m)$. By Proposition 3.4, we get

$$F(Y_k - (2k-1)Y_1) \in \mathbb{Z}\mathcal{S}_{3,1}(m), \quad 2 \leqslant k \leqslant m/2 + 1. \tag{30}$$

Note that $FY_{m/4+1} = FY_{m/4}$ in this case. So

$$F^2Y_1 = F\Big(Y_{m/4} - \Big(\frac{m}{2} - 1\Big)Y_1\Big) - F\Big(Y_{m/4} - \Big(\frac{m}{2} + 1\Big)Y_1\Big) \tag{31}$$

lies in $\mathbb{Z}\mathcal{S}_{3,1}(m/2)$. Hence F^2Y_k does since it equals

$$2F(Y_k - (2k-1)Y_1) + (2k-1)F^2Y_1. \tag{32}$$

Recalling the definition of $\mathcal{S}_{3,2}(m/2)$, we are done. □

Theorem 4.5. *For any $n \geqslant 2$, $\Delta^n(C_m)$ is the free abelian group based on*

$$\mathcal{S}_{n,0}\Big(\frac{m}{4}\Big) \cup \mathcal{S}_{n,1}\Big(\frac{m}{4}\Big). \tag{33}$$

Proof. We just need to show that (33) generates $\Delta^n(C_m)$ since it has cardinality $m/2$. Recall that, for any $n \geqslant 2$, $\Delta^n(C_m)$ is generated by

$$\{F^n\} \cup \mathcal{S}_{n,0}(m) \cup \mathcal{S}_{n,1}(m) \cup \cdots \cup \mathcal{S}_{n,n-1}(m). \tag{34}$$

Hence, due to Lemma 4.3 and Lemma 4.4, $\Delta^n(C_m)$ is generated by

$$\mathcal{S}_{n,0}(m) \cup \mathcal{S}_{n,1}(m) \tag{35}$$

Note that $Y_k = -Y_{m+1-k}$. So (35) can be replaced by

$$\mathcal{S}_{n,0}\left(\frac{m}{2}\right) \cup \mathcal{S}_{n,1}\left(\frac{m}{2}\right). \tag{36}$$

To finish the proof, recall that $Y_k = -Y_{m/2+1-k} + FY_k$. This implies

$$\mathcal{S}_{n,0}\left(\frac{m}{2}\right) \subset \mathbb{Z}\mathcal{S}_{n,0}\left(\frac{m}{4}\right) + \mathbb{Z}\mathcal{S}_{n+1,1}\left(\frac{m}{2}\right). \tag{37}$$

In addition, it is easy to verify that

$$\mathcal{S}_{n+1,1}\left(\frac{m}{2}\right) \subset \mathbb{Z}\mathcal{S}_{n,1}\left(\frac{m}{2}\right) = \mathbb{Z}\mathcal{S}_{n,1}\left(\frac{m}{4}\right). \tag{38}$$

It follows that $\Delta^n(C_m)$ is generated by (33). □

Theorem 4.6. *When m is a multiple of* 4,

$$Q_n(C_m) \cong C_2 \oplus C_m \tag{39}$$

for each natural number n.

Proof. Due to Proposition 3.4, we get

$$\mathbb{Z}\mathcal{S}_{n+1,0}\left(\frac{m}{4}\right) = \mathbb{Z}\left\{(Y_k - (2k-1)Y_1)Y_1^{n-1} | 2 \leqslant k \leqslant \frac{m}{4} + 1\right\}. \tag{40}$$

Note that

$$\left(Y_{m/4} - \left(\frac{m}{2} - 1\right)Y_1\right) + \left(Y_{m/4+1} - \left(\frac{m}{2} - 1\right)Y_1\right) = FY_{m/4} - mY_1. \tag{41}$$

So by Theorem 4.5,

$$\begin{aligned}\Delta^{n+1}(C_m) = \mathbb{Z}\left\{(Y_k - (2k-1)Y_1)Y_1^{n-1} | 2 \leqslant k \leqslant \frac{m}{4}\right\} + m\mathbb{Z}Y_1^n \\ + \mathbb{Z}\mathcal{S}_{n+1,1}\left(\frac{m}{4}\right).\end{aligned} \tag{42}$$

We compute $Q_1(C_m)$ first. Thanks to (28), we have

$$\begin{aligned}\Delta^2(C_m) &= \mathbb{Z}\left\{Y_k-(2k-1)Y_1 | 2\leqslant k\leqslant \frac{m}{4}\right\}+m\mathbb{Z}Y_1+\mathbb{Z}\mathcal{S}_{2,1}\left(\frac{m}{4}\right)\\ &= \mathbb{Z}\left\{(Y_k-(2k-1)Y_1) | 2\leqslant k\leqslant \frac{m}{4}\right\}+m\mathbb{Z}Y_1\\ &\quad +\mathbb{Z}\left\{FY_k | 2\leqslant k\leqslant \frac{m}{4}\right\}+2\mathbb{Z}F. \end{aligned}\tag{43}$$

Meanwhile, it is easy to verify that $\Delta(C_m)$ has the basis

$$\left\{Y_k-(2k-1)Y_1 | 2\leqslant k\leqslant \frac{m}{4}\right\}\cup\{Y_1\}\cup\left\{FY_k | 2\leqslant k\leqslant \frac{m}{4}\right\}\cup\{F\}.\tag{44}$$

Thus

$$Q_1(C_m)\cong \frac{\mathbb{Z}Y_1}{m\mathbb{Z}Y_1}\oplus\frac{\mathbb{Z}F}{2\mathbb{Z}F}\cong C_2\oplus C_m.\tag{45}$$

Secondly, by Proposition 3.4 and (31), we get, for any $n\geqslant 2$,

$$\begin{aligned}\mathbb{Z}\mathcal{S}_{n+1,1}\left(\frac{m}{4}\right) &= \mathbb{Z}\left\{F(Y_k-(2k-1)Y_1)Y_1^{n-2} | 2\leqslant k\leqslant \frac{m}{4}+1\right\}\\ &= \mathbb{Z}\left\{F(Y_k-(2k-1)Y_1)Y_1^{n-2} | 2\leqslant k\leqslant \frac{m}{4}\right\}+2\mathbb{Z}FY_1^{n-1}. \end{aligned}\tag{46}$$

Hence,

$$\begin{aligned}\Delta^{n+1}(C_m) &= \mathbb{Z}\left\{(Y_k-(2k-1)Y_1)Y_1^{n-1} | 2\leqslant k\leqslant \frac{m}{4}\right\}+m\mathbb{Z}Y_1^n\\ &\quad +\mathbb{Z}\left\{F(Y_k-(2k-1)Y_1)Y_1^{n-2} | 2\leqslant k\leqslant \frac{m}{4}\right\}+2\mathbb{Z}FY_1^{n-1}. \end{aligned}\tag{47}$$

Moreover, it is easy to see that $\Delta^n(C_m)$ has the basis

$$\begin{aligned}&\left\{(Y_k-(2k-1)Y_1)Y_1^{n-1} | 2\leqslant k\leqslant \frac{m}{4}\right\}\cup\{Y_1^n\}\cup\\ &\left\{F(Y_k-(2k-1)Y_1)Y_1^{n-2} | 2\leqslant k\leqslant \frac{m}{4}\right\}\cup\{FY_1^{n-1}\}. \end{aligned}\tag{48}$$

Therefore,

$$Q_n(C_m)\cong \frac{\mathbb{Z}Y_1^n}{m\mathbb{Z}Y_1^n}\oplus\frac{\mathbb{Z}FY_1^{n-1}}{2\mathbb{Z}FY_1^{n-1}}\cong C_2\oplus C_m.\tag{49}$$

Together (45) and (49) finish the proof. □

4.3 $m\equiv 2 \pmod 4$

This case is much more complex than the others.

Lemma 4.7. *When* $m \equiv 2 \pmod 4$, *we have*

$$Y_{(m+2)/4} \in \Delta^2(C_m), \quad \mathcal{S}_{2,1}(m) \subset \Delta^3(C_m). \tag{50}$$

Proof. Recall that $\bar{\eta}_{m/2} = 2\bar{\rho}_2$. So

$$Y_{(m+2)/4} + \sum_{k=1}^{(m-2)/4} FY_k = \sum_{k=1}^{m/2} Y_k = -2F = -F^2. \tag{51}$$

Then

$$Y_{(m+2)/4} = -F^2 - \sum_{k=1}^{(m-2)/4} FY_k \in \Delta^2(C_m) \tag{52}$$

Hence, by the fact $FY_{(m+2)/4} = 2Y_{(m+2)/4}$, we get

$$\begin{aligned} FY_k &= Y_k + Y_{m/2+1-k} - 2Y_{(m+2)/4} + FY_{(m+2)/4} \\ &= (\bar{\eta}_{(m+2)/4-k} - \bar{\eta}_0)Y_{(m+2)/4} + FY_{(m+2)/4} \\ &= -(\bar{\eta}_{(m+2)/4-k} - \bar{\eta}_0 + F)\left(F^2 + \sum_{k=1}^{(m-2)/4} FY_k\right) \\ &\in -F^3 + \mathbb{Z}\mathcal{S}_{3,1}(m/2) + \mathbb{Z}\mathcal{S}_{3,2}(m/2), \end{aligned} \tag{53}$$

as required. □

Lemma 4.8. *For any* $n \geqslant 2$, $\Delta^n(C_m)$ *is generated by*

$$\{F^{n-2}Y_{(m+2)/4}\} \cup \mathcal{S}_{n,0}\left(\frac{m-2}{4}\right) \cup \mathcal{S}_{n,1}\left(\frac{m+2}{4}\right) \tag{54}$$

as an abelian group.

Proof. Note that (34) still generates $\Delta^n(C_m)$. Due to (52), the generator F^n can be replaced by $F^{n-2}Y_{(m+2)/4}$. Moreover, thanks to (53), we get

$$\begin{aligned} \bigcup_{j=1}^{n-1} \mathcal{S}_{n,j}(m) &\subset \mathbb{Z}F^{n+1} + \mathbb{Z}\left(\bigcup_{j=1}^{n} \mathcal{S}_{n+1,j}(m)\right) \\ &= \mathbb{Z}F^{n-1}Y_{(m+2)/4} + \mathbb{Z}\left(\bigcup_{j=1}^{n} \mathcal{S}_{n+1,j}(m)\right). \end{aligned} \tag{55}$$

Hence

$$\bigcup_{j=1}^{n-1} \mathcal{S}_{n,j}(m) \subset \mathbb{Z}F^{n-1}Y_{(m+2)/4} + \mathbb{Z}\left(\bigcup_{j=1}^{2n-3} \mathcal{S}_{2n-2,j}(m)\right). \tag{56}$$

It is easy to see that

$$\mathcal{S}_{2n-2,j}(m) \subset \mathbb{Z}\mathcal{S}_{n,1}(m), \quad 1 \leqslant j \leqslant n-1. \tag{57}$$

For $n \leqslant j \leqslant 2n-3$, we have

$$\mathcal{S}_{2n-2,j}(m) = 2^{n-2}\mathcal{S}_{n,j-n+2}(m) \subset \mathbb{Z}\mathcal{S}_{n,1}(m), \tag{58}$$

since $2\mathcal{S}_{n,*+1}(m) \subset \mathcal{S}_{n,*}(m)$ in this case (like Lemma 4.4). So

$$\bigcup_{j=1}^{n-1} \mathcal{S}_{n,j}(m) \subset 2^{n-1}\mathbb{Z}Y_{(m+2)/4} + \mathbb{Z}\mathcal{S}_{n,1}(m). \tag{59}$$

Therefore, $\Delta^n(C_m)$ is generated by

$$\{F^{n-2}Y_{(m+2)/4}\} \cup \mathcal{S}_{n,0}(m) \cup \mathcal{S}_{n,1}(m) \tag{60}$$

Note that

$$\mathcal{S}_{n,0}(m) \subset \mathbb{Z}\mathcal{S}_{n,0}\Big(\frac{m}{2}\Big) \subset \mathbb{Z}\mathcal{S}_{n,0}\Big(\frac{m+2}{4}\Big) + \mathbb{Z}\mathcal{S}_{n,1}(m). \tag{61}$$

In addition, brief calculations show that

$$Y_{(m+2)/4}Y_1 = FY_{(m-2)/4} - FY_{(m+2)/4} \in \mathbb{Z}\mathcal{S}_{2,1}(m). \tag{62}$$

Then the lemma follows from the fact that

$$\mathcal{S}_{n,1}(m) \subset \mathbb{Z}\mathcal{S}_{n,1}\Big(\frac{m}{2}\Big) = \mathbb{Z}\mathcal{S}_{n,1}\Big(\frac{m+2}{4}\Big). \tag{63}$$

□

COROLLARY 4.9

For any $n \geqslant 2$, $\Delta^{n+1}(C_m)$ is generated by

$$\{F^{n-1}Y_{(m+2)/4}\} \cup \mathcal{S}_{n+1,0}\Big(\frac{m-2}{4}\Big) \cup \mathcal{S}_{n,1}\Big(\frac{m+2}{4}\Big). \tag{64}$$

as an abelian group.

Proof. It is easy to see that

$$\mathcal{S}_{n+1,1}\Big(\frac{m+2}{4}\Big) \subset \mathbb{Z}\mathcal{S}_{n,1}\Big(\frac{m+2}{4}\Big). \tag{65}$$

So by Lemma 4.8, we just need to show

$$\mathcal{S}_{n,1}\Big(\frac{m+2}{4}\Big) \subset \Delta^{n+1}(C_m), \tag{66}$$

which is a direct corollary of Lemma 4.7. □

Theorem 4.10. *For any $n \geqslant 2$, there exist three integers a_n, b_n, d_n with $2^{n-1}b_n + a_n d_n = 1$ such that $\Delta^n(C_m)$ is the free abelian group based on*

$$\{X_n\} \cup \mathcal{S}_{n,0}\Big(\frac{m-2}{4}\Big) \cup \mathcal{S}_{n,1}\Big(\frac{m-2}{4}\Big), \tag{67}$$

where $X_n = a_n F^{n-2} Y_{(m+2)/4} + b_n F Y_{(m+2)/4} Y_1^{n-2}$.

Proof. Note that (67) is contained in $\Delta^n(C_m)$ and has cardinality $m/2$. So we just need to show it generates $\Delta^n(C_m)$. Moreover, we only need to show it generates $F^{n-2}Y_{(m+2)/4}$ and $FY_{(m+2)/4}Y_1^{n-2}$ by comparing it with (54). The theorem is trivial for $n = 2$ by setting $b_2 = 0$ and $a_2 = d_2 = 1$. For $n \geqslant 3$, by (19), we get

$$4\begin{pmatrix} FY_1 \\ FY_2 \\ \vdots \\ FY_{\frac{m+2}{4}} \end{pmatrix} = MO^2_{\frac{m+2}{4}} \begin{pmatrix} FY_1Y_1 \\ FY_2Y_1 \\ \vdots \\ FY_{\frac{m+2}{4}}Y_1 \end{pmatrix}, \tag{68}$$

where

$$M = \begin{cases} \begin{pmatrix} 0 & -1 \\ 4 & -3 \end{pmatrix}, & \text{if } m = 6, \\ \begin{pmatrix} 0 & 0 \\ 4I_{\frac{m-2}{4}} & 0 \end{pmatrix} + \begin{pmatrix} 0 \cdots 0 & 1 & 0 & -1 \\ 0 \cdots 0 & 3 & 0 & -3 \\ \vdots \quad \vdots & \vdots & \vdots & \vdots \\ 0 \cdots 0 & \frac{m}{2} & 0 & -\frac{m}{2} \end{pmatrix}, & \text{if } m \geqslant 10. \end{cases} \tag{69}$$

From this it follows that

$$2^{n-1}\begin{pmatrix} F^{n-2}Y_1 \\ F^{n-2}Y_2 \\ \vdots \\ F^{n-2}Y_{\frac{m+2}{4}} \end{pmatrix} = \Big(MO^2_{\frac{m+2}{4}}\Big)^{n-2} \begin{pmatrix} FY_1Y_1^{n-2} \\ FY_2Y_1^{n-2} \\ \vdots \\ FY_{\frac{m+2}{4}}Y_1^{n-2} \end{pmatrix}. \tag{70}$$

Hence

$$2^{n-1}F^{n-2}Y_{\frac{m+2}{4}} = (0, \cdots, 0, 1)\left(MO^2_{\frac{m+2}{4}}\right)^{n-2}\begin{pmatrix} FY_1Y_1^{n-2} \\ FY_2Y_1^{n-2} \\ \vdots \\ FY_{\frac{m+2}{4}}Y_1^{n-2} \end{pmatrix}. \tag{71}$$

Short calculations show that

$$MO^2_{\frac{m+2}{4}} \equiv \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} \pmod{M_{\frac{m+2}{4}}(2\mathbb{Z})}. \tag{72}$$

Hence

$$\left(MO^2_{\frac{m+2}{4}}\right)^{n-2} \equiv \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} \pmod{M_{\frac{m+2}{4}}(2\mathbb{Z})}. \tag{73}$$

Denote by d_n the integer in the lower right hand corner of $\left(MO^2_{\frac{m+2}{4}}\right)^{n-2}$. Then there exist two integers a_n, b_n such that $2^{n-1}b_n + a_nd_n = 1$ since d_n is an odd number. Therefore, either

$$F^{n-2}Y_{(m+2)/4} = d_nX_n + b_nZ_n \tag{74}$$

or

$$FY_{(m+2)/4}Y_1^{n-2} = 2^{n-1}X_n - a_nZ_n \tag{75}$$

is generated by (67), where

$$Z_n = 2^{n-1}F^{n-2}Y_{(m+2)/4} - d_nFY_{(m+2)/4}Y_1^{n-2} \in \mathbb{Z}\mathcal{S}_{n,1}\left(\frac{m-2}{4}\right), \tag{76}$$

as required. □

Theorem 4.11. *When* $m \equiv 2 \pmod 4$, *we have*

$$Q_n(C_m) \cong C_2 \oplus C_{m/2} \cong C_m. \tag{77}$$

Proof. We compute $Q_1(C_m)$ first. By Theorem 4.10, we get

$$
\begin{aligned}
\Delta^2(C_m) &= \mathbb{Z}Y_{(m+2)/4} + \mathbb{Z}\mathcal{S}_{2,0}\Big(\frac{m-2}{4}\Big) + \mathbb{Z}\mathcal{S}_{2,1}\Big(\frac{m-2}{4}\Big) \\
&= \mathbb{Z}Y_{(m+2)/4} + \mathbb{Z}\left\{Y_k - (2k-1)Y_1 | 2 \leqslant k \leqslant \frac{m+2}{4}\right\} \\
&\quad + \mathbb{Z}\mathcal{S}_{2,1}\Big(\frac{m-2}{4}\Big) \\
&= \mathbb{Z}Y_{(m+2)/4} + \mathbb{Z}\left\{Y_k - (2k-1)Y_1 | 2 \leqslant k \leqslant \frac{m-2}{4}\right\} \\
&\quad + (m/2)\mathbb{Z}Y_1 + \mathbb{Z}\mathcal{S}_{2,1}\Big(\frac{m-2}{4}\Big).
\end{aligned} \tag{78}
$$

Hence, (51) implies

$$
\begin{aligned}
\Delta^2(C_m) &= 2\mathbb{Z}F + \mathbb{Z}\left\{Y_k - (2k-1)Y_1 | 2 \leqslant k \leqslant \frac{m-2}{4}\right\} \\
&\quad + (m/2)\mathbb{Z}Y_1 + \mathbb{Z}\mathcal{S}_{2,1}\Big(\frac{m-2}{4}\Big).
\end{aligned} \tag{79}
$$

Meanwhile, it is easy to verify that $\Delta(C_m)$ has the basis

$$
\{F\} \cup \left\{Y_k - (2k-1)Y_1 | 2 \leqslant k \leqslant \frac{m-2}{4}\right\} \cup \{Y_1\} \cup \mathcal{S}_{2,1}\Big(\frac{m-2}{4}\Big). \tag{80}
$$

So

$$
\begin{aligned}
Q_1(C_m) &\cong \frac{\mathbb{Z}F}{2\mathbb{Z}F} \oplus \frac{\mathbb{Z}Y_1}{(m/2)\mathbb{Z}Y_1} \\
&\cong C_2 \oplus C_{m/2}.
\end{aligned} \tag{81}
$$

Suppose $n \geqslant 2$. Due to Corollary 4.9, (62), (74), (75) and the fact that d_n is an odd number, we get

$$
\begin{aligned}
\Delta^{n+1}(C_m) &= \mathbb{Z}F^{n-1}Y_{(m+2)/4} + \mathbb{Z}\left\{(Y_k - (2k-1)Y_1)Y_1^{n-1} | 2 \leqslant k \leqslant \frac{m-2}{4}\right\} \\
&\quad + (m/2)\mathbb{Z}Y_1^n + \mathbb{Z}\mathcal{S}_{n,1}\Big(\frac{m-2}{4}\Big) + \mathbb{Z}FY_{(m+2)/4}Y_1^{n-2} \\
&= 2d_n\mathbb{Z}X_n + \mathbb{Z}\left\{(Y_k - (2k-1)Y_1)Y_1^{n-1} | 2 \leqslant k \leqslant \frac{m-2}{4}\right\} \\
&\quad + (m/2)\mathbb{Z}Y_1^n + \mathbb{Z}\mathcal{S}_{n,1}\Big(\frac{m-2}{4}\Big) + 2^{n-1}\mathbb{Z}X_n
\end{aligned}
$$

$$= 2\mathbb{Z}X_n + \mathbb{Z}\left\{(Y_k - (2k-1)Y_1)Y_1^{n-1} | 2 \leqslant k \leqslant \frac{m-2}{4}\right\}$$
$$+ (m/2)\mathbb{Z}Y_1^n + \mathbb{Z}\mathcal{S}_{n,1}\left(\frac{m-2}{4}\right). \tag{82}$$

Like (80), $\Delta^n(C_m)$ has the basis

$$\{X_n\} \cup \left\{(Y_k - (2k-1)Y_1)Y_1^{n-1} | 2 \leqslant k \leqslant \frac{m-2}{4}\right\} \cup \{Y_1^n\} \cup \mathcal{S}_{n,1}\left(\frac{m-2}{4}\right). \tag{83}$$

Hence,

$$Q_n(C_m) \cong \frac{\mathbb{Z}X_n}{2\mathbb{Z}X_n} \oplus \frac{\mathbb{Z}Y_1^n}{(m/2)\mathbb{Z}Y_1^n} \cong C_2 \oplus C_{m/2}. \tag{84}$$

Together (81) and (84) finish the proof. □

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