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Some polynomials associated with the r-Whitney numbers

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Abstract. In the present article, we study three families of polynomials associated with the r-Whitney numbers of the second kind. They are the r-Dowling polynomials, r-Whitney–Fubini polynomials and the r-Eulerian–Fubini polynomials. Then we derive several combinatorial results by using algebraic arguments (Rota's method), combinatorial arguments (set partitions) and asymptotic methods.

Keywords. Combinatorial identities; r-Whitney number; r-Dowling polynomial; r-Whitney–Fubini polynomial; r-Eulerian–Fubini polynomial.

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1. Introduction

The r-Whitney numbers of the first kind $w_{m,r}(n,k)$ and the second kind $W_{m,r}(n,k)$ were defined by Mező [29] as the connecting coefficients between some particular polynomials. Note that the r-Whitney numbers of the second kind are exactly the same numbers defined by Ruciński and Voigt [45] and the (r,β)-Stirling numbers defined by Corcino *et al.* [15].

For non-negative integers n, k and r with $n \geq k \geq 0$ and for any integer $m > 0$,

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k) x^{\underline{k}} \tag{1}$$

and

$$m^n x^{\underline{n}} = \sum_{k=0}^{n} w_{m,r}(n,k)(mx+r)^k, \tag{2}$$

where

$$x^{\underline{n}} = \begin{cases} x(x-1)\cdots(x-n+1), & \text{if } n \geq 1; \\ 1, & \text{if } n = 0. \end{cases}$$

The r-Whitney numbers of the first kind and the second kind satisfy the following recurrences, respectively [29]:

$$w_{m,r}(n,k) = w_{m,r}(n-1,k-1) + (m - nm - r)w_{m,r}(n-1,k), \tag{3}$$
$$W_{m,r}(n,k) = W_{m,r}(n-1,k-1) + (km+r)W_{m,r}(n-1,k). \tag{4}$$

Moreover, these numbers have the following rational generating function [10]:

$$\sum_{k=0}^{n} w_{m,r}(n,n-k)x^k = \prod_{k=0}^{n-1}(1-(r+mk)x), \tag{5}$$

$$\sum_{n\geqslant k} W_{m,r}(n,k)x^n = \frac{x^k}{(1-rx)(1-(r+m)x)\cdots(1-(r+mk)x)}. \tag{6}$$

Note that if $(m,r) = (1,0)$ we obtain the Stirling numbers [21], if $(m,r) = (1,r)$ we have the r-Stirling (or noncentral Stirling) numbers [7], and if $(m,r) = (m,1)$ we have the Whitney numbers [5,6]. See [3,10,16,38] for combinatorial interpretations of the r-Whitney numbers, [26–28] for their connections to elementary symmetric functions, [13,14,17] for asymptotic expansions of $W_{m,r}(n,k)$, [34,35] for their connections to matrix theory, [29,39] for their relations with the Bernoulli and generalized Bernoulli polynomials and [12,18,24,25,42,43] for their q and (p,q)-generalizations.

In this article, we study some families of combinatorial polynomials associated with the r-Whitney numbers of the second kind. They are the r-Dowling polynomials, r-Whitney–Fubini polynomials and the r-Eulerian–Fubini polynomials. They are defined by using generating functions in a similar way as in the classical cases. Then we derive several combinatorial results by using the algebraic method introduced by Rota in [44] and combinatorial arguments by means of (r,m,x)-partitions. Finally, we study the root structure of the r-Dowling polynomials. In particular, we analyse the leftmost zero of these polynomials and its asymptotic behavior. Additionally, we show that the sequence of r-Whitney–Fubini polynomials is a log-convex sequence and we prove that these polynomials have only negative real zeros in the interval $]-1,0[$.

2. The r-Dowling polynomials

Cheon and Jung [10] defined the r-Dowling polynomials of degree n by

$$\mathcal{D}_{m,r}(n,x) := \sum_{k=0}^{n} W_{m,r}(n,k)x^k.$$

They found some combinatorial identities by means of Riordan arrays. In this section, we use a technique from linear algebra introduced by Rota [44] to give an alternative proof of some results of Cheon and Jung.

Theorem 1. *The exponential generating function for the r-Dowling polynomials is*

$$\sum_{n=0}^{\infty} \mathcal{D}_{m,r}(n,x)\frac{z^n}{n!} = \exp\left(rz + x\frac{\mathrm{e}^{mz}-1}{m}\right). \tag{7}$$

Proof. Let V be the vector space of polynomials. It is clear that the following sequence is a basis of V:

$$\left(\frac{x-r}{m}\right)^{\underline{l}}, \quad l \geqslant 0.$$

Let $L_{m,r}$ be a linear transformation on V defined as

$$L_{m,r}\left(\left(\frac{x-r}{m}\right)^{\underline{l}}\right) = \frac{x^l}{m^l}, \quad l \geqslant 0.$$

The r-Whitney numbers of the second kind satisfy

$$x^n = \sum_{l=0}^{n}\left(\frac{x-r}{m}\right)^{\underline{l}} m^l W_{m,r}(n,l). \tag{8}$$

Then

$$\begin{aligned} L_{m,r}(x^n) &= L_{m,r}\left(\sum_{l=0}^{n}\left(\frac{x-r}{m}\right)^{\underline{l}} m^l W_{m,r}(n,l)\right) \\ &= \sum_{l=0}^{n} x^l W_{m,r}(n,l) = \mathcal{D}_{m,r}(n,x). \end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty}\mathcal{D}_{m,r}(n,x)\frac{z^n}{n!} = \sum_{n=0}^{\infty} L_{m,r}(x^n)\frac{z^n}{n!} = L_{m,r}\left(\sum_{n=0}^{\infty}\frac{(xz)^n}{n!}\right) = L_{m,r}(\mathrm{e}^{xz}).$$

Note that $\mathrm{e}^{xz} = \mathrm{e}^{rz}(\mathrm{e}^{mz})^{\frac{x-r}{m}} = \mathrm{e}^{rz}(1+u)^{\frac{x-r}{m}}$, where $u = \mathrm{e}^{mz}-1$. Then by the binomial theorem we get

$$L_{m,r}(\mathrm{e}^{xz}) = \mathrm{e}^{rz}L_{m,r}((1+u)^{\frac{x-r}{m}}) = \mathrm{e}^{rz}L_{m,r}\left(\sum_{j=0}^{\infty}\left(\frac{x-r}{m}\right)^{\underline{j}}\frac{u^j}{j!}\right)$$

$$= \mathrm{e}^{rz} \sum_{j=0}^{\infty} \frac{\left(\frac{xu}{m}\right)^j}{j!} = \mathrm{e}^{rz} \mathrm{e}^{\frac{xu}{m}}.$$

Then equation (7) follows. □

COROLLARY 2

The r-Dowling polynomials satisfy the relations:

- $\mathcal{D}_{m,r+1}(n,x) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_{m,r}(k,x)$.
- $\mathcal{D}_{m,r}(n,x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \mathcal{D}_{m,r+1}(k,x)$.

Proof. Since

$$\mathrm{e}^{z} \sum_{n=0}^{\infty} \mathcal{D}_{m,r}(n,x) \frac{z^n}{n!} = \exp\left((r+1)z + x \frac{\mathrm{e}^{mz-1}}{m}\right),$$

Cauchy's product implies the identities. □

From the above corollary, we obtain the following identities:

$$\mathcal{D}_{m,r+s}(n,x) = \sum_{k=0}^{n} \binom{n}{k} s^{n-k} \mathcal{D}_{m,r}(k,x),$$
$$\mathcal{D}_{m,r}(n,x) = \sum_{k=0}^{n} \binom{n}{k} (-s)^{n-k} \mathcal{D}_{m,r+s}(k,x),$$

for any $r, s \geq 0$.

This was proven in a particular case by Mihoubi and Belbachir [37] and before that in an even more special case by Mező [32]. The combinatorial description that we are going to introduce for $\mathcal{D}_{m,r}(n,x)$ makes the proof of this formula straightforward.

Now, we find an ordinary generating function of $\mathcal{D}_{m,r}(n,x)$. In order to determine the ordinary generating function we need some other notions. The rising factorial (a.k.a. Pochhammer symbol) is defined by

$$(x)_n \equiv x^{\overline{n}} = \begin{cases} x(x+1)(x+2)\cdots(x+n-1), & \text{if } n \geq 1, \\ 1, & \text{if } n = 0. \end{cases} \tag{9}$$

It is obvious that $(1)_n = n!$. Fitting our notations to the theory of hypergeometric functions defined below, we apply the notation $(x)_n$ instead of $x^{\overline{n}}$. The next transformation formula

$$x^{\underline{n}} = (-1)^n (-x)_n. \tag{10}$$

holds.

The hypergeometric function (or hypergeometric series) is defined by

$$ {}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} \,\middle|\, t\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{t^k}{k!}. $$

Note that the denominator of equation (6) can be transformed using the falling factorial:

$$ \begin{aligned} &(1 - rx)(1 - (r+m)x) \cdots (1 - (r+mk)x) \\ &\quad = \frac{(1-x)(1-2x)\cdots(1-(r+mk)x)}{\prod_{i=1}^{r-1}(1-ix)\prod_{i=1}^{m-1}(1-(r+i)x)\cdots\prod_{i=1}^{m-1}(1-(r+(k-1)m+i)x)} \\ &\quad = \frac{x^{r+mk+1}\left(\frac{1}{x}\right)^{\underline{r+mk+1}}}{x^{r+mk-k}\left(\frac{1}{x}\right)^{\underline{r}}\prod_{l=0}^{k-1}\left(\frac{1}{x} - r - lm - 1\right)^{\underline{m-1}}} \\ &\quad = \frac{x^{k+1}\left(\frac{1}{x}\right)^{\underline{r+mk+1}}}{\left(\frac{1}{x}\right)^{\underline{r}}\prod_{l=0}^{k-1}\left(\frac{1}{x} - r - lm - 1\right)^{\underline{m-1}}}. \end{aligned} $$

Hence

$$ \sum_{n=k}^{\infty} W_{m,r}(n,k)x^n = \frac{\frac{1}{x}\left(\frac{1}{x}\right)^{\underline{r}}}{\left(\frac{1}{x}\right)^{\underline{r+mk+1}}} \prod_{l=0}^{k-1}\left(\frac{1}{x} - r - lm - 1\right)^{\underline{m-1}}. $$

From (9) and (10), we get

$$ \begin{aligned} \left(\frac{1}{x}\right)^{\underline{r+mk+1}} &= (-1)^{r+mk+1}\left(-\frac{1}{x}\right)_{r+mk+1} \\ &= (-1)^{r+mk+1}\left(-\frac{1}{x}\right)_{r+1}\left(-\frac{1}{x} + r + 1\right)_{mk}. \end{aligned} $$

Therefore

$$ \sum_{n=k}^{\infty} W_{m,r}(n,k)x^n = \frac{(-1)^{r+mk+1}\frac{1}{x}\left(\frac{1}{x}\right)^{\underline{r}}}{\left(-\frac{1}{x}\right)_{r+1}\left(-\frac{1}{x} + r + 1\right)_{mk}} \prod_{l=0}^{k-1}\left(\frac{1}{x} - r - lm - 1\right)^{\underline{m-1}}. $$

Since

$$ \frac{\left(\frac{1}{x}\right)^{\underline{r}}}{\left(-\frac{1}{x}\right)_{r+1}} = (-1)^r \frac{x}{rx-1}, $$

and from equation (10) we have

$$ \sum_{n=k}^{\infty} W_{m,r}(n,k)x^n = \frac{-1}{rx-1}\frac{(-1)^k}{\left(\frac{xr+x-1}{x}\right)_{mk}} \prod_{l=0}^{k-1}\left(r + lm + 1 - \frac{1}{x}\right)_{m-1}. $$

We multiply both sides by z^k and take summation over the non-negative integers:

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{D}_{m,r}(n,z)x^n &= \frac{-1}{rx-1}\sum_{k=0}^{\infty}\frac{(-z)^k}{\left(\frac{xr+x-1}{x}\right)_{mk}}\prod_{l=0}^{k-1}\left(r+lm+1-\frac{1}{x}\right)_{m-1}\\ &= \frac{-1}{rx-1}\sum_{k=0}^{\infty}\frac{(-z)^k}{(v)_{mk}}\left(m^{(m-1)k}\left(\frac{v}{m}\right)_k\left(\frac{v+1}{m}\right)_k\cdots\left(\frac{v+m-2}{m}\right)_k\right)\\ &= \frac{-1}{rx-1}\sum_{k=0}^{\infty}\frac{(-z)^k}{(v)_{mk}}\left(\frac{(v)_{mk}}{m^k\left((v+m-1)/m\right)_k}\right)\\ &= \frac{-1}{rx-1}\sum_{k=0}^{\infty}\frac{(-z/m)^k}{((v+m-1)/m)_k},\end{aligned}$$

where $v=r+1-\frac{1}{x}$. Therefore we have the following theorem.

Theorem 3. *The ordinary generating function of r-Dowling polynomials is*

$$\sum_{n=0}^{\infty}\mathcal{D}_{m,r}(n,z)x^n = \frac{-1}{rx-1}{}_1F_1\left(\left.\begin{matrix}1\\ \frac{rx+mx-1}{mx}\end{matrix}\right| -z/m\right).$$

Note that if $m=1$, we obtain the ordinary generating function of r-Bell polynomials (see [32,37])

$$\sum_{n=0}^{\infty}\mathcal{D}_{r}(n,z)x^n = \frac{-1}{rx-1}{}_1F_1\left(\left.\begin{matrix}1\\ \frac{rx+x-1}{x}\end{matrix}\right| -z\right) = \frac{-1}{rx-1}\frac{1}{\mathrm{e}^z}\,{}_1F_1\left(\left.\begin{matrix}\frac{rx-1}{x}\\ \frac{rx+x-1}{x}\end{matrix}\right| z\right).$$

The last equality follows by Kummer's formula [1, p. 505].

Theorem 4. *The r-Dowling polynomials $\mathcal{D}_{m,r}(n,x)$ satisfy the recurrence relation*

$$\mathcal{D}_{m,r}(n+1,x) = r\mathcal{D}_{m,r}(n,x) + x\sum_{j=0}^{n}\binom{n}{j}m^{n-j}\mathcal{D}_{m,r}(j,x). \tag{11}$$

Proof. Note that

$$\left(\frac{x-r}{m}\right)^{\underline{l}} = \frac{x-r}{m}\left(\frac{x-r}{m}-1\right)^{\underline{l-1}}.$$

Then for any polynomial $P(x)$, we get

$$L_{m,r}\left((x-r)P(x-m)\right) = xL_{m,r}(P(x)).$$

Indeed, if $P(x)$ is a polynomial of degree n then it can be written as

$$P(x) = b_0\left(\frac{x-r}{m}\right)^{\underline{0}} + b_1\left(\frac{x-r}{m}\right)^{\underline{2}} + \cdots + b_n\left(\frac{x-r}{m}\right)^{\underline{n}}.$$

Therefore

$$\begin{aligned}
L_{m,r}((x-r)P(x-m)) &= L_{m,r}\left((x-r)\left(\sum_{i=0}^{n} b_i \left(\frac{x-m-r}{m}\right)^{\underline{i}}\right)\right)\\
&= L_{m,r}\left(m\left(\sum_{i=0}^{n} b_i \frac{(x-r)}{m}\left(\frac{x-r}{m}-1\right)^{\underline{i}}\right)\right)\\
&= L_{m,r}\left(m\left(\sum_{i=1}^{n+1} b_{i-1}\left(\frac{x-r}{m}\right)^{\underline{i}}\right)\right)\\
&= m\left(\sum_{i=1}^{n+1} b_{i-1}\frac{x^i}{m^i}\right) = x\left(\sum_{i=0}^{n} b_i \frac{x^i}{m^i}\right) = xL_{m,r}(P(x)).
\end{aligned}$$

In particular, if $P(x)=(x+m)^n$ we obtain

$$\begin{aligned}
&L_{m,r}(x^{n+1}-rx^n) = xL_{m,r}\left((x+m)^n\right)\\
&\Longrightarrow L_{m,r}(x^{n+1}) - rL_{m,r}(x^n) = xL_{m,r}\left(\sum_{j=0}^{n}\binom{n}{j}m^{n-j}x^j\right)\\
&\Longrightarrow L_{m,r}(x^{n+1}) = rL_{m,r}(x^n) + x\sum_{j=0}^{n}\binom{n}{j}m^{n-j}L_{m,r}\left(x^j\right)\\
&\Longrightarrow L_{m,r}(x^{n+1}) = rL_{m,r}(x^n) + x\sum_{j=0}^{n}\binom{n}{j}m^{n-j}\mathcal{D}_{m,r}(j,x).
\end{aligned}$$

Then equation (11) follows. □

2.1 *Combinatorial proof of Theorem* 4

It is not hard to find the combinatorial meaning of the values of the $\mathcal{D}_{m,r}(n,x)$ polynomials, at least when x is a positive integer.

Recall that a *partition* of a set A is a class of disjoint subsets of A such that the union of them covers A. The subsets are often called *blocks*. Any fixed partition can be written uniquely: we order the elements in the blocks in increasing order and we put the blocks into increasing order with respect to their first elements. This representation is called the partition's *standard form*.

For instance, the below partition is in standard form

$$\{1,7,9\}, \{2\}, \{3,4,5,8\}, \{6\}, \{10,11\}.$$

We introduce two more notions to describe our proof in a simpler form. Let $r, n \geq 0$ be integers, and let us consider the set

$$A_{n,r} := \{1, 2, \ldots, r, r+1, \ldots, n+r\}.$$

The elements $1, 2, \ldots, r$ will be called *distinguished elements* by us. A block of a partition of the above set is called *distinguished* if it contains a distinguished element. Then the above mentioned interpretation is as follows:

Let $n, r \geq 0$ and $m, x \geq 1$ be positive integers, and let us write all the partitions of the set $A_{n,r}$ into the standard form. Then $\mathcal{D}_{m,r}(n, x)$ is the number of the partitions of $A_{n,r}$ such that

- the elements $1, 2, \ldots, r$ are in distinct blocks (i.e., any distinguished block contains exactly one distinguished element),
- all the elements *but* the last one in non-distinguished blocks are coloured with one of m colours independently (note that the last element is also the maximal in the block, thanks to the standard form),
- the non-distinguished blocks are coloured with one of x colours independently,
- neither the elements in the distinguished blocks nor the distinguished blocks are coloured.

The partitions of $A_{n,r}$ satisfying the above assumptions will be called (r, m, x)-*partitions*.

To take an example, let $r = x = 2$ and $m = 3$. The $m = 3$ different colours of the elements will be fixed as blue, red and green, while the blocks will be coloured with $x = 2$ colors: yellow and cyan. Then a typical $(2, 3, 2)$-partition of $A_{11,2}$ looks like

$$\{\underline{1}, 7, 9\}, \{\underline{2}\}, \{3, 4, 5, 8\}, \{6\}, \{10, 11\}.$$

(The distinguished elements are underlined.) We are now ready to prove Theorem 4.

Second proof of Theorem 4. Let $n, m, x \geq 1$ and $r \geq 0$. Then a typical (r, m, x)-partition of $A_{n+1,r}$ can be constructed recursively as follows:

- If the last element $n + 1$ of the set $A_{n+1,r}$ happens to be in a distinguished block, then we have r possible such cases, because there are r distinguished blocks. The other elements previously goes to an (r, m, x) partition in $\mathcal{D}_{m,r}(n, x)$ ways. Altogether we have $r\mathcal{D}_{m,r}(n, x)$ possibilities.
- If the last element of the set $A_{n+1,r}$ is in a non-distinguished block (this can happen, because $n \geq 1$), then its block, say B, might contain other elements as well. Let us suppose that B contains j elements plus $n+1$. These j elements are non-distinguished, so $0 \leq j \leq n$. We have $\binom{n}{j}$ possibilities to fill up the block B. These j elements are not maximal, and hence are needed to be coloured with one of the m colours. To perform this colouring we have m^j possibilities. The remaining $n - j + r$ elements preliminarily must form an (r, m, x)-partition and this can be done in $\mathcal{D}_{m,r}(n - j, x)$ ways. One more step remains: since B is not distinguished, it must be coloured with one of the x colours. All of these together give $x\binom{n}{j}m^j\mathcal{D}_{m,r}(n - j, x)$ possibilities. Summing over the disjoint possibilities $j = 0, 1, \ldots, n$, we are done.

□

In [4], the authors consider the case when x is a real or complex number.

2.2 *Dobinski's formula*

There exist several formulas to calculate the Bell numbers. One of them is by using the Dobinski's formula [11,19,21,40]

$$B_n = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}.$$

The following theorem generalizes this identity to our case.

Theorem 5. *The r-Dowling polynomials satisfy the identity*

$$\mathcal{D}_{m,r}(n, mx) = \frac{1}{\mathrm{e}^x}\sum_{s=0}^{\infty}\frac{(ms+r)^n}{s!}x^s,$$

for s an integer. In particular,

$$\mathcal{D}_{m,r}(n, m) = \frac{1}{e}\sum_{s=0}^{\infty}\frac{(ms+r)^n}{s!}.$$

Proof. From equation (1) we have for any integer s,

$$(ms+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k)s^{\underline{k}} = \sum_{k=0}^{n} m^k W_{m,r}(n,k)\frac{s!}{(s-k)!}.$$

Then

$$\frac{(ms+r)^n}{s!} = \sum_{k=0}^{n}\frac{m^k W_{m,r}(n,k)}{(s-k)!}.$$

In the next step, we multiply both sides by x^s and sum from $m = 0$ to ∞. Then

$$\begin{aligned}\sum_{s=0}^{\infty}(ms+r)^n\frac{x^s}{s!} &= \sum_{s=0}^{\infty}\sum_{k=0}^{n} k!\frac{m^k}{s!}W_{m,r}(n,k)\binom{s}{k}x^s \\ &= \sum_{l=0}^{\infty}\sum_{k=0}^{n} k!\frac{m^k}{(l+k)!}W_{m,r}(n,k)\binom{l+k}{k}x^{l+k} \\ &= \mathrm{e}^x\sum_{k=0}^{n} m^k W_{m,r}(n,k)x^k = \mathrm{e}^x\mathcal{D}_{m,r}(n, mx).\end{aligned}$$

□

If $x = 1/m$ in the above theorem, then the r-Dowling numbers $\mathcal{D}_{m,r}(n,1) := \mathcal{D}_{m,r}(n)$ satisfy the identity

$$\mathcal{D}_{m,r}(n) = \frac{1}{\mathrm{e}^{1/m}}\sum_{s=0}^{\infty}\frac{(ms+r)^n}{m^s s!}.$$

2.3 *An integral representation*

In 1885, Cesàro [9] found a remarkable integral representation of the Bell numbers (see also [2,8]):

$$B_n = \frac{2n!}{\pi e}\,\mathrm{Im}\int_0^{\pi}\mathrm{e}^{\mathrm{e}^{\mathrm{e}^{i\theta}}}\sin(n\theta)\mathrm{d}\theta.$$

It is not hard to deduce the 'r-Dowling version'.

Theorem 6. *The r-Dowling numbers have the integral representation*

$$\mathcal{D}_{m,r}(n,1)=\mathcal{D}_{m,r}(n)=\frac{2n!}{\pi \mathrm{e}^{1/m}}\,\mathrm{Im}\int_0^{\pi}\mathrm{e}^{\frac{\mathrm{e}^{m\mathrm{e}^{i\theta}}}{m}}\mathrm{e}^{r\mathrm{e}^{i\theta}}\sin(n\theta)\mathrm{d}\theta.$$

Proof. We need the following identity [8]:

$$\mathrm{Im}\int_0^{\pi}\mathrm{e}^{j\mathrm{e}^{i\theta}}\sin(n\theta)\mathrm{d}\theta=\frac{\pi}{2}\frac{j^n}{n!}.$$

The r-Whitney numbers can be represented in closed form [29]

$$W_{m,r}(n,k)=\frac{1}{m^k k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(mj+r)^n. \tag{12}$$

From the above equation, we get

$$\begin{aligned}W_{m,r}(n,k)&=\frac{2n!}{\pi}\frac{1}{k!m^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\mathrm{Im}\int_0^{\pi}\mathrm{e}^{(mj+r)\mathrm{e}^{i\theta}}\sin(n\theta)\mathrm{d}\theta\\&=\frac{2n!}{\pi}\frac{1}{k!m^k}\mathrm{Im}\int_0^{\pi}(\mathrm{e}^{m\mathrm{e}^{i\theta}}-1)^k\mathrm{e}^{r\mathrm{e}^{i\theta}}\sin(n\theta)\mathrm{d}\theta.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{k=0}^{\infty}W_{m,r}(n,k)&=\frac{2n!}{\pi}\mathrm{Im}\int_0^{\pi}\left(\sum_{k=0}^{\infty}\frac{(\mathrm{e}^{m\mathrm{e}^{i\theta}}-1)^k}{m^k k!}\right)\mathrm{e}^{r\mathrm{e}^{i\theta}}\sin(n\theta)\mathrm{d}\theta\\&=\frac{2n!}{\pi\mathrm{e}^{1/m}}\mathrm{Im}\int_0^{\pi}\mathrm{e}^{\frac{\mathrm{e}^{m\mathrm{e}^{i\theta}}}{m}}\mathrm{e}^{r\mathrm{e}^{i\theta}}\sin(n\theta)\mathrm{d}\theta.\end{aligned}$$

□

2.4 *Spivey's formula*

Spivey [47] proved a formula for the $(n+m)$-th Bell number

$$B_{n+m}=\sum_{k=0}^{n}\sum_{j=0}^{m}j^{n-k}S(m,j)\binom{n}{k}B_k,$$

where $S(m,j)$ is the Stirling number of the second kind with parameters m and j. This identity was generalized by Xu in [50]. From Theorem 4 of [50], we obtain Spivey's formula to r-Dowling polynomials:

$$\mathcal{D}_{m,r}(n+h,x) = \sum_{k=0}^{n}\sum_{j=0}^{h}\binom{n}{k}\mathcal{D}_{m,r}(k,x)W_{m,r}(h,j)x^j j^{n-k}m^{n-k}.$$

Finding the combinatorial proof is an easy task by the generalization of Spivey's argument. Namely, $\mathcal{D}_{m,r}(n+h,x)$ calculates the (r,m,x)-partitions of $A_{n+h,r}$. Such a partition can be formed in the following way: We fix a $k \in \{0,1,\dots,n\}$ and make an (r,m,x)-partition of $A \subset A_{n,r}$ (there are $\binom{n}{k}$ different such subsets) in one of the $\mathcal{D}_{m,r}(k,x)$ ways and another (r,m,x)-partition B of $\{1,\dots,r,n+r+1,\dots,n+r+h\}$ with some, say $j+r$, blocks in $x^jW_{m,r}(h,j)$ ways. We consider the unification of A and B such that we take the *union* of the distinguished blocks, and the other blocks remain disjoint and separated. For example, if $n=10$, $h=5$ and $(r,m,x)=(2,2,2)$ with element colors blue and red and block colors yellow and cyan, then the unification of

$$A = \{\underline{1},5,9\}\cup\{\underline{2},4,11\}\cup\{7,8,12\}$$

and

$$B = \{\underline{1},13,15\}\cup\{\underline{2},14\}\cup\{16,17\}\cup\{18\}$$

is

$$\{\underline{1},5,9,13,15\}\cup\{\underline{2},4,11,14\}\cup\{7,8,12\}\cup\{16,17\}\cup\{18\}.$$

Then, continuing the construction, we take the $n-k$ elements stayed out from $A_{n,r}\setminus\{1,\dots,r\}$ and put down one by one into one of the j *non-distinguished* blocks of the partition of B in j^{n-k} ways. Note that our unification process is bijective but putting these elements into the distinguished blocks would lose the bijectivity.

At the end of this process we have to colour the former block-maximal elements of B in m^{n-k} ways.

Going back to our example, we have that $k=1$, $j=2$ and the elements $3,6,10\in A_{10,r}$ stayed out. We must put these into the last two blocks, so we get

$$\{\underline{1},5,9,13,15\}\cup\{\underline{2},4,11,14\}\cup\{7,8,12\}\cup\{3,10,16,17\}\cup\{6,18\}.$$

We must colour these three elements in $2^3=8$ ways to finalize the construction.

$$\{\underline{1},5,9,13,15\}\cup\{\underline{2},4,11,14\}\cup\{7,8,12\}\cup\{3,10,16,17\}\cup\{6,18\}.$$

Note that knowing the values k,n,h, we can decipher the original sets A and B and the set outlier set $\{3,6,10\}$, too.

2.5 *A congruence for r-Dowling numbers*

Gessel [20] introduced a method to study sequences defined by exponential generating functions. In particular, he proved that there exists a sequence $\{a_i\}_{i=0}^n$ such that

$$B_{m+n}+a_{n-1}B_{m+n-1}+\cdots+a_0B_m \equiv 0 \pmod{n!}.$$

Rahmani [41] used this method to find a congruence analogue to Dowling numbers. Using the same ideas it is not difficult to show the following theorem.

Theorem 7. *Let n, i be non-negative integers, we have*

$$\sum_{k=0}^{n} R_{n,k}^{(m,r)}(t)D_{m,r}(i+k,t) \equiv 0 \pmod{n!},$$

where

$$R_{n,m}^{(m,r)}(t) = \sum_{j=0}^{n}(-1)^j \binom{n}{j} w_{m,r}(n-j,k)t^j,$$

and $w_{m,r}(n,k)$ is a r-Whitney number of the first kind.

3. The root structure of the r-Dowling polynomials

In this section, we study the structure of zeros of the r-Dowling polynomials. It is known [10] that all the zeros of the r-Dowling polynomials are real and negative for any $r, m, n > 1$. Cheon and Jung [10] also proved that the zeros of two consecutive r-Dowling polynomials are interlacing. Here, we are going to study the asymptotic growth of the leftmost roots, i.e., the unique (negative) zero with maximal absolute value.

In particular, we will use the definition

$$\widehat{\mathcal{D}}_{m,r}(n,x) := \sum_{k=0}^{n} W_{m,r}(n,k)m^k x^k$$

in order to facilitate the calculations. Note that $\widehat{\mathcal{D}}_{m,r}(n,x) = \mathcal{D}_{m,r}(n,mx)$.

From this consideration it comes readily that the leftmost (unique) zeros of $\widehat{\mathcal{D}}_{m,r}(n,x)$ steadily grow as n grows. This leftmost zero of $\widehat{\mathcal{D}}_{m,r}(n,x)$ will be denoted by $z^*_{n,m,r}$. In this section, we would like to study how $z^*_{n,m,r}$ grows asymptotically.

Note that identity (12) can be expressed as

$$W_{m,r}(n,k) = m^{n-k}\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\left(j+\frac{r}{m}\right)^n = m^{n-k}\left\{ \begin{matrix} n+\frac{r}{m} \\ k+\frac{r}{m} \end{matrix} \right\}_{\frac{r}{m}}. \tag{13}$$

Here $\left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r$ is an r-Stirling number of the second kind [7] which has the following generating function:

$$\sum_{n=0}^{\infty}\left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r \frac{z^n}{n!} = \frac{\mathrm{e}^{rz}}{k!}(\mathrm{e}^z-1)^k.$$

This generating function of the r-Stirling numbers permits us to substitute rational numbers in place of r. We would think that for non-integer rational r the combinatorial description loses its meaning but – as (13) shows – via the r-Whitney numbers this combinatorial meaning is recovered.

Using the fact that

$$\left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r \sim \frac{(k+r)^n}{k!},$$

we have

$$m^k W_{m,r}(n,k) \sim \frac{(mk+r)^n}{k!} \quad \text{as } n \to \infty.$$

For a first approximation, we use Lagrange's estimation [36], which in our particular case, for zeros of r-Dowling polynomials says that

$$z^*_{n,m,r} \leq R_{m,r} + \rho_{m,r},$$

where $R_{m,r}$ and $\rho_{m,r}$ are respectively the maximal and second maximal elements of the set

$$\left\{ \left(\frac{(m(n-k)+r)^n}{(n-k)!} \right)^{\frac{1}{k}} : k = 1, 2, \ldots, n \right\}.$$

One can easily verify that the partial derivative

$$\frac{\partial}{\partial k} \sqrt[k]{\frac{(m(n-k)+r)^n}{(n-k)!}}$$

is always negative on $[1, n]$ and that $\sqrt[k]{\frac{(m(n-k)+r)^n}{(n-k)!}}$ is maximal at $k = 1$ and the second maximal element occurs at $k = 2$. It has been computed in [33] that

$$\left\{ \begin{matrix} n+r \\ n+r-1 \end{matrix} \right\}_r = \binom{n}{2} + rn,$$

$$\left\{ \begin{matrix} n+r \\ n+r-2 \end{matrix} \right\}_r = g(n,r),$$

where

$$g(n,r) := \frac{1}{2}\binom{n-1}{2}^2 + \left(r^2 + \left(r + \frac{1}{3} \right) n - \frac{1}{2} \right) \binom{n-1}{2} + r^2(n-1).$$

Hence,

$$\begin{aligned} R_{m,r} &= m^{n-1} W_{m,r}(n, n-1) = m^n \left\{ \begin{matrix} n+\frac{r}{m} \\ n+\frac{r}{m}-1 \end{matrix} \right\}_{\frac{r}{m}} \\ &= m^n \left(\binom{n}{2} + \frac{r}{m} n \right) = m^n \binom{n}{2} + m^{n-1} rn \\ \rho_{m,r} &= \sqrt{m^{n-2} W_{m,r}(n, n-2)} = \sqrt{m^n \left\{ \begin{matrix} n+\frac{r}{m} \\ n+\frac{r}{m}-2 \end{matrix} \right\}_{\frac{r}{m}}} = \sqrt{m^n g\left(n, \frac{r}{m}\right)}. \end{aligned}$$

Thus a rough estimate for the zeros of r-Dowling polynomials is given by

$$z^*_{n,m,r} \leq m^n \binom{n}{2} + m^{n-1} rn + \sqrt{m^n g\left(n, \frac{r}{m}\right)}.$$

However, Samuelson's result [46] states that all the zeros in a given polynomial

$$p(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

are contained in the interval $[x_-, x_+]$, where

$$x_\pm = -\frac{a_1}{n} \pm \frac{n-1}{n}\sqrt{a_1^2 - \frac{2n}{n-1}a_2}.$$

Using this result, we obtain the following improved estimation.

Theorem 8. *For the leftmost zero $z^*_{n,m,r}$ of the r-Dowling polynomials $\widehat{\mathcal{D}}_{m,r}(n,x)$, we have that*

$$|z^*_{n,m,r}| < \frac{1}{n}R_{m,r} + \frac{n-1}{n}\sqrt{R^2_{m,r} - \frac{2n}{n-1}g\left(n, \frac{r}{m}\right)}.$$

Note that, after some algebra, from this result it follows that

$$|z^*_{n,m,r}| = O(m^n n^2).$$

4. A definition of the r-Whitney–Fubini polynomials

DEFINITION 9

The r-Whitney–Fubini polynomials of degree n are defined as

$$\mathcal{F}_{m,r}(n,x) := \sum_{k=0}^{n} W_{m,r}(n,k)(k+r)!(mx)^k.$$

Note that the r-Dowling and r-Whitney–Fubini polynomials are connected by the relation

$$\int_0^{+\infty} \mathcal{D}_{m,r}(n, mxt)\mathrm{e}^{-t}t^r \mathrm{d}t = \mathcal{F}_{m,r}(n,x).$$

Note that if $(m,r) = (1,0)$, we recover the Fubini polynomials (see, e.g. [48]). If $(m,r) = (1,r)$, we obtain the r-Fubini polynomials [30,31] and if $(m,r) = (m,1)$ we obtain a slight variation of the Whitney–Fubini polynomials defined by Benoumhani in [5].

Having the notion of (r,m,x)-partitions in mind, it is easy to interpret $\mathcal{F}_{m,r}(n,x)$: for positive integer x, the positive integer number $\mathcal{F}_{m,r}(n,x)$ is the number of (r,m,x)-partitions of $A_{n,r}$ such that

- the blocks are *ordered*, thanks to the factor $(k+r)!$,
- the last elements in the non-distinguished blocks are *coloured*. This is so by the presence of the factor m^k: there are k non-distinguished blocks and their last (maximal) elements will be coloured with one of the m colours.

Such partitions will be called *ordered* (r,m,x)-partitions.

Theorem 10. *The exponential generating function for the r-Whitney–Fubini polynomials is*

$$\sum_{n=0}^{\infty} \mathcal{F}_{m,r}(n,x)\frac{z^n}{n!} = \frac{r!\mathrm{e}^{rz}}{(1-x(\mathrm{e}^{mz}-1))^{r+1}}. \tag{14}$$

Proof. Let $T_{m,r}$ be a linear transformation on V defined as

$$T_{m,r}\left(\left(\frac{x-r}{m}\right)^{\underline{l}}\right) = (l+r)!x^l, \quad l \geqslant 0.$$

Applying $T_{m,r}$ to (8), we obtain

$$\begin{aligned} T_{m,r}(x^n) &= T_{m,r}\left(\sum_{l=0}^{n}\left(\frac{x-r}{m}\right)^{\underline{l}} m^l W_{m,r}(n,l)\right) \\ &= \sum_{l=0}^{n}(l+r)!(mx)^l W_{m,r}(n,l) = \mathcal{F}_{m,r}(n,x). \end{aligned}$$

Then

$$\sum_{n=0}^{\infty}\mathcal{F}_{m,r}(n,x)\frac{z^n}{n!} = T_{m,r}(\mathrm{e}^{xz}) = \mathrm{e}^{rz}T_{m,r}(1+u)^{\frac{x-r}{m}},$$

where $u = \mathrm{e}^{mz}-1$. Then, by the binomial theorem,

$$\begin{aligned} \sum_{n=0}^{\infty}\mathcal{F}_{m,r}(n,x)\frac{z^n}{n!} &= \mathrm{e}^{rz}T_{m,r}\left(\sum_{j=0}^{\infty}\left(\frac{x-r}{m}\right)^{\underline{j}}\frac{u^j}{j!}\right) \\ &= r!\mathrm{e}^{rz}\sum_{j=0}^{\infty}\frac{(j+r)!}{r!j!}(xu)^j = r!\mathrm{e}^{rz}\sum_{j=0}^{\infty}\binom{j+r}{j}(xu)^j \\ &= r!\mathrm{e}^{rz}\frac{1}{(1-xu)^{r+1}}. \end{aligned}$$

Then equation (14) follows. □

Theorem 11. *The following equality holds for any real* $x \neq -1$:

$$\mathcal{F}_{m,r}(n,x) = \frac{r!}{(1+x)^{r+1}}\sum_{k=0}^{\infty}\binom{r+k}{r}\left(\frac{x}{1+x}\right)^k(mk+r)^n. \tag{15}$$

Proof. Using the exponential generating function of $\mathcal{F}_{m,r}(n,x)$, we have

$$\begin{aligned} \sum_{n=0}^{\infty}\mathcal{F}_{m,r}(n,x)\frac{z^n}{n!} &= \frac{r!\mathrm{e}^{rz}}{(1+x-x\mathrm{e}^{mz})^{r+1}} = \frac{r!\mathrm{e}^{rz}}{(1+x)^{r+1}}\cdot\frac{1}{\left(1-\frac{x\mathrm{e}^{mz}}{1+x}\right)^{r+1}} \\ &= \frac{r!}{(1+x)^{r+1}}\sum_{i=0}^{\infty}\frac{(rz)^i}{i!}\sum_{k=0}^{\infty}\binom{r+k}{k}\left(\frac{x}{1+x}\right)^k\mathrm{e}^{kmz} \\ &= \frac{r!}{(1+x)^{r+1}}\sum_{k=0}^{\infty}\binom{r+k}{k}\left(\frac{x}{1+x}\right)^k\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\frac{r^i(km)^l}{i!l!}z^{l+i} \end{aligned}$$

$$
\begin{aligned}
&= \frac{r!}{(1+x)^{r+1}} \sum_{k=0}^{\infty} \binom{r+k}{k} \left(\frac{x}{1+x}\right)^k \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{r^i (km)^{n-i}}{i!(n-i)!} z^n \\
&= \frac{r!}{(1+x)^{r+1}} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \binom{r+k}{k} \left(\frac{x}{1+x}\right)^k \sum_{i=0}^{n} \binom{n}{i} r^i (km)^{n-i} \right) \frac{z^n}{n!} \\
&= \frac{r!}{(1+x)^{r+1}} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \binom{r+k}{k} \left(\frac{x}{1+x}\right)^k (km+r)^n \right) \frac{z^n}{n!}.
\end{aligned}
$$

Then equation (15) follows. □

We can easily have a recursion of $\mathcal{F}_{m,r}(n,x)$ with respect to the parameter r.

Theorem 12. *The following equality holds*:

$$
\mathcal{F}_{m,r+1}(n,x) = (r+1) \sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} \mathcal{F}_{m,0}(l,x) \mathcal{F}_{m,r}(k-l,x).
$$

Proof. Using the exponential generating function of $\mathcal{F}_{m,r}(n,x)$, we get that

$$
\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{F}_{m,r+1}(n,x) \frac{z^n}{n!} &= \frac{(r+1)\mathrm{e}^z}{1 - x(\mathrm{e}^{mz}-1)} \frac{r!\mathrm{e}^{rz}}{(1 - x(\mathrm{e}^{mz}-1))^{r+1}} \\
&= (r+1) \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \left(\sum_{n=0}^{\infty} \mathcal{F}_{m,0}(n,x) \frac{z^n}{n!} \right) \left(\sum_{n=0}^{\infty} \mathcal{F}_{m,r}(n,x) \frac{z^n}{n!} \right) \\
&= (r+1) \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \mathcal{F}_{m,0}(l,x) \mathcal{F}_{m,r}(n-l,x) \right) \frac{z^n}{n!} \\
&= (r+1) \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \left(\sum_{l=0}^{k} \binom{k}{l} \mathcal{F}_{m,0}(l,x) \mathcal{F}_{m,r}(k-l,x) \right) \frac{z^n}{n!}.
\end{aligned}
$$

Comparing the coefficients, the result follows. □

The combinatorial proof is equally easy to present.

Second proof of Theorem 12. To consider all of the $(r+1,m,x)$-partitions calculated by $\mathcal{F}_{m,r+1}(n,x)$, we can do the following: we choose one distinguished element from $r+1$ and we choose $n-k$ elements from n going to its block. The remaining k elements will go to (1) non-distinguished blocks or (2) distinguished blocks. Supposing that from k elements l will go to non-distinguished blocks. Then, we have $\binom{k}{l}\mathcal{F}_{m,0}(l,x)\mathcal{F}_{m,r}(k-l,x)$ cases. Summing over l and k, we are done. □

4.1 *Log-convex property*

Let $(a_n)_{n\geq 0}$ be a sequence of nonnegative real numbers. We say that the sequence is log-concave if $a_n a_{n+2} \leq a_{n+1}^2$ for all $n \geq 0$. It is called log-convex if $a_n a_{n+2} \geq a_{n+1}^2$ for all $n \geq 0$.

COROLLARY 13

The sequence $\mathcal{F}_{m,r}(n, x)$ is log-convex in n for any positive x, because it is a convolution of these [23]. *This means that, in particular, the following inequality holds for the r-Whitney–Fubini numbers $\mathcal{F}_{m,r}(n) = \mathcal{F}_{m,r}(n, 1)$*:

$$\mathcal{F}_{m,r}(n-1)\mathcal{F}_{m,r}(n+1) \geq \mathcal{F}_{m,r}^2(n).$$

Theorem 14. *The following equality holds*:

$$\mathcal{F}_{m,r}(n, x) = (mx(r+1)+r)\mathcal{F}_{m,r}(n-1, x) + mx(x+1)\mathcal{F}'_{m,r}(n-1, x).$$

Proof. From the recursion (4) of r-Whitney numbers of the second kind, we get

$$\begin{aligned}
\mathcal{F}_{m,r}(n, x) &= \sum_{k=0}^{n} W_{m,r}(n, k)(k+r)!(mx)^k \\
&= \sum_{k=0}^{n} W_{m,r}(n-1, k-1)(k+r)!(mx)^k \\
&\quad + \sum_{k=0}^{n}(km+r)W_{m,r}(n-1, k)(k+r)!(mx)^k
\end{aligned}$$

For the first sum,

$$\begin{aligned}
\sum_{k=0}^{n} W_{m,r}(n-1, k-1)(k+r)!(mx)^k &= \sum_{k=0}^{n-1} W_{m,r}(n-1, k)(k+r+1)!(mx)^{k+1} \\
&= \frac{1}{x^{r-1}}\left(\sum_{k=0}^{n-1} W_{m,r}(n-1, k)(k+r)!m^{k+1}x^{k+r+1}\right)' \\
&= \frac{1}{x^{r-1}}\left(mx^{r+1}\sum_{k=0}^{n-1} W_{m,r}(n-1, k)(k+r)!m^k x^k\right)' \\
&= \frac{1}{x^{r-1}}(mx^{r+1}\mathcal{F}_{m,r}(n-1, x))' \\
&= (r+1)mx\mathcal{F}_{m,r}(n-1, x) + mx^2\mathcal{F}'_{m,r}(n-1, x),
\end{aligned}$$

while for the second

$$\sum_{k=0}^{n}(km+r)W_{m,r}(n-1, k)(k+r)!(mx)^k$$

$$
\begin{aligned}
&= m\sum_{k=0}^{n-1} k(k+r)!W_{m,r}(n-1,k)(mx)^k + r\mathcal{F}_{m,r}(n-1,x)\\
&= m\sum_{k=0}^{n-1}(k+r)(k+r)!W_{m,r}(n-1,k)(mx)^k\\
&\quad -mr\sum_{k=0}^{n-1}(k+r)!W_{m,r}(n-1,k)(mx)^k + r\mathcal{F}_{m,r}(n-1,x)\\
&= \frac{m}{x^{r-1}}\left(x^r\sum_{k=0}^{n-1}(k+r)!W_{m,r}(n-1,k)(mx)^k\right)' + r(1-m)\mathcal{F}_{m,r}(n-1,x)\\
&= \frac{m}{x^{r-1}}\left(x^r\mathcal{F}_{m,r}(n-1,x)\right)' + r(1-m)\mathcal{F}_{m,r}(n-1,x)\\
&= mr\mathcal{F}_{m,r}(n-1,x) + mx\mathcal{F}'_{m,r}(n-1,x) + r(1-m)\mathcal{F}_{m,r}(n-1,x)\\
&= mx\mathcal{F}'_{m,r}(n-1,x) + r\mathcal{F}_{m,r}(n-1,x).
\end{aligned}
$$

□

4.2 *Combinatorial proof of Theorem* 14

Now we are going to present a combinatorial justification of the recursion in Theorem 14.

Proof. By its definition

$$\mathcal{F}_{m,r}(n,x) := \sum_{k=0}^{n} W_{m,r}(n,k)(k+r)!m^k x^k$$

is the total number of ordered (r,m,x)-partitions of the formerly defined set $A_{n,r}$. This means that the order of the blocks count in the individual partitions and any elements in the non-distinguished blocks are coloured. To construct all such partitions, we have the below possible cases:

- The last element n alone is in the first position. This offers $mx\mathcal{F}_{m,r}(n-1,x)$ possibilities.
- The element n is not the first but it has no other elements in its block. Such a partition can be constructed in such a way that we construct an arbitrary ordered (r,m,x)-partition on $n-1$ elements with $k+r$ blocks and we put down n *after* a block and we colour n and its block as well. In total, we have

  $$W_{m,r}(n-1,k)(k+r)!m^k x^k \cdot mx(k+r)$$

  cases for any fixed $k = 0, 1, \ldots, n-1$. Summing over k, we have that the number of possibilities can be expressed by the derivative of the r-Whitney–Fubini number

  $$
  \begin{aligned}
  &\sum_{k=0}^{n-1} W_{m,r}(n-1,k)(k+r)!m^k x^k \cdot mx(k+r)\\
  &\quad = mxr\sum_{k=0}^{n-1} W_{m,r}(n-1,k)(k+r)!m^k x^k
  \end{aligned}
  $$

$$
\begin{aligned}
&+mx\sum_{k=0}^{n-1} W_{m,r}(n-1,k)(k+r)!km^k x^k\\
&= mxr\mathcal{F}_{m,r}(n-1,x) + mx^2\mathcal{F}'_{m,r}(n-1,x).
\end{aligned}
$$

- The element n is put into a non-empty block. First, we construct an ordered (r, m, x)-partition on $n-1$ elements with given $k+r$ blocks, and then put down n into one of the $k+r$ blocks. Up to this step, we have

$$
\begin{aligned}
&W_{m,r}(n-1,k)(k+r)!m^k x^k \cdot (k+r)\\
&\quad = W_{m,r}(n-1,k)(k+r)!m^k x^k \cdot r + W_{m,r}(n-1,k)(k+r)!m^k x^k k
\end{aligned}
$$

possibilities. We still have to consider the colouring. For that we have two different cases: n goes to a distinguished block (these cases are counted by the first sum above), then we do not colour n. Or, n goes to a non-distinguished block and it must be coloured. Therefore, for the second sum above we multiply it by m. Summing over k, we get that the number of cases is

$$
\begin{aligned}
&W_{m,r}(n-1,k)(k+r)!m^k x^k \cdot r + mW_{m,r}(n-1,k)(k+r)!x^k k\\
&\quad r\mathcal{F}_{m,r}(n-1,x) + mx\mathcal{F}'_{m,r}(n-1,x).
\end{aligned}
$$

Summing all of the particular cases above and rearranging the sum we have that

$$
\begin{aligned}
\mathcal{F}_{m,r}(n,x) &= mx\mathcal{F}_{m,r}(n-1,x) + mxr\mathcal{F}_{m,r}(n-1,x) + r\mathcal{F}_{m,r}(n-1,x)\\
&\quad + mx^2\mathcal{F}'_{m,r}(n-1,x) + mx\mathcal{F}'_{m,r}(n-1,x)\\
&= (mx(1+r)+r)\mathcal{F}_{m,r}(n-1,x) + mx(x+1)\mathcal{F}'_{m,r}(n-1,x).
\end{aligned}
$$

□

5. The real zero property of the r-Whitney–Fubini polynomials

A sequence $\{a_0, a_1, \ldots, a_n\}$ of the coefficients of a polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$ of degree n with only real zeros is called the Pólya Frequency sequence (PF). We are going to prove that the sequence $m^k(k+r)!W_{m,r}(n,k)$ is a PF-sequence. To reach this aim, we first prove an equally interesting fact.

Theorem 15. *The following equality holds*:

$$
mx^{1-\frac{r}{m}}(1+x)^{(\frac{1}{m}-1)r}[(x^{\frac{r}{m}}(1+x)^{1+r(1-\frac{1}{m})})F_{m,r}(n-1,x)]' = F_{m,r}(n,x).
$$

Proof. Let $H(x) = x^{\frac{r}{m}}(1+x)^{1+r(1-\frac{1}{m})}$ and $G(x) = mx^{1-\frac{r}{m}}(1+x)^{(\frac{1}{m}-1)r}$. It is not difficult to show that

$$
H'(x) = \frac{mx(r+1)+r}{mx(x+1)}H(x) \qquad \text{and} \qquad G(x) = \frac{mx(x+1)}{H(x)}.
$$

Therefore from Theorem 14, we have

$$
\begin{aligned}
\mathcal{F}_{m,r}(n,x) &= (mx(r+1)+r)\mathcal{F}_{m,r}(n-1,x) + mx(x+1)\mathcal{F}'_{m,r}(n-1,x)\\
&= \frac{mx(x+1)}{H(x)}\mathcal{F}_{m,r}(n-1,x)\left(\frac{mx(r+1)+r}{mx(x+1)}H(x)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{mx(x+1)}{H(x)}\mathcal{F}'_{m,r}(n-1,x)H(x)\\
&= G(x)\mathcal{F}_{m,r}(n-1,x)H'(x)+G(x)\mathcal{F}'_{m,r}(n-1,x)H(x)\\
&= G(x)(\mathcal{F}_{m,r}(n-1,x)H(x))'.
\end{aligned}
$$

□

COROLLARY 16

For all $r, m, n \geqslant 1$, the r-Whitney–Fubini polynomials $\mathcal{F}_{m,r}(n,x)$ of degree n have only negative real zeros in the interval $]-1,0[$. That is, the sequence $(W_{m,r}(n,k)(k+r)!m^k)_{k=0}^n$ is a PF sequence.

Proof. For

$$\mathcal{F}_{m,r}(1,x) = rr! + xr!m(1+r),$$

the statement is true for any integers $r, m > 0$. We proceed by induction, and suppose that the statement is true till $n-1$.

From the left-hand side of our theorem, we see that the term $mx^{1-\frac{r}{m}}(1+x)^{\left(\frac{1}{m}-1\right)r}$ can have zeros only at $x=0$ or $x=-1$. We will see that these are not zeros since they are cancelled out by some factors of

$$[(x^{\frac{r}{m}}(1+x)^{1+r(1-\frac{1}{m})})\mathcal{F}_{m,r}(n-1,x)]'.$$

Under the derivative the function has zeros at $x=0$, $x=-1$ and, by the induction hypothesis, it also has $n-1$ zeros in the interval $]-1,0[$. So, by Rolle's theorem, the derivative must have $n-2$ zeros in between the zeros of $\mathcal{F}_{m,r}(n-1,x)$, and a zero between -1 and the leftmost zero of $\mathcal{F}_{m,r}(n-1,x)$, and between 0 and the rightmost zero of $\mathcal{F}_{m,r}(n-1,x)$). These altogether are n zeros, all in the interval $]-1,0[$. Since on the right-hand side we have an n degree polynomial, we get that these are all the zeros. And, in addition, as we mentioned, the other zeros at $x=0$ and $x=-1$ cancels out. □

The particular case of r-Fubini polynomials $\mathcal{F}_{1,r}(n,x)$ was proven by Mező [30,31]. We remark that the proof of the corollary actually provide more information than what is stated. It also shows that the polynomials $\mathcal{F}_{m,r}(n,x)$ and $\mathcal{F}_{m,r}(n-1,x)$ are interlacing in the following sense.

Let $(r_i)_{i\in\mathbb{N}}$ and $((s_j)_{j\in\mathbb{N}}$ be the sequences of the real zeros of polynomials f of degree n and g of degree $n-1$ in nonincreasing order respectively. We say that g interlaces f [22], denoted by $g \preccurlyeq f$, if

$$r_n \leqslant s_{n-1} \leqslant \cdots \leqslant s_2 \leqslant r_2 \leqslant s_1 \leqslant r_1.$$

So, by using the argument of the proof, we can state that

$$\mathcal{F}_{m,r}(n-1,x) \preccurlyeq \mathcal{F}_{m,r}(n,x).$$

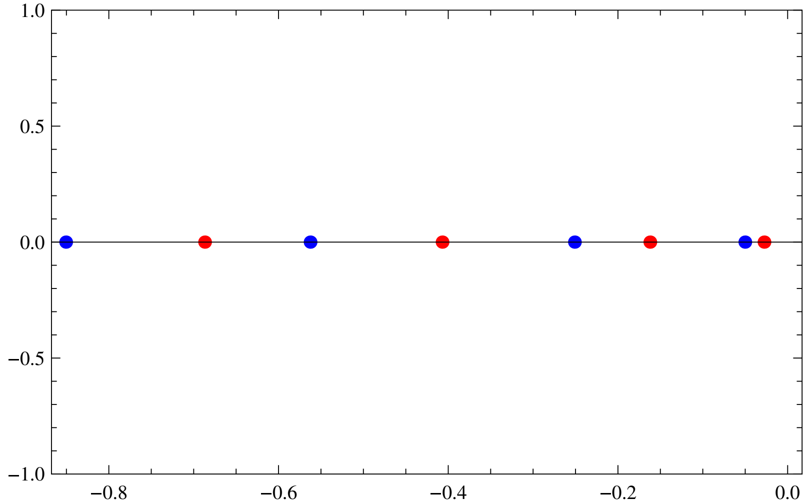

Figure 1. Zeros of $\mathcal{F}_{2,3}(4, x)$ (red) and $\mathcal{F}_{2,3}(5, x)$ (blue).

Example 17. In figure 1, we show the zeros of the polynomials $\mathcal{F}_{2,3}(4, x)$ and $\mathcal{F}_{2,3}(5, x)$ as

$$\mathcal{F}_{2,3}(4, x) = 80640x^4 + 138240x^3 + 73920x^2 + 13056x + 486,$$
$$\mathcal{F}_{2,3}(5, x) = 1290240x^5 + 2822400x^4 + 2131200x^3 + 648000x^2 + 69168x + 1458.$$

6. A definition of the r-Eulerian–Fubini polynomials

DEFINITION 18

The r-Eulerian–Fubini polynomials $\mathcal{A}_{m,r}(n, x)$ are defined as

$$\mathcal{A}_{m,r}(n, x) = \sum_{k=0}^{n} W_{m,r}(n, k)(k + r)!m^k (x - 1)^{n-k} \tag{16}$$
$$= (x - 1)^n \mathcal{F}_{m,r}\left(n, \frac{1}{x - 1}\right). \tag{17}$$

Note that

$$\mathcal{A}_{m,r}(n, x) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{n-k}{j} W_{m,r}(n, k)(k + r)!m^k x^j$$
$$= \sum_{j=0}^{n} \sum_{k=0}^{n} (-1)^{n-k-j} \binom{n-k}{j} W_{m,r}(n, k)(k + r)!m^k x^j.$$

Then, we define the r-Eulerian–Fubini numbers $a_{m,r}(n, j)$ by

$$a_{m,r}(n, j) := \sum_{k=0}^{n} (-1)^{n-k-j} \binom{n-k}{j} (k+r)! m^k W_{m,r}(n, k).$$

Moreover, if $(m, r) = (1, 0)$ we recover the classical Eulerian polynomials.

Theorem 19. *The exponential generating function for the r-Eulerian–Fubini polynomials is*

$$\sum_{n=0}^{\infty} \mathcal{A}_{m,r}(n, x) \frac{z^n}{n!} = \frac{r!(x-1)^{r+1} \mathrm{e}^{r(x-1)z}}{(x - \mathrm{e}^{m(x-1)z})^{r+1}}. \tag{18}$$

Proof. The exponential generating function for the r-Whitney numbers of the second kind is [29]

$$\sum_{n=k}^{\infty} W_{m,r}(n, k) \frac{z^n}{n!} = \frac{\mathrm{e}^{rz}}{k!} \left(\frac{\mathrm{e}^{mz} - 1}{m} \right)^k .$$

Then we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{A}_{m,r}(n, x) \frac{z^n}{n!} &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} W_{m,r}(n, k)(k+r)! m^k (x-1)^{n-k} \frac{z^n}{n!} \\
&= \sum_{k=0}^{\infty} \frac{(k+r)! m^k}{(x-1)^k} \sum_{n=k}^{\infty} W_{m,r}(n, k)(x-1)^n \frac{z^n}{n!} \\
&= \sum_{k=0}^{\infty} \frac{(k+r)! m^k}{(x-1)^k} \cdot \frac{\mathrm{e}^{r(x-1)z}}{k!} \cdot \left(\frac{\mathrm{e}^{m(x-1)z} - 1}{m} \right)^k \\
&= r! \mathrm{e}^{r(x-1)z} \sum_{k=0}^{\infty} \binom{k+r}{r} \left(\frac{\mathrm{e}^{m(x-1)z} - 1}{x-1} \right)^k .
\end{aligned}$$

Then equation (18) follows. □

Theorem 20. *The r-Whitney–Fubini polynomials satisfy the identity*

$$\mathcal{F}_{m,r}(n, x) = \sum_{j=0}^{n} a_{m,r}(n, j)(1+x)^j x^{n-j}. \tag{19}$$

Proof. From (17), we obtain

$$\mathcal{F}_{m,r}(n, x) = x^n \mathcal{A}_{m,r} \left(n, \frac{x+1}{x} \right) = x^n \sum_{j=0}^{n} a_{m,r}(n, j) \left(\frac{x+1}{x} \right)^j ,$$

which results in (19). □

Theorem 21. *The r-Eulerian–Fubini polynomials are represented by the following infinite sum*:

$$\frac{\mathcal{A}_{m,r}(n,x)}{(1-x)^n} = r!(1-x)^{r+1}\sum_{j=0}^{\infty} x^j \binom{r+j}{j} ((j+r+1)m-r)^n \,. \tag{20}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\mathcal{A}_{m,r}(n,x)}{(1-x)^n}\frac{z^n}{n!} &= \sum_{n=0}^{\infty} \mathcal{A}_{m,r}(n,x)\frac{(\frac{z}{1-x})^n}{n!} = \frac{r!(x-1)^{r+1}\mathrm{e}^{-rz}}{(x-\mathrm{e}^{-mz})^{r+1}} \\
&= r!\mathrm{e}^{(m(r+1)-r)z}\left(\frac{1-x}{1-x\mathrm{e}^{mz}}\right)^{r+1} \\
&= r!(1-x)^{r+1}\mathrm{e}^{(m(r+1)-r)z}\sum_{j=0}^{\infty}\binom{r+j}{j}(x\mathrm{e}^{mz})^j \\
&= r!(1-x)^{r+1}\sum_{j=0}^{\infty}\binom{r+j}{j}x^j\sum_{n=0}^{\infty}\frac{(((j+r+1)m-r)z)^n}{n!} \\
&= r!(1-x)^{r+1}\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{j=0}^{\infty}x^j\binom{r+j}{j}((j+r+1)m-r)^n.
\end{aligned}$$

So statement (20) follows. □

COROLLARY 22

The Eulerian polynomials satisfy the identity

$$\frac{\mathcal{A}(n,x)}{(1-x)^{n+1}} = \sum_{j=0}^{\infty} x^j \,(j+1)^n \,.$$

In [35,49], the authors introduced a different family of Eulerian polynomials related to r-Whitney numbers.

Theorem 23. *The following equality holds*:

$$\begin{aligned}
a_{m,r}(n,i) &= (m(1+r)-r)a_{m,r}(n-1,j) + ra_{m,r}(n-1,j-1) \\
&\quad + m\sum_{k=0}^{n-1} k(-1)^{(n-1)-k-(j-1)}(k+r)!m^k W_{m,r}(n-1,k)\binom{n-k-1}{j-1},
\end{aligned}$$

where $a_{m,r}(n,i)$ are the r-Eulerian–Fubini numbers.

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