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A generalization of zero divisor graphs associated to commutative rings

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Abstract. Let R be a commutative ring with a nonzero identity element. For a natural number n, we associate a simple graph, denoted by Γ_R^n, with $R^n\backslash\{0\}$ as the vertex set and two distinct vertices X and Y in R^n being adjacent if and only if there exists an $n \times n$ lower triangular matrix A over R whose entries on the main diagonal are nonzero and one of the entries on the main diagonal is regular such that $X^T AY = 0$ or $Y^T AX = 0$, where, for a matrix B, B^T is the matrix transpose of B. If $n = 1$, then Γ_R^n is isomorphic to the zero divisor graph $\Gamma(R)$, and so Γ_R^n is a generalization of $\Gamma(R)$ which is called a generalized zero divisor graph of R. In this paper, we study some basic properties of Γ_R^n. We also determine all isomorphic classes of finite commutative rings whose generalized zero divisor graphs have genus at most three.

Keywords. Zero divisor graph; lower triangular matrix; genus; complete graph.

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1. Introduction

Let R be a commutative ring with a nonzero identity element, and let $Z(R)$ be the set of all zero divisors of R. Set $Z^*(R) := Z(R)\backslash\{0\}$. We call an element $a \in R\backslash Z(R)$ a regular element. The zero divisor graph of R, which is denoted by $\Gamma(R)$, is an undirected graph whose vertices are elements of $Z^*(R)$ with two distinct vertices x and y which are adjacent if and only if $xy = 0$. The concept of the zero divisor graph of a commutative ring was first introduced by Beck [6], but this work was mostly concerned with coloring of rings. The zero divisor graph of commutative rings has been studied extensively by several authors (cf. [1–5]).

In this paper, we provide a generalization of the zero divisor graphs by using matrix theory. We associate a simple graph to a commutative ring R which is called a generalized zero divisor graph of R and it is denoted by Γ_R^n, with $R^n\backslash\{0\}$ as the vertex set and two distinct vertices X and Y being adjacent if and only if there exists an $n \times n$ lower triangular matrix A over R whose entries on the main diagonal are nonzero and one of the entries on the main diagonal is regular such that $X^T AY = 0$ or $Y^T AX = 0$. In this situation, we say that the vertices X and Y are adjacent by applying the matrix A. Note that in the case $n = 1$, the graph Γ_R^n is isomorphic to the zero divisor graph $\Gamma(R)$. Hence, Γ_R^n is a generalization of the zero divisor graph $\Gamma(R)$.

In this paper, we study some properties of the graph Γ_R^n such as planarity, genus and crosscap of the graph. Also, we investigate the independence, dominion and chromatic numbers of Γ_R^n, in the case that R is a finite field.

Now, we recall some definitions of graph theory which are necessary in this paper from [7]. A *simple graph* is a pair $G = (V, E)$, where $V(G)$ and $E(G)$ are the sets of vertices and edges of G, respectively. A *subgraph* of a graph G is a graph G' such that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. In a graph G, the *distance* between two distinct vertices x and y, denoted by $\mathrm{d}(x, y)$, is the length of the shortest path connecting x and y, if such a path exists, otherwise, we set $\mathrm{d}(x, y) := \infty$. The *diameter* of a graph G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(x, y) \ : \ x$ and y are distinct vertices of $G\}$. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of the shortest cycle in G, if G contains a cycle; otherwise, $\mathrm{gr}(G) := \infty$. Also, for two distinct vertices x and y in G, the notation $x - y$ means that x and y are adjacent. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if each pair of distinct vertices is joined by an edge. Also, a graph G is called *planar* if it can be drawn in the plane without any edge crossing. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$. A *clique* of a graph is a complete subgraph of it and the number of vertices in a largest clique of G is called the *clique number* of G and is denoted by $\omega(G)$. An *independent set* of G is a subset of the vertices of G such that no two distinct vertices in the subset represents an edge. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of the largest independent set. Also a *dominating set* of a graph G is a subset of the vertex set, say S, such that every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\lambda(G)$, is the cardinality of the smallest dominating set. A *cutset* in a graph G is a set of vertices whose deletion increases the number of connected components of G. The *vertex connectivity* of a connected graph G is the minimum number of vertices in a vertex cutset, and is denoted by $K_0(G)$.

2. Basic properties

Throughout the paper, R is a commutative ring with a nonzero identity and n is a positive integer with $n > 1$. Also, $\mathbb{Z}_p$ is a finite field of order p, where p is a prime integer. We denote the set of zero divisors and unit elements of R by $Z(R)$ and $U(R)$, respectively. Also, we denote $R\backslash Z(R)$ by $\mathrm{Reg}(R)$. Clearly in a finite ring R we have $\mathrm{Reg}(R) = U(R)$. A generalized zero divisor graph associated to R, denoted by Γ_R^n with $R^n\backslash\{0\}$ as the vertex set and two distinct vertices X and Y in R^n being adjacent if and only if there exists an $n \times n$ lower triangular matrix A over R whose entries on the main diagonal are nonzero such that $X^T AY = 0$ or $Y^T AX = 0$.

In the rest of the paper, for an integer i with $1 \leq i \leq n$, we use the notation E_i to denote the vertex whose ith component is 1 and other components are zero.

We begin with the following lemma.

Lemma 2.1. *Assume that $X = E_i$ and $Y = E_j$ for some $1 \leq i, j \leq n$. Then X is adjacent to Y for all $1 \leq i \neq j \leq n$.*

Proof. Consider the $n \times n$ lower triangular matrix A whose entries satisfy the following equations:

$$\begin{aligned} a_{kl} &= 0, \quad \text{for } k = i \text{ and } l = j, \\ a_{kl} &= 1, \quad \text{for } k \geq l,\ k \neq i \text{ and } l \neq j. \end{aligned}$$

Then all entries on the main diagonal of A are nonzero and one of the entries on the main diagonal is regular. It is easy to see that $X^T AY = 0$, and so the result follows. □

Lemma 2.2. *Let X and Y be two distinct vertices in Γ_R^n such that x_l and y_k are units for $1 \leq l, k \leq n$. Also, suppose that one of the following conditions hold*:

(*i*) *$l \neq k$.*
(*ii*) *$l = k$ and one of the vertices X or Y has at least two unit components.*

Then X and Y are adjacent.

Proof. If (i) holds, then one can easily see that X and Y are adjacent. Now, suppose that (ii) holds. Assume that x_l, x_m, y_k are distinct unit elements for $1 \leq l, k, m \leq n$. Consider the $n \times n$ lower triangular matrix A whose entries satisfy the following equations:

$$\begin{aligned} a_{mk} &= -x_m^{-1} y_k^{-1} \left(\sum\nolimits_{i,j=1}^{n} x_i y_j \right), \quad \text{for } i \neq m \text{ and } j \neq k, \\ a_{ij} &= 1, \quad \text{for } i \neq m \text{ and } j \neq k. \end{aligned}$$

Then all entries on the main diagonal of A are nonzero and one of the entries on the main diagonal is regular. It is easy to see that $X^T AY = 0$, and so X and Y are adjacent. □

By using the method similar to that used in the proof of Lemma 2.1, one can obtain the following corollary.

COROLLARY 2.3

Let $X^T = (0, \ldots, 0, x_i, 0, \ldots, 0)$ and $Y^T = (0, \ldots, 0, y_i, 0, \ldots, 0)$ be distinct vertices such that x_i and y_i are units. Then X and Y are not adjacent.

The following theorem follows from Lemmas 2.1 and 2.2.

Theorem 2.4. *Let R be a field. Then we have the following statements*:

(*i*) *If* $|R| = 2$, *then* $\operatorname{diam}(\Gamma_R^n) = 1$.
(*ii*) *If* $|R| > 2$, *then* $\operatorname{diam}(\Gamma_R^n) = 2$.

Proof.

(i) If $|R| = 2$, then $R \cong \mathbb{Z}_2$. Now, by Lemmas 2.1 and 2.2, we have that Γ_R^n is a complete graph. Hence $\operatorname{diam}(\Gamma_R^n) = 1$.

(ii) Assume that $|R| > 2$. Then there exists at least two unit elements. If X or Y has at least two unit components, then, by Lemma 2.2, we have $\mathrm{d}(X, Y) = 1$. Now, suppose that both vertices X and Y have exactly one unit component, say x_i and y_j, for $1 \le i, j \le n$. If $i \neq j$, then by Lemma 2.2, X and Y are adjacent. If $i = j$, then one can consider the path

$$X - (0, \ldots, 0, z_i, 0, \ldots, 0, z_j, 0, \ldots, 0) - Y,$$

where $z_i, z_j \in U(R)$. Hence $\mathrm{d}(X, Y) \le 2$. Therefore we have $\mathrm{diam}(\Gamma_R^n) \le 2$. Also, by Corollary 2.3, the vertices $(0, \ldots, 0, x_i, 0, \ldots, 0)$ and $(0, \ldots, 0, y_i, 0, \ldots, 0)$ are not adjacent, where x_i and y_i are distinct unit elements in R. Thus Γ_R^n is not complete, and so in this case, we have $\mathrm{diam}(\Gamma_R^n) = 2$. □

Lemma 2.5. *If* $|\mathrm{Reg}(R)| > 1$, *then* $\mathrm{diam}(\Gamma_R^n) = 2$.

Proof. By Theorem 2.4, the result holds. □

COROLLARY 2.6

If R is an integral domain with $|R| > 2$, then $\mathrm{diam}(\Gamma_R^n) = 2$.

In the following proposition, we study the girth of Γ_R^n.

PROPOSITION 2.7

In the generalized zero divisor graph Γ_R^n, we have $\mathrm{gr}(\Gamma_R^n) = 3$.

Proof. If $n \ge 3$, then by Lemma 2.2, the vertices $(1, 1, 1, 0, \ldots, 0)$, $(1, 1, 0, \ldots, 0)$ and $(1, 0, \ldots, 0)$ form a triangle. Suppose that $n = 2$. If $R = \mathbb{Z}_2$, then clearly Γ_R^n is isomorphic to K_3. If $R \neq \mathbb{Z}_2$, then one can consider the triangle

$$(1, 0) - (1, 1) - (1, a) - (1, 0)$$

in Γ_R^n, where $a \notin \{0, 1\}$. Therefore we have $\mathrm{gr}(\Gamma_R^n) = 3$. □

In the following theorem, we investigate the planarity of Γ_R^n.

Theorem 2.8. *The graph Γ_R^n is planar if and only if $n = 2$ and $R = \mathbb{Z}_2$.*

Proof. First suppose that Γ_R^n is planar. We have the following cases:

Case 1. $n \ge 5$. Then, by Lemma 2.1, the vertices of the set $\{E_1, E_2, \ldots, E_5\}$ form a subgraph isomorphic to K_5. This is impossible.

Case 2. $n = 4$. Then, by Lemmas 2.1 and 2.2, the vertices of the set

$$\{E_1, E_2, E_3, E_4, (1, 1, 1, 1)\}$$

form a subgraph isomorphic to K_5. This is impossible.

Case 3. $n = 3$. Then, by Lemmas 2.1 and 2.2, the vertices of the set

$$\{E_1, E_2, E_3, (1, 1, 1), (1, 1, 0)\}$$

form a subgraph isomorphic to K_5. This is impossible.

Case 4. $n = 2$. If $|R| \geq 3$, then, by Lemmas 2.1 and 2.2, the vertices of the set $\{(0, 1), (1, 0), (1, 1), (1, a), (a, 1)\}$ with $a \notin \{0, 1\}$ form the complete graph K_5, which is impossible. Hence we have $|R| = 2$. In this situation, Γ_R^n is isomorphic to K_3, which is planar.

The converse statement is clear. □

In the following theorem, we study the clique number of Γ_R^n.

Theorem 2.9. *If R is a finite ring, then*

$$\omega(\Gamma_R^n) \geq |U(R)|^n + n + \sum_{i=2}^{n-1} \binom{|U(R)|}{i}.$$

Proof. By Lemma 2.2, all vertices with at least two unit components form a clique. Hence we have a clique A of size $|U(R)|^n + \sum_{i=2}^{n-1} \binom{|U(R)|}{i}$, where $k = |U(R)|$. Now, by Lemmas 2.1 and 2.2, the set $A \cup \{E_i \mid 1 \leq i \leq n\}$ forms a complete subgraph of Γ_R^n. Hence the result holds. □

Example 2.10. If $R = \mathbb{Z}_2$, then $\omega(\Gamma_R^n) = 2^n - 1$.

Example 2.11. If $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\omega(\Gamma_R^2) = 15$.

Proof. Put

$$\begin{aligned}
&X_1 = ((1, 1), (1, 0)), \quad X_2 = ((1, 1), (0, 1)), \quad X_3 = ((1, 1), (0, 0)),\\
&X_4 = ((1, 1), (1, 1)), \quad X_5 = ((0, 1), (1, 0)), \quad X_6 = ((0, 1), (0, 1)),\\
&X_7 = ((0, 1), (0, 0)), \quad X_8 = ((0, 1), (1, 1)), \quad X_9 = ((1, 0), (1, 0)),\\
&X_{10} = ((1, 0), (0, 1)), \quad X_{11} = ((1, 0), (0, 0)), \quad X_{12} = ((1, 0), (1, 1)),\\
&X_{13} = ((0, 0), (1, 0)), \quad X_{14} = ((0, 0), (0, 1)), \quad X_{15} = ((0, 0), (1, 1)).
\end{aligned}$$

By Lemma 2.2, X_4 is adjacent to the vertices X_1, X_2, X_3, X_8, X_{12} and X_{15}. Set

$$A = \begin{pmatrix} (1,0) & (0,0) \\ (1,0) & (1,0) \end{pmatrix} \quad B = \begin{pmatrix} (1,0) & (0,0) \\ (1,0) & (0,1) \end{pmatrix} \quad C = \begin{pmatrix} (0,1) & (0,0) \\ (1,0) & (1,0) \end{pmatrix}$$

$$D = \begin{pmatrix} (0,1) & (0,0) \\ (1,0) & (0,1) \end{pmatrix} \quad E = \begin{pmatrix} (1,0) & (0,0) \\ (0,1) & (0,1) \end{pmatrix} \quad F = \begin{pmatrix} (1,0) & (0,0) \\ (0,0) & (1,0) \end{pmatrix}$$

Now, one can easily check the following adjacencies:

$X_1 - X_i$ and $X_1 - X_j$, for $i = 2, 3, 6, 7, 10, 11, 13, 14$ and $j = 5, 8, 9, 12, 15$, by applying matrices A and B.

$X_2 - X_i$ and $X_2 - X_j$, for $i = 3, 5, 6, 7, 8, 9, 13, 14, 15$ and $j = 10, 11, 12$, by applying matrices A and C.
$X_3 - X_i$, $X_3 - X_j$ and $X_3 - X_{13}$, for $i = 5, 6, 7, 8, 14, 15$ and $j = 9, 10, 11, 12$, by applying matrices A, C and E.
$X_4 - X_i$, $X_4 - X_j$, $X_4 - X_5$ and $X_4 - X_9$, for $i = 6, 7, 13, 14$ and $j = 10, 11$, by applying matrices A, E, B and C.
$X_5 - X_i$, $X_5 - X_j$, $X_5 - X_k$ and $X_5 - X_l$, for $i = 6, 7, 14$, $j = 8, 13, 15$, $k = 9, 12$ and $l = 10, 11$, by applying matrices A, B, E and F.
$X_6 - X_i$, $X_7 - X_j$, $X_{11} - X_k$ and $X_{14} - X_{15}$, for $i = 7, 8, 9, 10, 11, 12, 13, 14, 15$, $j = 8, 9, 10, 11, 12, 13, 14, 15$ and $k = 12, 13, 14, 15$, by applying matrix A.
$X_8 - X_{14}$, $X_8 - X_i$ and $X_8 - X_j$, for $i = 9, 11, 13, 15$ and $j = 10, 12$, by applying matrices A, E and F.
$X_9 - X_{11}$, $X_9 - X_i$ and $X_9 - X_{10}$, for $i = 12, 13, 14, 15$, by applying matrices A, B and E.
$X_{10} - X_i$ and $X_{10} - X_j$, for $i = 13, 14, 15$ and $j = 11, 12$, by applying matrices A and C.
$X_{12} - X_{13}$, $X_{12} - X_{14}$ and $X_{12} - X_{15}$, by applying matrices A, B and E.
$X_{13} - X_{14}$ and $X_{13} - X_{15}$, by applying matrices A and B. So, Γ_R^n is a complete graph. Hence $\omega(\Gamma_R^2) = 15$. □

PROPOSITION 2.12

If $|U(R)| > 1$, *then the graph* Γ_R^n *is not complete.*

Proof. Assume that a and b are two unit elements in R. By Corollary 2.3, we know that $X = (0, \ldots, 0, x_i = a, 0, \ldots, 0)$ and $Y = (0, \ldots, 0, y_i = b, 0, \ldots, 0)$ are not adjacent for $1 \leq i \leq n$. Thus Γ_R^n is not complete. □

In the following proposition, we study some adjacencies in the graph Γ_R^n, when R is an infinite ring.

PROPOSITION 2.13

Let $X^T = (x_1, x_2, \ldots, x_n)$ *and* $Y^T = (y_1, y_2, \ldots, y_n)$ *be two vertices such that all components are not units and nonzero divisor elements of an infinite ring* R. *Then* X *and* Y *are adjacent.*

Proof. Consider the following two cases:

Case 1. n is even. Then we consider the $n \times n$ lower triangular matrix A whose entries satisfy the following equations:

$$a_{11} = x_2 y_2,\ a_{22} = -x_1 y_1,\ a_{33} = x_4 y_4,\ a_{44} = -x_3 y_3, \ldots,\ a_{n-1n-1} = x_n y_n,$$
$$a_{nn} = -x_{n-1} y_{n-1} \quad \text{and} \quad a_{ij} = 0 \ \text{ for } \ i \neq j.$$

Case 2. n is odd. Then we suppose that A is a matrix which is satisfied in the following equations:

$$a_{11}=x_2y_2,\ a_{22}=-x_1y_1,\ a_{33}=x_4y_4,\ a_{44}=-x_3y_3,\ \ldots,\ a_{n-1n-1}=x_ny_n,$$
$$a_{nn}=-x_{n-1}y_{n-1},\quad \text{and}\quad a_{ij}=0,\ \text{ for } i\neq j \text{ and } i\neq 2,\ j\neq 1.$$

Therefore we have $X^TAY=0$, which means that X and Y are adjacent. □

3. On the genus and crosscap numbers of Γ_R^n

In this section, we denote by S_g the surface formed by a connected sum of g tori, and by N_k the one formed by a connected sum of k projective planes. The number g is called the genus of the surface S_g and k is called the crosscap of N_k. When considering the orientability, the surfaces S_g and sphere are among the orientable class and the surfaces N_k are among the non-orientable one.

A simple graph which can be embedded in S_g but not in S_{g-1} is called a graph of genus g. Similarly, if it can be embedded in N_k but not in N_{k-1}, then we call it a graph of crosscap k. The notations $\gamma(G)$ and $\overline{\gamma}(G)$ are denoted for the genus and crosscap of a graph G, respectively.

Recall that, for a rational number q, $\lceil q\rceil$ is the first integer number greater or equal than q. It is easy to see that $\gamma(H)\leq\gamma(G)$ and $\overline{\gamma}(H)\leq\overline{\gamma}(G)$, for every subgraph H of G.

The following lemma states some well-known formulas for genus of a graph from [9].

Lemma 3.1. *The following statements hold*:

(*i*) *For* $n\geq 3$, *we have* $\gamma(K_n)=\lceil\frac{1}{12}(n-3)(n-4)\rceil$.
(*ii*) *For* $m,n\geq 2$, *we have* $\gamma(K_{m,n})=\lceil\frac{1}{4}(m-2)(n-2)\rceil$.

Clearly, we have $\gamma(K_n)=0$ for $1\leq n\leq 4$ and $\gamma(K_n)=1$ for $5\leq n\leq 7$, and, for other values of n, $\gamma(K_n)\geq 2$.

In the following theorem, we determine all isomorphic classes of finite commutative rings R whose Γ_R^n has genus at most three.

Theorem 3.2. *The following statements hold*:

(*i*) $\gamma(\Gamma_R^n)=0$ *if and only if* $R=\mathbb{Z}_2$ *and* $n=2$.
(*ii*) $\gamma(\Gamma_R^n)=1$ *if and only if* $R=\mathbb{Z}_2$ *and* $n=3$ *or* $R=\mathbb{Z}_3$ *and* $n=2$.
(*iii*) $\gamma(\Gamma_R^n)=3$ *if and only if* $R=\frac{\mathbb{Z}_2[x]}{\langle x^2\rangle}$ *and* $n=2$, *or* $R=\mathbb{Z}_4$ *and* $n=2$.
(*iv*) *There is no ring* R *with* $\gamma(\Gamma_R^n)=2$.

Proof. We consider the following cases:

Case 1. $n\geq 4$. If $R=\mathbb{Z}_2$, then by Example 2.10, $\omega(\Gamma_R^n)=2^n-1\geq 15$ and so, by Lemma 3.1, $\gamma(\Gamma_R^n)\geq 11$. If $|R|\geq 3$, then by Theorem 2.9, $\omega(\Gamma_R^n)\geq 30$ and so, by Lemma 3.1, $\gamma(\Gamma_R^n)\geq 37$.

Case 2. $n=3$. If $|R|\geq 3$, then, by Theorem 2.9, $\omega(\Gamma_R^n)\geq 14$ and so, by Lemma 3.1, $\gamma(\Gamma_R^n)\geq 10$. If $R=\mathbb{Z}_2$, then $\omega(\Gamma_R^n)=7$ and so, by Lemma 3.1, $\gamma(\Gamma_R^n)=1$.

Case 3. $n = 2$. If $|R| \geq 5$, then by Theorem 2.9, $\omega(\Gamma_R^n) \geq 18$ and so, by Lemma 3.1, $\gamma(\Gamma_R^n) \geq 18$.

If R is local with $|R| = 4$, then R is a field or R is one of the rings $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$. If $R = \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, then $|U(R)| = 2$. Hence $\omega(\Gamma_R^n) = 9$, and so, by Lemma 3.1, $\gamma(\Gamma_R^n) = 3$.

If R is a field with $|R| = 4$, then $|U(R)| = 3$. Now, by Theorem 2.9, $\omega(\Gamma_R^n) \geq 11$ and so, by Lemma 3.1, $\gamma(\Gamma_R^n) \geq 5$. If R is not local and $|R| = 4$, then $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Now, by Example 2.11, Γ_R^n is a complete graph with 15 vertices. Hence, by Lemma 3.1, $\gamma(\Gamma_R^n) = 11$.

If $|R| = 3$, then $\omega(\Gamma_R^n) = 6$ and so, by Lemma 3.1, $\gamma(\Gamma_R^n) = 1$.

If $|R| = 2$, then $\Gamma_R^n \cong K_3$. Hence $\gamma(\Gamma_R^n) = 0$.

Now by considering the above cases, the results hold. □

The following lemma which is from [8] is needed for the proof of the next theorem.

Lemma 3.3. *The following statements hold*:

$$\overline{\gamma}(K_n) = \begin{cases} \frac{1}{6}(n-3)(n-4), & \text{if } n \geq 3 \text{ and } n \neq 7, \\ 3, & \text{if } n = 7. \end{cases}$$

$$\overline{\gamma}(K_{m,n}) = \left\lceil \frac{1}{2}(m-2)(n-2) \right\rceil.$$

By slight modifications in the proof of Theorem 3.2, in conjunction with Lemma 3.3, one can prove the following theorem.

Theorem 3.4. *The following statements hold*:

(i) $\overline{\gamma}(\Gamma_R^n) = 0$ *if and only if* $R = \mathbb{Z}_2$ *and* $n = 2$.
(ii) $\overline{\gamma}(\Gamma_R^n) = 1$ *if and only if* $R = \mathbb{Z}_3$ *and* $n = 2$.
(iii) $\overline{\gamma}(\Gamma_R^n) = 2$ *if and only if* $R = \mathbb{Z}_2$ *and* $n = 3$.
(iv) *There is no ring* R *with* $\gamma(\Gamma_R^n) = 3$.

4. Structure of the graph $\Gamma_{\mathbb{Z}_p}^n$

Throughout this section, we assume that $R = \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of integers modulo the prime integer p. In the sequel of the paper, we study Hamiltonian cycle, Eulerian cycle, independence number and domination number of the generalized zero divisor graph Γ_R^n.

Recall that a graph is Eulerian if there exists a closed trail containing every edge. By [7, Theorem 4.1], a connected graph is Eulerian if and only if all the degrees of vertices are even. A Hamiltonian cycle in a graph G is a cycle that contains all vertices of G. Moreover, G is called Hamiltonian if it contains a Hamiltonian cycle.

Remark 4.1. Let R be a finite ring and $|R| = p$, where p is a prime number. Then we have the following statements:

(i) If $|R| = 2$, then $\deg(X) = 2^n - 2$ for all $X \in V(\Gamma_R^n)$.
(ii) If $|R| \geq 3$, then $\deg(X) = |R|^n - 2$, where X has at least two unit components and $\deg(X) = |R|^n - |U(R)| - 1$, where X has one unit component.

Theorem 4.2. *The generalized zero divisor graph $\Gamma^n_{\mathbb{Z}_p}$ is an Eulerian graph if and only if $p = 2$.*

Proof. Consider the following two cases:

Case 1. $p = 2$. Then, by Proposition 2.12, Γ^n_R is a complete graph. By Remark 4.1(i), we have $\deg(X) = 2^n - 2$ for every $X \in V(\Gamma^n_R)$. Since $2^n - 2$ is an even number, we have that Γ^n_R is an Eulerian graph.

Case 2. $p \geq 3$. Put

$$V_1 = \{X \in V(\Gamma^n_R) \mid X \text{ has at least two unit components}\}$$

and

$$V_2 = \{Y \in V(\Gamma^n_R) \mid Y \text{ has exactly one unit component}\}.$$

Now, by Remark 4.1(ii), we have $\deg(X) = p^n - 2$ for every $X \in V_1$. Since $p^n - 2$ is an odd number, we have that Γ^n_R is not Eulerian.

Now, by considering the above cases, the result holds. □

In the following theorem, we study the Hamiltonian generalized zero divisor graphs.

Theorem 4.3. *Let $|R| = p$, where p is a prime number. Then Γ^n_R is Hamiltonian.*

Proof. If $|R| = p$, where p is a prime number, then $R \cong \mathbb{Z}_p$. So R has $p-1$ unit elements. Put

$$V_1 = \{X \in V(\Gamma^n_R) \mid X \text{ has at least two unit components}\}$$

and

$$V_2 = \{Y \in V(\Gamma^n_R) \mid Y \text{ has exactly one unit component}\}.$$

Suppose that $V_1 = \{X_1, \ldots, X_n\}$ and $V_2 = \{Y_1, \ldots, Y_m\}$. Clearly, $m \leq n$. Now, by Lemma 2.2, we have the following path:

$$P: \; Y_1 - X_1 - Y_2 - X_2 - \cdots - Y_m - X_m - X_{m+1} - X_m.$$

So one can easily see that Γ^n_R contains a Hamiltonian cycle. Thus, the result holds. □

For any vertex x of a connected graph G, the *eccentricity* of x, denoted by $\varepsilon(x)$, is the maximum of the distances from x to the other vertices of G, and the minimum value of the eccentricity of vertices is the *radius* of G, which is denoted by $r(G)$. In the next lemma, we compute the eccentricity of Γ^n_R.

Lemma 4.4. *If R is a finite ring and $|R| = p$, then $\varepsilon(X) \in \{1, 2\}$ for each $X \in V(\Gamma^n_R)$.*

Proof. We consider the following cases:

Case 1. If $|R| = 2$, then by Theorem 2.4, $\Gamma_R^n \cong K_{2^n-2}$, and so $\varepsilon(X) = 1$ for every $X \in V(\Gamma_R^n)$.

Case 2. If $|R| > 2$, then R has at least two unit elements. By Lemma 2.3, the vertices $X = (0, \ldots, 0, x_i, 0, \ldots, 0)$ and $Y = (0, \ldots, 0, y_i, 0, \ldots, 0)$, where $x_i, y_i \in U(R)$ are not adjacent. By Theorem 2.4, $\mathrm{d}(X, Y) \leq 2$. Hence $\varepsilon(X) = 2$. Other kinds of vertices are adjacent.

Now, by considering the above cases, the result holds. □

The following corollary follows easily from Lemma 4.4.

COROLLARY 4.5

If R is a finite ring and $|R| = p$, then $r(\Gamma_R^n) = 1$.

A *split graph* is a graph in which the vertices can be partitioned into a clique and an independent set. In the following lemma, we show that Γ_R^n is a split graph if and only if $|R| = 2$.

PROPOSITION 4.6

Suppose that $|R| = p$. Then Γ_R^n is a split graph if and only if $|R| = 2$.

Proof. First suppose that Γ_R^n is split. We have the following two cases:

Case 1. Assume that $|R| = 2$. Then Γ_R^n is a complete graph, which is split.

Case 2. $|R| > 2$. Suppose that $|U(R)| = m$, when $m \geq 2$. We put

$$\begin{aligned}
V_1 &= \{(x_i, 0, \ldots, 0) \mid x_i \in U(R),\ \ 1 \leq i \leq m\},\\
V_2 &= \{(0, x_i, \ldots, 0) \mid x_i \in U(R),\ \ 1 \leq i \leq m\},\\
&\ \vdots\\
V_m &= \{(0, \ldots, 0, x_i) \mid x_i \in U(R),\ \ 1 \leq i \leq m\}.
\end{aligned}$$

Since Γ_R^n is split, we have $V(\Gamma_R^n) = K \cup S$, when the induced subgraph with vertex set K is complete and S is an independent set. Now, one can easily check that $S \subseteq V_i$ for some i with $1 \leq i \leq m$. Also, we have $V_j \subseteq K$ for all $j \in \{1, \ldots, m\} \backslash \{i\}$. This is impossible, since each V_i for $1 \leq i \leq n$ forms an independent set. Therefore, if Γ_R^n is split, then $|R| = 2$.

The converse statement is clear. □

We end this section with the following proposition.

PROPOSITION 4.7

Let $|R| = p$. The following statements hold:

(*i*) *If* $p = 2$, *then* $K_0(\Gamma_R^n) = 2^n - 1$.
(*ii*) *If* $p > 2$, *then* $\alpha(\Gamma_R^n) = |U(R)|$ *and* $\lambda(\Gamma_R^n) = 1$.

Proof.

(i) If $p = 2$, then Γ_R^n is a complete graph with $2^n - 1$ vertices. Therefore $K_0(\Gamma_R^n) = 2^n - 1$.
(ii) Let $p > 2$. Then

$$\begin{aligned} V_1 &= \{(x_i, 0, \ldots, 0) \mid x_i \in U(R), \quad 1 \le i \le m\}, \\ V_2 &= \{(0, x_i, \ldots, 0) \mid x_i \in U(R), \quad 1 \le i \le m\}, \\ &\;\;\vdots \\ V_m &= \{(0, \ldots, 0, x_i) \mid x_i \in U(R), \quad 1 \le i \le m\}. \end{aligned}$$

By Lemma 2.4, the sets V_j form an independent set for $1 \le j \le n$. Also, one can easily see that $|U(R)|$ is the size of the largest independent set. Hence we have $\alpha(\Gamma_R^n) = |U(R)|$. Also, by Lemma 2.2, we know that each vertex with at least two unit components is adjacent to other vertices. Therefore $\lambda(\Gamma_R^n) = 1$. □

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