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ON DOMINIONS OF THE RATIONALS IN NILPOTENT GROUPS

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Abstract: The dominion of a subgroup H of a group G in a class M is the set of all $a \in G$ that have the same images under every pair of homomorphisms, coinciding on H from G to a group in M. A group H is n-closed in M if for every group $G = \mathrm{gr}(H, a_1, \dots, a_n)$ in M that includes H and is generated modulo H by some n elements, the dominion of H in G (in M) is equal to H. We prove that the additive group of the rationals is 2-closed in every quasivariety of torsion-free nilpotent groups of class at most 3.

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Introduction

In this paper, we complete the study that was started in [1, 2] of the 2-closedness of the additive group of the rationals in quasivarieties of torsion-free nilpotent groups of class at most 3.

Let $\mathscr{M}$ be an arbitrary quasivariety of groups. Given a group G in $\mathscr{M}$ and a subgroup H of G, we define the dominion $\mathrm{dom}_G^{\mathscr{M}}(H)$ of H in G (in $\mathscr{M}$) as follows:

$$\mathrm{dom}_G^{\mathscr{M}}(H) = \{a \in G \mid \forall M \in \mathscr{M}\ \forall f, g : G \to M, \ \text{ if } f\mid_H = g\mid_H, \text{ then } f(a) = g(a)\}.$$

Here, as usual, $f, g : G \to M$ denote homomorphisms from G to M and $f\mid_H$ stands for the restriction of f to H.

The dominions are of interest primarily because they are closely connected with the free constructions in quasivarieties of universal algebras and amalgams (see [3] and the references in [4]). It is not difficult to see that $\mathrm{dom}_G^{\mathscr{M}}(-)$ is a closure operator on the subgroup lattice of G. This leads to the study of closed subgroups of a given group (with respect to $\mathscr{M}$).

A group H is said to be *n-closed* in $\mathscr{M}$ if $\mathrm{dom}_G^{\mathscr{M}}(H) = H$ for every group $G = \mathrm{gr}(H, a_1, \dots, a_n)$ in $\mathscr{M}$ that includes H and is generated modulo H by some n elements. It was shown in [5] that the study of dominions can be reduced to the study of n-closed groups (a more detailed treatment of this connection is provided in [6]). This explains our interest to the n-closed groups.

Dominions have been studied extensively in the quasivarieties of abelian groups [7–10]. Dominions in the class of nilpotent groups have also been investigated in an extensive series of papers among which we mention [11–13]. In recent years, a particular attention was paid to dominions of metabelian groups [14–18]. The n-closed groups in quasivarieties of nilpotent groups were studied in [5, 6, 13].

Suppose that $\mathscr{M}$ is a quasivariety of torsion-free nilpotent groups of class at most 3. We prove that the additive group of the rationals is 2-closed in $\mathscr{M}$.

Throughout the paper, we use the notation and facts about groups and quasivarieties of groups from [19, 20] and [21] respectively.

§ 1. Preliminaries

We begin by recalling some notions and definitions.

We write $\mathscr{N}_c$ for the variety of all nilpotent groups of class at most c, while $\mathscr{N}_{c,\infty}$ stands for the class of torsion-free groups in $\mathscr{N}_c$; and $F_r(\mathscr{N})$, for the free group in $\mathscr{N}$ of rank r. Throughout the paper, $F_2 = F_2(\mathscr{N}_3)$ is the free group of rank 2 in $\mathscr{N}_3$ generated by x and y.

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We write $\mathrm{gr}(S)$ for the group generated by a set S, while (a) stands for the cyclic group generated by an element a; G', for the derived group of a group G; and $Z(G)$, for the center of G. As usual, $\gamma_1(G) = G$ and $\gamma_{c+1}(G) = [\gamma_c(G), G]$. If x and y are some elements of a group G, then $[x, y] = x^{-1}y^{-1}xy$, and e denotes the identity of G. By $H \leq G$ we mean that H is a subgroup of G. An element gN of a factor group G/N is sometimes denoted by $\bar{g}$. The additive group of the rationals is denoted by Q, and we use a multiplicative notation for the group operation in Q.

The commutators of the form $[x_i, x_j, x_k]$, with $i > j \leq k$, are called *basic commutators of weight* 3.

As usual, $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of naturals and integers respectively.

We will need the following well-known commutator identities which are valid in every nilpotent group of class at most 3

$$[a, bc] = [a, c][a, b][a, b, c], \quad [b^{-1}, a] = b[a, b]b^{-1},$$

$$[ab, c] = [a, c][a, c, b][b, c], \quad [b, a^{-1}] = a[a, b]a^{-1},$$

$$[a, b, c][b, c, a][c, a, b] = e,$$

and which imply that the commutator $[x, y, z]$ of a nilpotent group of class 3 is linear in each argument. Using them, we can easily derive the identities that are valid in all groups in $\mathscr{N}_3$:

$$[b^n, a^k] = [b, a]^{nk}[b, a, a]^{n\frac{k^2-k}{2}}[b, a, b]^{k\frac{n^2-n}{2}}, \tag{1}$$

$$[xy, ab] = [x, b][x, a][y, a][y, b][x, a, b][x, a, y][x, b, y][y, a, b]. \tag{2}$$

Also, we will use the notation

$$z_1 = [x_4, x_1, x_1],\ z_2 = [x_4, x_1, x_3],\ z_3 = [x_4, x_3, x_3],\ z_4 = [x_3, x_2, x_3],\ z_5 = [x_2, x_1, x_3].$$

An *embedding of a group A into a group B* is a homomorphism $\varphi : A \to B$ that is an isomorphism from A onto $\varphi(A)$. If there is an embedding of A into B, then we say that A can be *embedded into B*.

The quasivariety generated by a group G is denoted by qG.

We will use the following Dyck Theorem [22].

Lemma 1. *Let a group G has the presentation*

$$G = \mathrm{gr}(\{x_i \ \mid \ i \in I\} \ \| \ \{r_j(x_{j_1}, \ldots, x_{j_{l(j)}}) = e \mid j \in J\})$$

in a quasivariety $\mathscr{N}$. Suppose that $H \in \mathscr{N}$ and H includes a set $\{g_i \mid i \in I\}$ such that $r_j(g_{j_1}, \ldots, g_{j_{l(j)}}) = e$ in H for all $j \in J$. Then the mapping $x_i \to g_i$ $(i \in I)$ can be extended to a homomorphism from G to H.

Also we need the following results.

Lemma 2 [13, Theorem 3]. *The additive group Q of the rationals is* 3*-closed in every quasivariety of torsion-free nilpotent groups of class ≤ 2.*

Lemma 3 [13, Lemma 2]. *Suppose that $\mathscr{M}$ is a quasivariety of groups, $G \in \mathscr{M}$, $H \leq G$, H is a complete group, and $H \cap G' = (e)$. Then $\mathrm{dom}_G^{\mathscr{M}}(H) = H$.*

Lemma 4 [1]. *Suppose that G is a torsion-free* 2*-generated nilpotent group of class* 3*. Then either $G \cong F_2(\mathscr{N}_3)$ or $G \cong A$, where A has the presentation in $\mathscr{N}_3$ as follows: $A = \mathrm{gr}(x, y \parallel [y, x, y] = e)$.*

Lemma 5 [2]. *Suppose that G is a torsion-free nilpotent group of class* 3*, $\mathscr{M}$ is a quasivariety of groups, $G = \mathrm{gr}(F_2, Q) \in \mathscr{M}$, and $F_2 \cap Q = ([x_2, x_1]^\delta[x_2, x_1, x_1]^s[x_2, x_1, x_2]^t)$. Then $\mathrm{dom}_G^{\mathscr{M}}(Q) = Q$.*

Lemma 6 [2]. *Suppose that G is a torsion-free nilpotent group of class* 3*, $\mathscr{M}$ is a quasivariety of groups, $G = \mathrm{gr}(F_2, Q) \in \mathscr{M}$, and $F_2 \cap Q = ([x_2, x_1, x_1])$. Then $\mathrm{dom}_G^{\mathscr{M}}(Q) = Q$.*

Throughout the paper, A denotes the group that is defined in Lemma 4.

§ 2. The Group H

Lemma 7. *Suppose that H is the group defined in $\mathscr{N}_{3,\infty}$ by the generators x_1, x_2, x_3, x_4 and the relations*

$$[x_2, x_1]^\delta [x_2, x_1, x_1]^\gamma = [x_4, x_3]^\delta [x_4, x_3, x_3]^\gamma, \quad [x_2, x_1, x_2] = e,$$

$$[x_3, x_2, x_2] = [x_3, x_2, x_4] = [x_3, x_1, x_1] = [x_3, x_1, x_2] = [x_3, x_1, x_3] = [x_3, x_1, x_4] = e,$$

$$[x_4, x_1, x_2] = [x_4, x_1, x_4] = [x_4, x_2, x_2] = [x_4, x_2, x_3] = [x_4, x_2, x_4] = [x_4, x_3, x_4] = e,$$

where δ and γ are fixed integers, $\delta \geq 1$, and $\gcd(\delta, \gamma) = 1$. Then $H \in qA$.

Proof. Since the factor group of a torsion-free nilpotent group by the center of the latter is also torsion-free [19, 16.2.10], it follows that $[x_2, x_1][x_4, x_3]^{-1} \in Z(H)$. Clearly, $H/\operatorname{gr}([x_2, x_1][x_4, x_3]^{-1}, \gamma_3(H))$ has the presentation in $\mathscr{N}_2$ as follows:

$$H/\operatorname{gr}([x_2, x_1][x_4, x_3]^{-1}, \gamma_3(H)) = \operatorname{gr}(x_1, x_2, x_3, x_4 \parallel [x_2, x_1] = [x_4, x_3]).$$

The nilpotent groups of class 2 with one defining relation (in $\mathscr{N}_2$) were studied in [23]. In particular, it is known [23, Lemma 3] that such a group belongs to $qF_2(\mathscr{N}_2)$ and every element of this group is uniquely written as

$$x_1^k x_2^m x_3^n x_4^p [x_2, x_1]^{r_1} [x_3, x_1]^{r_2} [x_4, x_1]^{r_3} [x_3, x_2]^{r_4} [x_4, x_2]^{r_5},$$

$k, m, n, p, r \in \mathbb{Z}$.

We claim that H can be approximated by the groups belonging to qA. To demonstrate this, we take a nonidentity element $g \in H$ and show that there is a homomorphism from H to a suitable group in qA that sends g to a nonidentity element. By the above, we can assume that $g \in \operatorname{gr}([x_2, x_1][x_4, x_3]^{-1}, \gamma_3(H))$.

Using commutator identities, we derive the following relations in H:

$$[x_2, x_1, x_1]^\delta = [x_4, x_3, x_1]^\delta = [x_3, x_1, x_4]^{-\delta} [x_4, x_1, x_3]^\delta = [x_4, x_1, x_3]^\delta,$$

$$[x_2, x_1, x_2]^\delta = [x_4, x_3, x_2]^\delta = [x_3, x_2, x_4]^{-\delta} [x_4, x_2, x_3]^\delta = e,$$

$$[x_2, x_1, x_3]^\delta = [x_4, x_3, x_3]^\delta, \quad [x_2, x_1, x_4]^\delta = [x_4, x_3, x_4]^\delta = e.$$

Since H and $H/Z(H)$ are torsion-free, this yields

$$[x_2, x_1, x_1] = [x_4, x_1, x_3] = z_2, \quad [x_2, x_1, x_2] = e, \tag{3}$$

$$[x_2, x_1, x_3] = [x_4, x_3, x_3] = z_3, \quad [x_2, x_1, x_4] = e. \tag{4}$$

We have $[x_2, x_1][x_4, x_3]^{-1} \in Z(H)$, and so g can be written as

$$g = [x_2, x_1]^r [x_4, x_3]^{-r} \prod_{i=1}^{4} z_i^{l_i}.$$

Hence g^δ has the form

$$g^\delta = \prod_{i=1}^{4} z_i^{k_i}.$$

Define the mapping $\tau : \{x_1, x_2, x_3, x_4\} \to A$ by setting

$$\tau : x_2 \to y^l [x, y]^{g_1}, \ x_1 \to x^m, \ x_4 \to y^p [x, y]^{g_2}, \ x_3 \to x^q,$$

where l, m, p, q, g_1, and g_2 are arbitrary integers such that

$$lm = pq, \quad g_1 = \frac{m-1}{2} l + \frac{ml\gamma}{\delta} - l\frac{\gamma}{\delta}, \quad g_2 = \frac{q-1}{2} p + \frac{qp\gamma}{\delta} - p\frac{\gamma}{\delta}.$$

Using (1) and (2), we calculate that

$$[\tau(x_2),\tau(x_1)]^\delta[\tau(x_2),\tau(x_1),\tau(x_1)]^\gamma$$
$$=[y,x]^{lm\delta}[y,x,x]^{\frac{m^2-m}{2}l\delta-g_1m\delta+lm^2\gamma}=([y,x]^\delta[y,x,x]^\gamma)^{lm}.$$

Similarly, $[\tau(x_4),\tau(x_3)]^\delta[\tau(x_4),\tau(x_3),\tau(x_3)]^\gamma=([y,x]^\delta[y,x,x]^\gamma)^{pq}$. Furthermore,

$$[\tau(x_3),\tau(x_1)]=[\tau(x_4),\tau(x_2)]=e,$$
$$e=[\tau(x_2),\tau(x_1),\tau(x_2)]=[\tau(x_4),\tau(x_1),\tau(x_2)]$$
$$=[\tau(x_4),\tau(x_1),\tau(x_4)]=[\tau(x_4),\tau(x_3),\tau(x_4)]$$
$$=[\tau(x_3),\tau(x_2),\tau(x_2)]=[\tau(x_3),\tau(x_2),\tau(x_4)].$$

Now, we consider the cases $l=z$, $m=z^{10}$, $p=z^3$, and $q=z^8$ for every z divisible by 2δ. By Dyck's Theorem, τ can be extended to a homomorphism $\varphi_z : H\to A$. Observe that $\varphi_z(g^\delta)=[y,x,x]^{pq^2k_3+pmqk_2+pm^2k_1-q^2lk_4}$.

If $\varphi_z(g^\delta)=e$ for all z under consideration, then $k_3z^{19}+k_2z^{21}+k_1z^{23}-k_4z^{17}=0$. Since a nonzero polynomial has only finitely many roots, we conclude that $k_1=k_2=k_3=k_4=0$, i.e. $g^\delta=e$, whence $g=e$. The proof is complete.

Demonstrating Lemma 7, we established the uniqueness of decomposition of the elements of $\gamma_3(H)$. It is not difficult to see that the more general result holds:

Corollary 1. *Every element of H is uniquely written as*

$$x_1^kx_2^mx_3^nx_4^p[x_2,x_1]^{r_1}[x_3,x_1]^{r_2}[x_4,x_1]^{r_3}[x_3,x_2]^{r_4}[x_4,x_2]^{r_5}[x_4,x_3]^{r_6}\prod_{i=1}^{4}z_i^{k_i},$$

where $k,m,n,p,r_i,k_j\in\mathbb{Z}$, $0\le r_6<\delta$.

Proof. It suffices to prove that the identity can be uniquely decomposed as claimed. We noted above that the decomposition of elements of $H/\operatorname{gr}([x_2,x_1][x_4,x_3]^{-1},\gamma_3(H))$ is unique, and so

$$e=[x_2,x_1]^m[x_4,x_3]^r\prod_{i=1}^{4}z_i^{k_i},\quad 0\le r<\delta.$$

Considering the homomorphism $\psi : H\to A$ such that $\psi(x_1)=x$, $\psi(x_2)=y$, $\psi(x_3)=x$, and $\psi(x_4)=y$, we conclude that $m=-r$. Also, by (3) and (4) $e^\delta=z_1^{\delta k_1}z_2^{\gamma r+\delta k_2}z_3^{-\gamma r+\delta k_3}z_4^{\delta k_4}$. Using the uniqueness of decomposition for the elements of $\gamma_3(H)$, we see that $k_1=0$, $k_4=0$, $\gamma r+\delta k_2=0$, and $-\gamma r+\delta k_3=0$. Since $\gcd(\delta,\gamma)=1$, it follows that δ divides r. But $r<\delta$, and so $r=0$, which yields $k_2=k_3=0$. The proof is complete.

The following is an easy consequence of Corollary 1:

Corollary 2. $\operatorname{gr}(x_1,x_2)\cap\operatorname{gr}(x_3,x_4)=([x_2,x_1]^\delta[x_2,x_1,x_1]^\gamma)$ *in the group H.*

Proof. Take $g\in\operatorname{gr}(x_1,x_2)\cap\operatorname{gr}(x_3,x_4)$. Since $H/H'=(\bar{x}_1)\times(\bar{x}_2)\times(\bar{x}_3)\times(\bar{x}_4)$, it follows that g can be written as (we use that $[x_2,x_1,x_1]=z_2$ by (3)):

$$g=[x_2,x_1]^{r_1}[x_2,x_1,x_1]^{l_1}=[x_2,x_1]^{r_1}z_2^{l_1},\quad g=[x_4,x_3]^{r_2}z_3^{l_2},$$

i.e., $e=[x_2,x_1]^{r_1}z_2^{l_1}[x_4,x_3]^{-r_2}z_3^{-l_2}$. By Corollary 1, we see that δ divides r_2. Let $r_2=\delta s$. Then

$$g=[x_4,x_3]^{\delta s}z_3^{l_2}=\left([x_2,x_1]^\delta z_2^\gamma z_3^{-\gamma}\right)^s z_3^{l_2}=[x_2,x_1]^{r_2}z_2^{\gamma s}z_3^{-s\gamma+l_2}.$$

Again applying Corollary 1, we deduce that $r_1=r_2$, $l_1=s\gamma$, $-s\gamma+l_2=0$. This yields

$$g=[x_2,x_1]^{\delta s}[x_2,x_1,x_1]^{\gamma s}\in([x_2,x_1]^\delta[x_2,x_1,x_1]^\gamma),$$

and the proof is complete.

Lemma 8. *Suppose that H_d is defined in $\mathscr{N}_{3,\infty}$ by the generators x_1, x_2, x_3, x_4, t and the relations of H together with the relation*

$$t^d = [x_2, x_1]^\delta [x_2, x_1, x_1]^\gamma = [x_4, x_3]^\delta [x_4, x_3, x_3]^\gamma.$$

Then the mapping

$$\tau : t \to [x_2, x_1]^\delta [x_2, x_1, x_1]^\gamma, x_2 \to x_2^l [x_1, x_2]^{f_1}, x_1 \to x_1^m, x_4 \to x_4^p [x_3, x_4]^{f_2}, x_3 \to x_3^q,$$

where d, l, m, p, q, f_1, and f_2 are integers such that

$$d = lm = pq > 0, \quad f_1 = \frac{m-1}{2} l + \frac{ml\gamma}{\delta} - \frac{l\gamma}{\delta}, \quad f_2 = \frac{q-1}{2} p + \frac{qp\gamma}{\delta} - \frac{p\gamma}{\delta},$$

d is a multiple of δ, and $\gcd(\delta, \gamma) = 1$, can be extended to an embedding of H_d into H.

Proof. As in the proof of Lemma 7, we check that this mapping can be extended to a homomorphism $\varphi : H_d \to H$. Now we calculate $\ker \varphi$. To this end, we first prove some properties of H_d.

The relation $t^d = [x_2, x_1]^\delta [x_2, x_1, x_1]^\gamma$ implies that $[t^d, u, v] = [t, u, v]^d = e$ for all $u, v \in H_d$. But H_d is torsion-free, and so $[t, u, v] = e$, in particular, $[t, x_1], [t, x_3] \in Z(H_d)$. The relation $[x_2, x_1, x_2] = [x_4, x_3, x_4] = e$ yields $[t^d, x_2] = [t, x_2]^d = e$, i.e., $[t, x_2] = e$. Similarly, $[t, x_4] = e$.

Suppose that $e \neq g \in \ker \varphi$. If $g = t^k x_1^u x_2^r x_3^n x_4^f c, c \in H_d'$, then $\varphi(g) = x_1^{mu} x_2^{lr} x_3^{qn} x_4^{pf} c' = e$, $c' \in H'$, whence $u = r = n = f = 0$. Thus g can be written as follows:

$$g = t^k [t, u] [x_2, x_1]^{r_1} [x_3, x_1]^{r_2} [x_4, x_1]^{r_3} [x_3, x_2]^{r_4} [x_4, x_2]^{r_5} [x_4, x_3]^{r_6} c,$$

$c \in \gamma_3(H_d)$, $0 \leq k < d$, $0 \leq r_6 < \delta$.

Let σ_{ij} be the sum of the exponents of $[x_i, x_j]$ in the factorization of $\varphi(g)$. Then

$$\sigma_{31} = qmr_2, \quad \sigma_{41} = pmr_3, \quad \sigma_{32} = qlr_4, \quad \sigma_{42} = plr_5.$$

Since $\varphi(g) = e$, Corollary 1 shows that all these exponents are equal to 0. This yields $r_i = 0$, $2 \leq i \leq 5$. As $[t, x_i]^w = [t^w, x_i]$ for all integer w, $t^d \in \operatorname{gr}(x_1, x_2)'$ and d is a multiple of δ, it follows that g^d has the form $g^d = [x_2, x_1]^v c$, $c \in \gamma_3(H_d)$. Also $e = \varphi(g^d) = [x_2, x_1]^{lmv} c_1$ for some suitable $c_1 \in \gamma_3(H)$, and so $v = 0$. Thus we can write g^d as

$$g^d = \prod_{i=1}^{4} z_i^{k_i}.$$

Using the uniqueness of factorization of $\varphi\big(\prod_{i=1}^4 z_i^{k_i}\big)$ in H, we conclude that all k_i are equal to 0, and so $g^d = e$, i.e., $g = e$. Thus $\ker \varphi = (e)$, and the proof is complete.

The proof of the above lemma implies

Corollary 3. *Suppose that $d = lm = pq > 0$, $\gcd(\delta, \gamma) = 1$ and d is a multiple of δ. Then every element of $\operatorname{gr}(x_1, x_2, x_3, x_4) \leq H_d$ can be uniquely written as in Corollary 1.*

Corollary 4. *In the group H_d defined in Lemma 8, we have $\operatorname{gr}(x_1, x_2, t) \cap \operatorname{gr}(x_3, x_4, t) = (t)$.*

Proof. Take $g \in \operatorname{gr}(x_1, x_2, t) \cap \operatorname{gr}(x_3, x_4, t)$. The embedding φ defined in Lemma 8 sends g to $\varphi(g) \in \operatorname{gr}(x_1, x_2) \cap \operatorname{gr}(x_3, x_4)$. By Corollary 2,

$$\operatorname{gr}(x_1, x_2) \cap \operatorname{gr}(x_3, x_4) = ([x_2, x_1]^\delta [x_2, x_1, x_1]^\gamma) = (\varphi(t)).$$

It follows that $\varphi(g) = \varphi(t)^k$ for some k. This shows, as φ is an embedding, that $g = t^k$ which completes the proof.

Lemma 9. *The mapping $\tau : x_i \to x_i$ ($i = 1, 2, 3, 4$), $t \to t^c$, can be extended to an embedding $\varphi : H_d \to H_{dc}$, where $d = lm = pq > 0$, $\gcd(\delta, \gamma) = 1$, and d is a multiple of δ.*

Proof. The existence of a homomorphism φ follows from Dyck's Theorem. If $g \in \ker \varphi$, then $g^d \in \mathrm{gr}(x_1, x_2, x_3, x_4)$. By Corollary 3, the canonical decompositions of g^d and $\varphi(g^d)$ coincide. Thus $g^d = e$, i.e., $\ker \varphi = (e)$, and this completes the proof.

Lemma 10. *Suppose that G is a torsion-free nilpotent group of class 3, $\mathscr{M}$ is an arbitrary quasivariety of groups, and A has the presentation in $\mathscr{N}_3$ as follows:*

$$A = \mathrm{gr}(x_1, x_2 \parallel [x_2, x_1, x_2] = 1),\ G = \mathrm{gr}(A, Q) \in \mathscr{M},\ A \cap Q = ([x_2, x_1]^\delta [x_2, x_1, x_1]^\gamma).$$

Then $\mathrm{dom}_G^{\mathscr{M}}(Q) = Q$.

Proof. Take a generating set S of G to consist of x_1, x_2 and all elements of Q. Let Σ consist of the relations

$$[x_2, x_1, x_2] = e, \quad q = [x_2, x_1]^\delta [x_2, x_1, x_1]^\gamma,$$

and all defining relations of Q, where q is a suitable element of Q. We show first that $\mathrm{gr}(S \parallel \Sigma)$ is some presentation of G in the class $\mathscr{N}_{3,\infty}$.

In the class of torsion-free groups, the relations

$$[x_2, x_1, x_2] = e, \quad q = [x_2, x_1]^\delta [x_2, x_1, x_1]^\gamma$$

yield $[q, x_2] = e$ and $[q, x_1, x_1] = e$. Let $t \in Q$ be a nonidentity element. Since $t^n = q^m$ for some nonzero integers n and m, we can deduce in $\mathscr{N}_{3,\infty}$ from the above relations that $[t, x_2] = e$ and $[t, x_1, x_1] = e$. Furthermore, using commutator identities, we find that $[t^m, x_1] = [t, x_1]^m$ for all $t \in Q$, $m \in \mathbb{Z}$.

Now consider an arbitrary defining relation of G. By the above, we can assume that it has the form

$$t_1 x_1^k x_2^n [x_2, x_1]^l [x_2, x_1, x_1]^p [t_2, x_1] = e,$$

where $t_1, t_2 \in Q$, $k, n, l, p \in \mathbb{Z}$, $|t_1| < |q|, |t_2| < |q|$. It is clear that $Z(G) \cap A = ([x_2, x_1, x_1])$ and $q \notin Z(G)$. Since $G/Z(G)$ is torsion-free, $Q \cap Z(G) = (e)$. Hence $G/Z(G)$ is a direct product with amalgamation of its subgroups; i.e.,

$$G/Z(G) \cong A/([x_2, x_1, x_1]) \times Q([\bar{x}_1, \bar{x}_2] = \sqrt[\delta]{q}).$$

In $G/Z(G)$ the relation under consideration takes the form

$$t_1 \bar{x}_1^k \bar{x}_2^n [\bar{x}_2, \bar{x}_1]^l = e(|t_1| < |q|).$$

It follows from the structure of the above direct product with amalgamation that $t_1 = e$, $k = 0$, $n = 0$, and $l = 0$; i.e., the relation has the form

$$[x_2, x_1, x_1]^p [t_2, x_1] = e(|t_2| < |q|).$$

Suppose that $t_2 \neq e$. Choose nonzero integers r and m ($r > 0$) such that $t_2^r = q^m$. Then the derived relation is equivalent in the class $\mathscr{N}_{3,\infty}$ to the relation $[x_2, x_1, x_1]^{pr} [t_2, x_1]^r = e$. We have

$$\begin{aligned} e &= [x_2, x_1, x_1]^{pr} \left[t_2^r, x_1\right] = [x_2, x_1, x_1]^{pr} [q^m, x_1] \\ &= [x_2, x_1, x_1]^{pr} [q, x_1]^m = [x_2, x_1, x_1]^{pr + m\delta}. \end{aligned}$$

Thus our relation is equivalent in $\mathscr{N}_{3,\infty}$ to the relation $[x_2, x_1, x_1]^{pr+m\delta} = e$. Since $[x_2, x_1, x_1] \neq e$ in G, it follows that $pr + m\delta = 0$. Hence the last relation is equivalent in $\mathscr{N}_{3,\infty}$ to the trivial relation $[x_2, x_1, x_1]^0 = e$. This yields $t_2 = e$, and so $p = 0$. Thus we proved that every relation of G is trivial, i.e., it follows in $\mathscr{N}_{3,\infty}$ from Σ.

By Lemma 9, if k is sufficiently large ($k > \delta + 1$), we can assume that

$$H_{k!} \subseteq H_{(k+1)!} \subseteq H_{(k+2)!} \subseteq \cdots .$$

Furthermore, we can choose a set $S_{n!}$ of generators and a set $\Sigma_{n!}$ of relations for $H_{n!}$ so that

$$S_{k!} \subseteq S_{(k+1)!} \subseteq S_{(k+2)!} \subseteq \cdots, \Sigma_{k!} \subseteq \Sigma_{(k+1)!} \subseteq \Sigma_{(k+2)!} \subseteq \cdots .$$

Note that $\Sigma_{n!}$ consists of the defining relations of H and the relations

$$t_1 = [x_2, x_1]^{\delta}[x_2, x_1, x_1]^{\gamma}, \quad t_i^i = t_{i-1}, \quad i = 2, 3, \dots, n!.$$

It is known [20, Problem 16, Section 3] that in this situation the group $K = \bigcup_{n\geq k} H_{n!}$ has the presentation $K = \operatorname{gr}\left(\bigcup_{n\geq k} S_{n!} \parallel \bigcup_{n\geq k} \Sigma_{n!}\right)$. By Lemmas 7 and 8, $H_n \in qA$ for all $n \geq k$. Hence $K \in qA$. By Corollary 4,

$$\operatorname{gr}(x_1, x_2, t_1, t_2, \dots) \cap \operatorname{gr}(x_3, x_4, t_1, t_2, \dots) = \operatorname{gr}(t_1, t_2, \dots).$$

Let $q_1, q_2, q_3, \dots$ be elements of the group $Q \leq G$ such that

$$q_1 = q,\ q_2^2 = q_1, \dots, q_{k+1}^{k+1} = q_k, \dots .$$

Applying Dyck's Theorem, we see that the mappings $x_1 \to x_1$, $x_2 \to x_2$, $q_i \to t_i$ ($i \in \mathbb{N}$) and $x_1 \to x_3$, $x_2 \to x_4$, $q_i \to t_i$ ($i \in \mathbb{N}$) can be extended to homomorphisms from G to K, which we denote by λ and ρ respectively.

Let g be an arbitrary element of G and suppose that $\lambda(g) = \rho(g)$. We can write g as (since $[q_i, x_2] = e$):

$$g = q_i^k [q_i, x_1]^r [x_2, x_1]^l [x_2, x_1, x_1]^m$$

for some i. Observing that $q = [x_2, x_1]^{\delta}[x_2, x_1, x_1]^{\gamma}$, we have that some power of g has the form

$$e \neq g^f = [x_2, x_1]^r [x_2, x_1, x_1]^p.$$

Then

$$\lambda(g^f) = [x_2, x_1]^r [x_2, x_1, x_1]^p = \rho(g^f) = [x_4, x_3]^r [x_4, x_3, x_3]^p.$$

We can assume that $\lambda(g^f)$ belongs to some subgroup H_d, and by Corollary 3 $\operatorname{gr}(x_1, x_2, x_3, x_4)$ is equal to H. Since $\lambda(g^f) \in \operatorname{gr}(x_1, x_2) \cap \operatorname{gr}(x_3, x_4)$, Corollary 2 implies that

$$\lambda(g^f) = [x_2, x_1]^r [x_2, x_1, x_1]^p = ([x_2, x_1]^{\delta}[x_2, x_1, x_1]^{\gamma})^w$$

for some suitable w. This yields $r = \delta w$ and $p = \gamma w$. Hence

$$g^f = ([x_2, x_1]^{\delta}[x_2, x_1, x_1]^{\gamma})^w \in A \cap Q.$$

By the uniqueness of root extraction in a torsion-free nilpotent group, we conclude that $g \in Q$.

Thus $K \in qA$, and λ, ρ coincide exactly on Q. By the definition of dominion, $\operatorname{dom}_G^{qA}(Q) = Q = \operatorname{dom}_G^{\mathscr{M}}(Q)$. This completes the proof.

§ 3. The Group T

Lemma 11. *Let T be the group defined in $\mathscr{N}_3$ by the generators x_1, x_2, x_3, x_4 and the relations*

$$[x_2, x_1, x_1] = [x_4, x_3, x_3], \quad [x_2, x_1, x_2] = [x_2, x_1, x_4] = e,$$

$$[x_3, x_2, x_2] = [x_3, x_2, x_4] = [x_3, x_1, x_1] = [x_3, x_1, x_2] = [x_3, x_1, x_3] = [x_3, x_1, x_4] = e,$$

$$[x_4, x_1, x_2] = [x_4, x_1, x_4] = [x_4, x_2, x_2] = [x_4, x_2, x_3] = [x_4, x_2, x_4] = [x_4, x_3, x_4] = e.$$

Then $T \in qA$.

Proof. It is clear that $T/\gamma_3(T)$ is a $\mathscr{N}_2$-free group, and so it belongs to $qF_2(\mathscr{N}_2) \subseteq qA$.

We argue that T can be approximated by groups in qA. To this end, suppose that g is a nonidentity element of T and show that there exists a homomorphism from T to a suitable group in qA which sends g to a nonidentity element. By the above, we can assume that $g \in \gamma_3(T)$.

Thus g can be written as

$$g = \prod_{i=1}^{5} z_i^{k_i}.$$

Consider the mapping $\tau : x_2 \to y^l$, $x_1 \to x^m$, $x_4 \to y^p$, $x_3 \to x^q$, where $lm^2 = pq^2$. Easy calculation shows that

$$[\tau(x_3), \tau(x_1)] = [\tau(x_4), \tau(x_2)] = e,$$

$$[\tau(x_2), \tau(x_1), \tau(x_1)] = [\tau(x_4), \tau(x_3), \tau(x_3)],$$

$$\begin{aligned}[\tau(x_2), \tau(x_1), \tau(x_2)] &= [\tau(x_2), \tau(x_1), \tau(x_4)] = [\tau(x_3), \tau(x_2), \tau(x_2)] \\ &= [\tau(x_3), \tau(x_2), \tau(x_4)] = [\tau(x_4), \tau(x_1), \tau(x_2)] \\ &= [\tau(x_4), \tau(x_1), \tau(x_4)] = [\tau(x_4), \tau(x_3), \tau(x_4)] = e.\end{aligned}$$

Now consider the case $l = z$, $m = z^{10}$, $p = z^3$, and $q = z^9$ for every integer z. By Dyck's Theorem, τ can be extended to a homomorphism $\varphi_z : H \to A$. We have

$$\varphi_z(g) = [y, x, x]^{pm^2k_1 + pmqk_2 + pq^2k_3 - q^2lk_4 + lmqk_5}.$$

If $\varphi_z(g) = e$ for every z under consideration, then

$$k_1 z^{23} + k_2 z^{22} + k_3 z^{21} - k_4 z^{19} + k_5 z^{20} = 0.$$

Since a nonzero polynomial has only finitely many roots, $k_1 = k_2 = k_3 = k_4 = k_5 = 0$. This means that $g = e$, as required.

The proof of Lemma 11 yields

Corollary 5. *Every element of T is uniquely presented in the form*

$$x_1^k x_2^m x_3^n x_4^p \prod_{1 \le i < j \le 4} [x_j, x_i]^{r_{ji}} \prod_{i=1}^{5} z_i^{k_i},$$

where $k, m, n, p, r_{li}, k_j \in \mathbb{Z}$.

Corollary 6. $\mathrm{gr}(x_1, x_2) \cap \mathrm{gr}(x_3, x_4) = ([x_2, x_1, x_1])$ *in the group T.*

Proof. Given an arbitrary element $g \in \mathrm{gr}(x_1, x_2) \cap \mathrm{gr}(x_3, x_4)$, we can write

$$g = x_1^k x_2^m [x_2, x_1]^{r_{21}} [x_2, x_1, x_1]^{l_1} = x_3^n x_4^p [x_4, x_3]^{r_{43}} [x_4, x_3, x_3]^{k_1}.$$

Since $T/\gamma_3(T)$ is a free nilpotent group of class 2 and rank 4, it follows that $k = m = r_{21} = 0$, and hence $g = [x_2, x_1, x_1]^{l_1} \in ([x_2, x_1, x_1])$. This completes the proof.

Lemma 12. *Suppose that T_d is defined in $\mathscr{N}_{3,\infty}$ by the generators x_1, x_2, x_3, x_4, t, the defining relations of the group T, and the relation $t^d = [x_2, x_1, x_1]$. Then the mapping*

$$\tau : t \to [x_2, x_1, x_1],\ x_2 \to x_2^l,\ x_1 \to x_1^m,\ x_4 \to x_4^p,\ x_3 \to x_3^q,$$

where $d = lm^2 = pq^2 > 0$, can be extended to an embedding of T_d into T.

Proof. We check that this mapping can be extended to a homomorphism $\varphi : T_d \to T$ by reasoning as in the proof of Lemma 11. Now we calculate $\ker \varphi$. To this end, observe first that $t \in Z(T_d)$ because the factor group of a torsion-free nilpotent group by its center is also torsion-free.

Let $e \neq g \in \ker\varphi$. If $g = x_1^k x_2^r x_3^n x_4^f c, c \in T'_d$, then

$$\varphi(g) = x_1^{mk} x_2^{lr} x_3^{qn} x_4^{pf} c' = e, \quad c' \in T',$$

and so $k = r = n = f = 0$. Hence we can write g in the form

$$g = t^k [x_2, x_1]^{r_1} [x_3, x_1]^{r_2} [x_4, x_1]^{r_3} [x_3, x_2]^{r_4} [x_4, x_2]^{r_5} [x_4, x_3]^{r_6} c_1,$$

$c_1 \in \gamma_3(T_d)$, $0 \leq k < d$. Let σ_{ij} be the sum of the exponents of the commutator $[x_i, x_j]$ in the factorization of $\varphi(g)$. Then

$$\sigma_{21} = lmr_1, \quad \sigma_{31} = qmr_2, \quad \sigma_{41} = pmr_3, \quad \sigma_{32} = qlr_4, \quad \sigma_{42} = plr_5, \quad \sigma_{43} = pqr_6.$$

Since $\varphi(g) = e$, Corollary 5 shows that all these exponents are equal to 0. This yields $r_i = 0$, $i \geq 5$. Thus

$$g = t^k \prod_{i=1}^{5} z_i^{k_i}, \quad 0 \leq k < d.$$

Let σ_{iju} be the sum of the exponents of the commutator $[x_i, x_j, x_u]$ in the decomposition of $\varphi(g)$. Noting that $[x_2, x_1, x_1] = [x_4, x_3, x_3]$, we have

$$\sigma_{433} = pq^2 k_3 + k, \quad \sigma_{413} = pmqk_2, \quad \sigma_{411} = pm^2 k_1, \quad \sigma_{323} = q^2 lk_4, \quad \sigma_{213} = lmqk_5.$$

By the uniqueness of the factorization of elements in T, we conclude that $k_1 = k_2 = k_4 = k_5 = 0$ and $pq^2 k_3 + k = 0$, which implies that k is a multiple of pq^2. But $pq^2 = d$, with $0 \leq k < d$, and so $k = 0$, and then $k_3 = 0$. Thus $g = e$, which means that $\ker\varphi = (e)$, as required.

The proof of this lemma yields

Corollary 7. *Every element of T_d is uniquely presented as*

$$x_1^l x_2^r x_3^n x_4^f t^k [x_2, x_1]^{r_1} [x_3, x_1]^{r_2} [x_4, x_1]^{r_3} [x_3, x_2]^{r_4} [x_4, x_2]^{r_5} [x_4, x_3]^{r_6} \prod_{i=1}^{5} z_i^{k_i},$$

where $0 \leq k < d$, $k, l, m, n, p, r_i, k_j \in \mathbb{Z}$.

Corollary 8. $\mathrm{gr}(x_1, x_2, t) \cap \mathrm{gr}(x_3, x_4, t) = (t)$ *in the group T_d.*

Proof. Let $g \in \mathrm{gr}(x_1, x_2, t) \cap \mathrm{gr}(x_3, x_4, t)$. The embedding φ of Lemma 12 sends g to $\varphi(g) \in \mathrm{gr}(x_1, x_2) \cap \mathrm{gr}(x_3, x_4)$. Corollary 6 yields $\mathrm{gr}(x_1, x_2) \cap \mathrm{gr}(x_3, x_4) = ([x_2, x_1, x_1]) = (\varphi(t))$. It follows that $\varphi(g) = \varphi(t)^k$ for some k. Hence $g = t^k$ since φ is an embedding, and this completes the proof.

Lemma 13. *The mapping $\tau : x_i \to x_i$ $(i = 1, 2, 3, 4)$, $t \to t^c$ can be extended to an embedding $\varphi : T_d \to T_{dc}$.*

Proof. The existence of a homomorphism φ is guaranteed by Dyck's Theorem. It remains to calculate $\ker\varphi$. If a nonidentity element $g \in T_d$ has the form as in Corollary 7, then $\varphi(g) \neq e$; i.e., $\ker\varphi = (e)$ as required.

Lemma 14. *Let G be a torsion-free nilpotent group of class 3 and let $\mathscr{M}$ be a quasivariety of groups. Suppose that A has the presentation in $\mathscr{N}_3$ as follows:*

$$A = \mathrm{gr}(x_1, x_2 \parallel [x_2, x_1, x_2] = 1), \quad G = \mathrm{gr}(A, Q) \in \mathscr{M}, \quad A \cap Q = ([x_2, x_1, x_1]).$$

Then $\mathrm{dom}_G^{\mathscr{M}}(Q) = Q$.

Proof. We take the elements x_1, x_2 and all elements of Q as a generating set S of G. Let Σ be the set of relations $[x_2, x_1, x_2] = e$ and $q = [x_2, x_1, x_1]$, and all defining relations of Q, where q is a suitable element of Q. We show first that $\mathrm{gr}(S \parallel \Sigma)$ is a presentation of G in $\mathscr{N}_{3,\infty}$.

Take a nonzero element $t \in Q$. Since $t^n = q^m$ for some nonzero integers n and m, the above relations imply in $\mathscr{N}_{3,\infty}$ the relations $[t, x_2] = e$ and $[t, x_1] = e$.

Suppose that we have an arbitrary defining relation of G. By the above, we can assume that the relation has the form

$$t_1 x_1^k x_2^n [x_2, x_1]^l [x_2, x_1, x_1]^p = e,$$

where $k, n, l, p \in \mathbb{N}$, $|t_1| < |q|$. It is clear that $Z(G) \cap A = ([x_2, x_1, x_1])$ and $q \in Z(G)$. In $G/Z(G)$, the relation becomes $\bar{x}_1^k \bar{x}_2^n [\bar{x}_2, \bar{x}_1]^l = e$. This yields $k = 0$, $n = 0$, and $l = 0$; i.e., the relation has the form $t_1 [x_2, x_1, x_1]^p = e(|t_1| < |q|)$.

Suppose that $t_1 \neq e$. Let r and m be nonzero integers such that $t_1^r = q^m$. Since $|t_1| < |q|$, we have $|r| > |m|$. Hence

$$e = t_1^r [x_2, x_1, x_1]^{pr} = q^m [x_2, x_1, x_1]^{pr} = [x_2, x_1, x_1]^{m+pr}.$$

It follows that $m + pr = 0$, and so r divides m. This yields $|\frac{m}{r}| \geq 1$, which is not the case. Hence $t_1 = e$, and therefore $p = 0$. Thus we proved that any relation of G is trivial, i.e., it follows from Σ in the class $\mathscr{N}_{3,\infty}$.

By Lemma 13, we can assume that $T_{1!} \subseteq T_{2!} \subseteq T_{3!} \subseteq T_{4!} \subseteq \cdots$. Moreover, a generating set $S_{n!}$ and a set $\Sigma_{n!}$ of defining relations of the group $T_{n!}$ can be chosen so that

$$S_{1!} \subseteq S_{2!} \subseteq S_{3!} \subseteq S_{4!} \subseteq \cdots, \quad \Sigma_{1!} \subseteq \Sigma_{2!} \subseteq \Sigma_{3!} \subseteq \Sigma_{4!} \subseteq \cdots.$$

Observe that $\Sigma_{n!}$ consists of the defining relations of T and the relations $t_1 = [x_2, x_1, x_1]$, $t_i^i = t_{i-1}$, $i = 2, 3, \dots, n$. It is well known that in this situation the group $K = \bigcup_{n \in \mathbb{N}} T_{n!}$ has the presentation

$$K = \operatorname{gr}\Big(\bigcup_{n \in \mathbb{N}} S_{n!} \parallel \bigcup_{n \in \mathbb{N}} \Sigma_{n!}\Big).$$

By Lemmas 11 and 12, $T_n \in qA$ for all n. Hence $K \in qA$. By Corollary 8,

$$\operatorname{gr}(x_1, x_2, t_1, t_2, \dots) \cap \operatorname{gr}(x_3, x_4, t_1, t_2, \dots) = \operatorname{gr}(t_1, t_2, \dots).$$

We take some elements $q_1, q_2, q_3, q_4, \dots$ of the group $Q \leq G$ such that $q_1 = q$, $q_2^2 = q_1, \dots, q_k^k = q_{k-1}, \dots$. Applying Dyck's Theorem, we conclude that the mappings $x_1 \to x_1$, $x_2 \to x_2$, $q_i \to t_i$ $(i \in \mathbb{N})$ and $x_1 \to x_3$, $x_2 \to x_4$, $q_i \to t_i$ $(i \in \mathbb{N})$ can be extended to homomorphisms respectively λ and ρ from G to K.

Let g be an arbitrary element of G and suppose that $\lambda(g) \in \operatorname{gr}(t_1, t_2, \dots)$. We can write g as $g = q_i^k [x_2, x_1]^{r_1}$ for some i. Then $\lambda(g) = t_i^k [x_2, x_1]^{r_1} \in \operatorname{gr}(t_1, t_2, \dots)$, i.e., $t_i^k [x_2, x_1]^{r_1} = t_j^m$ for suitable j and m. We can assume that all elements in this equality belong to some subgroup T_d. Applying Corollary 8 to this subgroup, we have $r_1 = 0$, which means that $g \in \operatorname{gr}(q_1, q_2, \dots)$. Similarly, if $\rho(g) \in \operatorname{gr}(t_1, t_2, \dots)$, then $g \in \operatorname{gr}(q_1, q_2, \dots)$.

Thus $K \in qA$, while λ and ρ coincide exactly on Q. By the definition of dominion, $\operatorname{dom}_G^{qA}(Q) = Q = \operatorname{dom}_G^{\mathscr{M}}(Q)$. The proof is complete.

§ 4. The Main Result

Theorem 1. *Let $\mathscr{M}$ be an arbitrary quasivariety of torsion-free nilpotent groups of class at most 3 and let Q be the additive group of the rationals. Then Q is 2-closed in $\mathscr{M}$.*

PROOF. Suppose that $G = \operatorname{gr}(x_1, x_2, Q)$ and let $H = \operatorname{gr}(x_1, x_2)$.

CASE 1. H is a nilpotent group of class 3 and $Q \cap G' \neq (e)$.

By Lemma 4, $H \cong F_2$ or $H \cong A$. Fix some nonidentity element $q \in Q \cap G'$. We claim that $[Q, G'] = (e)$. Indeed, let $q_1 \in Q$, $q_1 \neq e$, and let $g \in G'$. Since Q is a locally cyclic group, $q_1^m = q^n$ for suitable integers m and n. But $\big[q_1^m, g\big] = [q^n, g] = e$. It is known [19, 16.2.9] that the elements of a torsion-free nilpotent group commute whenever some nonidentity powers of these elements commute. Thus $[q_1, g] = e$, which means that $[Q, G'] = (e)$.

We argue now that $[Q, G] \leq Z(G)$. Take arbitrary elements $q_1 \in Q, g, f \in G$ and nonzero integers m and n such that $q_1^m = q^n$. Then $[q^n, f] \in Z(G)$ since $q \in G'$. This yields $[q_1, f, g]^m = \left[q_1^m, f, g\right] = [q^n, f, g] = e$. Hence $[q_1, f, g] = e$, and thus $[Q, G] \leq Z(G)$.

By the above, since $q \in G'$, q is a product of basic commutators, and so q can be written as

$$q = [x_2, x_1]^k [x_2, x_1, x_1]^{l_1} [x_2, x_1, x_2]^{l_2} [x_1, q_1]^l [x_2, q_2]^s$$

for suitable integers k, l_1, l_2, l, and s and nonidentity elements $q_1, q_2 \in Q$ (possibly with $l = 0$ or $s = 0$).

Let r, p, and t be nonzero integers such that $q_1^r = q^p$ and $q_2^r = q^t$. Observing that $\left[x_1, q_1^r\right] = [x_1, q_1]^r$ and $\left[x_2, q_2^r\right] = [x_2, q_2]^r$, we have

$$\begin{aligned} e \neq q^r &= [x_2, x_1]^{kr} [x_2, x_1, x_1]^{l_1 r} [x_2, x_1, x_2]^{l_2 r} \left[x_1, q_1^r\right]^l \left[x_2, q_2^r\right]^s \\ &= [x_2, x_1]^{kr} [x_2, x_1, x_1]^{l_1 r} [x_2, x_1, x_2]^{l_2 r} [x_1, q^p]^l [x_2, q^t]^s \\ &= [x_2, x_1]^{kr} [x_2, x_1, x_1]^{l_1 r} [x_2, x_1, x_2]^{l_2 r} [x_1, [x_2, x_1]]^{pkl} [x_2, [x_2, x_1]]^{tks} \in H'. \end{aligned}$$

Thus $H' \cap Q \neq (e)$. Since Q is a locally cyclic group, $H \cap Q = (w)$. Then $w^m \in H'$ for a suitable nonzero integer m, and so $w \in H'$.

If $w \notin \gamma_3(H)$, then w has the form

$$w = ([x_2, x_1]^\delta [x_2, x_1, x_1]^s [x_2, x_1, x_2]^t)^r \quad (\delta \neq 0).$$

Let $w \in \gamma_3(H)$. We can write w as $w = [x_2, x_1, x_2]^k [x_2, x_1, x_1]^m$ for some integers k and m with $\gcd(k, m) = 1$. Now we choose new generators of H as in the proof of Lemma 1 in [1]. Namely, let u and v be integers such that $mv + ku = 1$. Then

$$\left[x_2^v x_1^{-u}, x_2^k x_1^m, x_2^k x_1^m\right] = [x_2, x_1, x_2]^{k(mv+uk)} [x_2, x_1, x_1]^{m(mv+uk)} = [x_2, x_1, x_2]^k [x_2, x_1, x_1]^m = w.$$

We take $y_2 = x_2^v x_1^{-u}$ and $y_1 = x_2^k x_1^m$ as the required generators of H. With respect to these generators, we have $w = [y_2, y_1, y_1]$. Since a torsion-free nilpotent group is a group with unique root extraction, $r = \pm 1$. Lemmas 5, 6, 10, and 14 imply that $\operatorname{dom}_G^{\mathscr{M}}(Q) = Q$.

Case 2. $Q \cap G' = (e)$. The result is immediate from Lemma 3.

Case 3. $Q \cap G' \neq (e)$ and H is a nilpotent group of class at most 2. Reasoning as in Case 1, we have $[Q, G'] = (e)$ and $[Q, G] \leq Z(G)$. We claim that all commutators of the form $[u, v, w]$, $u, v, w \in Q \cup \{x_1, x_2\}$ are equal to the identity.

Given $q_1, q_2 \in Q$, we have $[u, q_1, q_2] = [q_1, q_2, u]^{-1} [q_2, u, q_1]^{-1} = e$ since $[q_2, u] \in Z(G)$. This shows that $[q_1, u, q_2] = [u, q_1, q_2]^{-1} = e$ for $q_1 \in Q$, $u, v \in \{x_1, x_2\}$, $[u, v, q_1] = e$, $[u, q_1, v] = e$, and $[q_1, v, u] = e$ because $[Q, G'] = (e)$ and $[q_1, u] \in Z(G)$.

Thus G is a nilpotent group of class at most 2. By Lemma 2, the group Q is closed in G in any quasivariety of nilpotent groups of class at most 2 (for example, in qG). By the definition of dominion, Q is closed in each quasivariety that includes qG, and in particular in $\mathscr{M}$. We considered all possibilities, and so the proof is complete.

In fact, we proved the following

Corollary 9. *Let $\mathscr{M}$ be an arbitrary quasivariety of groups and let $G = \operatorname{gr}(x_1, x_2, Q)$ be a torsion-free nilpotent group of class 3 in $\mathscr{M}$ generated modulo Q by two elements. Then $\operatorname{dom}_G^{\mathscr{M}}(Q) = Q$.*

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