

Alternating groups as a quotient of $PSL(2, \mathbb{Z}[i])$

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Abstract. In this study, we developed an algorithm to find the homomorphisms of the Picard group $PSL(2, \mathbb{Z}[i])$ into a finite group G . This algorithm is helpful to find a homomorphism (if it is possible) of the Picard group to any finite group of order less than $15!$ because of the limitations of the GAP and computer memory. Therefore, we obtain only five alternating groups A_n , where $n = 5, 6, 9, 13$ and 14 are quotients of the Picard group. In order to extend the degree of the alternating groups, we use coset diagrams as a tool. In the end, we prove our main result with the help of three diagrams which are used as building blocks and prove that, for $n \equiv 1, 5, 6 \pmod{8}$, all but finitely many alternating groups A_n can be obtained as quotients of the Picard group $PSL(2, \mathbb{Z}[i])$. A code in Groups Algorithms Programming (GAP) is developed to perform the calculation.

Keywords. Bianchi group; fragment; orbits; groups algorithms programming.

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1. Introduction

An extensive part of the existing knowledge of combinatorial group theory is about the investigation of subgroups of a single group $PSL(2, \mathbb{C})$, where \mathbb{C} is the ring of complex numbers. The study of the projective special linear group $PSL(2, \mathbb{C}) = PGL(2, \mathbb{C})$ comprised of all linear fractional transformations (LFTs), with complex coefficients, and was one of the main stream topics of mathematics in the 19th century and played a vital role in the evolution of hyperbolic geometry. An important class of discrete subgroups of $PSL(2, \mathbb{C})$ consists of groups of the form $PSL(2, O_d)$, where O_d denotes the ring of integers in the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ and d is a square-free positive integer. It is known that O_d has a Euclidean algorithm only if $d = 1, 2, 3, 7, 11$ while only O_1 and O_3 have units $\neq \pm 1$. The groups $\Gamma_d = PSL(2, O_d)$ with d and O_d as above are called Bianchi groups. The Picard group is denoted by Γ_1 and contains all the linear fractional transformations of the form $\frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{Z}[i]$ and $ad - bc = 1$ (see [1, 6]). The Picard group is an important subgroup of $PSL(2, \mathbb{C})$. This group acts on hyperbolic three space H^3 [5] through linear fractional transformations. The center of this action is $\langle \pm 1 \rangle$, and we define $PSL(2, O_1) = SL(2, O_1) / \langle \pm 1 \rangle$. In [7], it is shown that the Picard group is generated by

$$A = \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This leads to the well-known finite presentation $\Gamma_1 = \langle A, B, C, D; A^3 = B^2 = C^3 = D^2 = (AC)^2 = (AD)^2 = (BC)^2 = (BD)^2 = 1 \rangle$. The Picard group Γ_1 is a free product with amalgamation of the following form: $\Gamma_1 = G_1 *_M G_2$ with $G_1 = S_3 *_Z A_4$, $G_2 = S_3 *_Z D_2$, where M is the modular group $PSL(2, Z)$ (see [2, 6]). The modular group plays a significant role in determining subgroups of the Picard group because of this decomposition. It is a Fuchsian subgroup of Γ_1 and is not normal. A classic method of investigating the structure of a group G (say) is to consider homomorphism of G into a specific group in which the calculations are relatively easy. In this way, homomorphism of G into a linear group or linear representations arose via a primary tool in the theory of abstract group. Schur (1904) was the first to realize that a new kind of representation had to be introduced in order to study the relation between the linear representation of a group and its factor groups. The study of the modular group and other similar groups using coset diagrams began with the work of O. Schreier. The actions of a finitely generated group can be conveniently studied using these diagrams. The A. G. Kurosh subgroup theorem provides the structure of a subgroup of an amalgamated free product. W. W. Stothers was one of the early mathematicians who used the coset diagrams and obtained recursive formulae for the number of subgroups of a given index in the modular group.

In this paper, we define an algorithm to find the homomorphisms of the Picard group into a finite group G , provided that $|G| < 15!$. Of the 26 sporadic groups, 12 of them have non-trivial outer automorphism group. The full automorphism groups of these simple groups are

$$M_{12}.2 \quad M_{22}.2 \quad HS.2 \quad Suz.2 \quad McL.2 \quad J_2.2 \\ He.2 \quad Fi_{22}.2 \quad O'N.2 \quad HN.2 \quad Fi_{24} \quad J_3.2.$$

We refer these 12 groups as sporadic almost simple groups. We use that algorithm to the class of 12 sporadic almost simple groups and find that only $M_{12}.2$ with permutation representation on 24 points and $J_2.2$ with permutation representation on 100 points, are quotients of the Picard group. Moreover, we see that $PSL(2, p)$ such that $p \pm a$ and $p \pm b$ are prime numbers, where a and b are distinct positive integers with $a < b < p$, are also the quotients of the Picard group. Furthermore, we apply the algorithm for the class of alternating groups and identify quotients of the Picard group up to degree 15. To extend the degree of alternating groups, we use coset diagrams (see [3, 4]) as a tool, in order to construct A_n for all possible n such that these alternating groups are homomorphic images of the Picard group. We use some fundamental coset diagrams and named these fundamental diagrams as ‘building blocks’. Additionally, these building blocks satisfy all the defining relations of Picard group as well. We develop a technique known as stitching of diagrams for alternating groups and draw coset diagrams for all building blocks which are essential for this construction. In the end, we prove our main result with the help of three diagrams which are used as building blocks and prove that for $n \equiv 1, 5, 6 \pmod{8}$, all but finitely many alternating groups A_n can be obtained as quotients of the Picard group $PSL(2, Z[i])$.

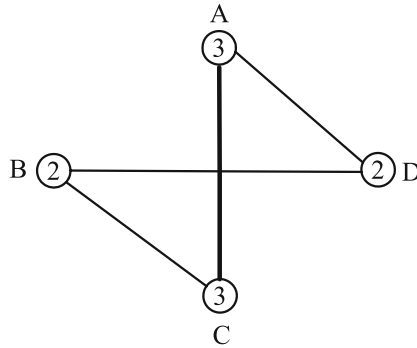


Figure 1. Relations of picard group.

Algorithm 1. Since the Picard group has four generators A , B , C and D with eight defining relations, where the generator D normalizes A and B normalizes C , as can be observed in figure 1, hence we are left with A and C such that $A^3 = C^3 = (AC)^2 = 1$, which is isomorphic to A_4 .

Now we define an algorithm to find homomorphisms of the Picard group into any group G . The algorithm consists of the following four steps:

- (1) Identify A, C in G up to conjugation, such that $A^3 = C^3 = (AC)^2 = 1$.
- (2) Go through $N_G(\langle A \rangle)$, look for the involution of D which inverts A , such that it is a set of permutations N .
- (3) Go through $N_G(\langle C \rangle)$, look for the involution of B which inverts C , such that it is a set of permutations M .
- (4) Then select all pairs $(D, B) \in N \times M$, such that B and D commute.

This gives us A, B, C and D , the generators of the Picard group in G , which are in fact normal closures of A^3, B^2, C^3 and D^2 . Working out each set of generators, we develop a code in group algorithm programming (GAP), which is given in the Appendix to find orbits and conjugacy classes and hence gives us homomorphic images of the Picard group as an output.

2. Alternating groups of small degree

The implications of Algorithm 1 gives that alternating groups A_n , where $n = 5, 6, 9, 13, 14$, are the quotients of the Picard group $PSL(2, \mathbb{Z}[i])$. In table 1, we present the computation which is applied in identifying the generators of the Picard group.

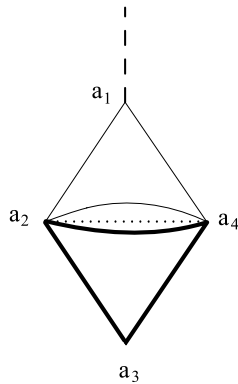
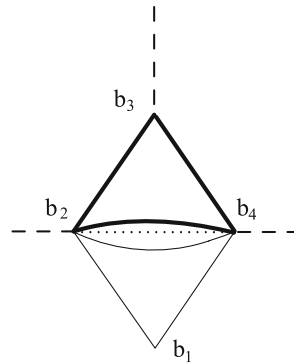
It may not be possible to find a homomorphism into alternating groups for $n > 14$ because of the limitations of computer memory. We use coset diagrams as a tool to extend the degree of alternating groups which are the quotients of the Picard group $PSL(2, \mathbb{Z}[i])$.

3. Construction of alternating groups of higher degree

In order to construct A_n for all possible n , such that these alternating groups are homomorphic images of the Picard group, by using graphs, we need some fundamental coset

Table 1. Alternating groups of small degree.

n	G	$3A$	$3C$	$\langle A, C \rangle$	$N_G(\langle A \rangle)$	$N_G(\langle C \rangle)$	$\langle A, B, C, D \rangle$
5	A_5	1	2	6	2	2	2
6	A_6	2	1	9	9	9	8
9	A_9	3	1	18	90	90	410
13	A_{13}	4	1	30	12420	12420	642384
14	A_{14}	4	1	33	52140	52140	4976501

**Figure 2.** Connector.**Figure 3.** Adopter.

diagrams. We call these fundamental diagrams building blocks. Additionally, these building blocks satisfy all the defining relations of the Picard group. To construct A_n for all possible n , we connect the diagrams in a special way such that each preserves all the properties of A_n . By connecting smaller diagrams representing groups of small degree, we can obtain the bigger diagrams representing a group of large degree. Any two or more coset diagrams can be joined together to obtain a coset diagram of an arbitrary size provided that each diagram is connected in a special way. So, the resulting diagram thus produced still preserves all the properties and satisfies all the defining relations of the Picard group. In this construction, we have three basic diagrams or building blocks. These diagrams are in fact three alternating groups, namely A_6 , A_9 and A_{13} . To connect their coset diagrams, one needs diagrams containing the following fragments of the coset diagram.

We call the fragment displayed in figure 2 as ‘connector’ and denote it by Λ . By connector Λ , we mean a fragment of the diagram containing the five vertices a_1 , a_2 , a_3 , a_4 and a_5 such that $D(a_1) = a_5$, cycle (a_1, a_4, a_2) occurs in the permutations of C and (a_3, a_4, a_2) occurs in the permutations of A . This fragment is present in all the coset diagrams which are used as building blocks.

We call the fragment displayed in figure 3 as an ‘adopter’ and denote it by ∇ . So by an adopter ∇ , we mean a fragment of the diagram containing the four vertices b_1 , b_2 , b_3 and b_4 such that cycle (b_1, b_2, b_4) occurs in the permutations of C while (b_1, b_2, b_4) occurs in

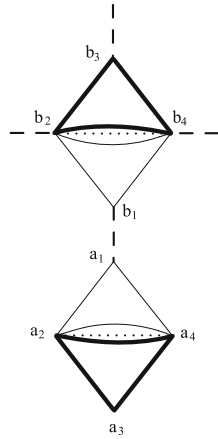


Figure 4. Stitching of connector and adopter.

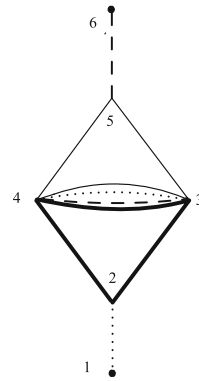


Figure 5. Alternating group A_6 .

the permutations of A . This fragment is present in a coset diagram which represents the alternating group A_9 .

The method for stitching together the coset diagrams is explained as follows.

Let us consider two coset diagrams for the quotients of the Picard group that contains a connector Λ and an adopter ∇ . Place the two coset diagrams on a common axis of symmetry, one above the other and merge the D -edges in this way, such that vertex a_5 omits and D connects vertex a_1 with the vertex b_1 . Symbolically, stitching together coset diagrams which contains a connector Λ and an adopter ∇ through D -edges is defined as $a_1 \stackrel{D}{\sim} b_1$, as shown in figure 4. The fragments stitched together in this way and forming a bigger fragment, still satisfy all the relations of $PSL(2, \mathbb{Z}[i])$.

3.1 Basic coset diagrams

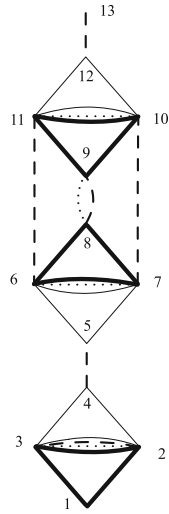
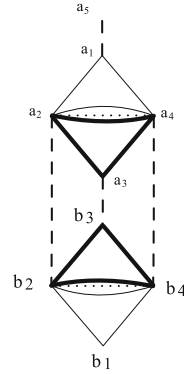
To draw coset diagrams of alternating group A_n for all possible n , we need just three diagrams. In fact, these coset diagrams represent three alternating groups, namely A_6 , A_9 and A_{13} . The coset diagram for A_9 performs a key role in the entire construction because several copies of A_9 can be connected periodically. We explain each building block separately, that describes the extra relations satisfied by the diagram.

In figure 5, the coset diagram represents A_6 and we call it B_1 with its defining relations

$$\begin{aligned} A^3 &= B^2 = C^3 = D^2 = (A^{-1}C^{-1})^2 = (BC^{-1})^2 = (A^{-1}D)^2 \\ &= (BD)^2 = (A^{-1}B)^3 = (C^{-1}D)^3 = 1. \end{aligned}$$

The coset diagram B_1 contain only one connector Λ , so it can connect only once with the adopters in the other diagrams. In this coset diagram, if we begin tracing the path by starting from the vertex labelled as 1 (from bottom), then there exists a word (say) $w = ba^{-1}cd$, which connects 1 with the vertex 6 (on top).

The coset diagram displayed in figure 6 is for alternating group A_{13} , which depicts transitive action of the Picard group on a set of nine vertices. We call it B_2 and it also

Figure 6. Alternating group A_{13} .Figure 7. Alternating group A_9 .

contains only one connector Λ , hence it can be linked with other coset diagrams only once. According to the diagram, there exists a word $w = a^{-1}cdc^{-1}ada^{-1}cd$ which connects 1 with the vertex 13.

This coset diagram represents A_9 . We call this coset diagram B_3 . It is a special one because it contains a connector Λ and an adopter ∇ . So, a number of copies of B_3 can be connected with themselves as well as with the other diagrams through ∇ and Λ .

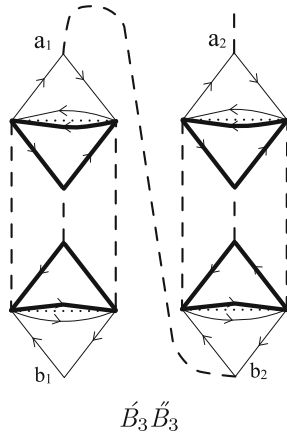
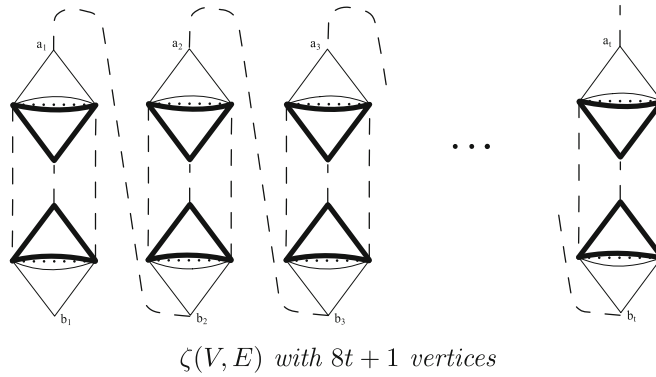
Theorem 2. $PSL(2, \mathbb{Z}[i])$ acts transitively on n vertices of a coset diagram $\zeta(V, E)$, where $n \equiv 1 \pmod{8}$.

Proof. We already have the coset diagram for $n = 9$, which is displayed in figure 7 for use here. Let $w = c^{-1}ada^{-1}cd$ be a word connecting b_1 by passing through each vertex with a_5 . It means that there exists a path between b_1 and the top vertex a_5 . Since B_3 (see figure 7) contains a connector Λ as well as an adopter ∇ , a number of copies can be stitched with each other by placing on a common axis of symmetry. Let \check{B}_3 and $\check{\check{B}}_3$ be the two copies of B_3 . That is, it may form a composition $\check{B}_3 \check{\check{B}}_3$ of \check{B}_3 and $\check{\check{B}}_3$ by placing \check{B}_3 and $\check{\check{B}}_3$ on a common axis of symmetry and by stitching the connector of \check{B}_3 with the adopter of $\check{\check{B}}_3$, as defined in section 3, such that $a_1 \stackrel{D}{\sim} b_2$, where a_1 and b_2 are the vertices of \check{B}_3 and $\check{\check{B}}_3$ respectively (see figure 8).

Firstly, the relations

$$A^3 = B^2 = C^3 = D^2 = (AC)^2 = (AD)^2 = (BC)^2 = (BD)^2 = 1$$

of the Picard group are still satisfied. Also word $w = c^{-1}ada^{-1}cd$ that passes through each vertex of the diagrams in $\check{B}_3 \check{\check{B}}_3$ becomes

**Figure 8.** $\check{B}_3 \check{B}_3$.**Figure 9.** $\zeta(V, E)$ with $8t + 1$ vertices.

$$w^2 = ww = (c^{-1}ada^{-1}cd)(c^{-1}ada^{-1}cd) = (c^{-1}ada^{-1}cd)^2$$

such that

$$(b_1)w^2 = a_2$$

while all other cycles remain unaffected. The resulting diagram depicts the transitive permutation representation for quotient of the Picard group $PSL(2, \mathbb{Z}[i])$ of degree $17 = 8 \cdot 2 + 1 = 9 + 9 - 1$ (as defined in section 3, the generator D merged one vertex, i.e. $a_1 \stackrel{D}{\sim} b_2$, where a_1 and b_2 are the vertices of \check{B}_3 and \check{B}_3 respectively).

For any $t \in \mathbb{N}$, we consider t copies of B_3 containing Λ and ∇ . We put these copies of B_3 adjacent to each other along a common axis of symmetry and then we add t copies of D -edges connecting Λ_{a_m} and $\nabla_{b_{m+1}}$, such that $a_m \stackrel{D}{\sim} b_{m+1}$, where $m = 1, 2, \dots, t$.

Thus the resulting coset diagram depicts the transitive permutation representation for quotient of the Picard group $PSL(2, \mathbb{Z}[i])$ of degree $8t + 1$ and the defining relations

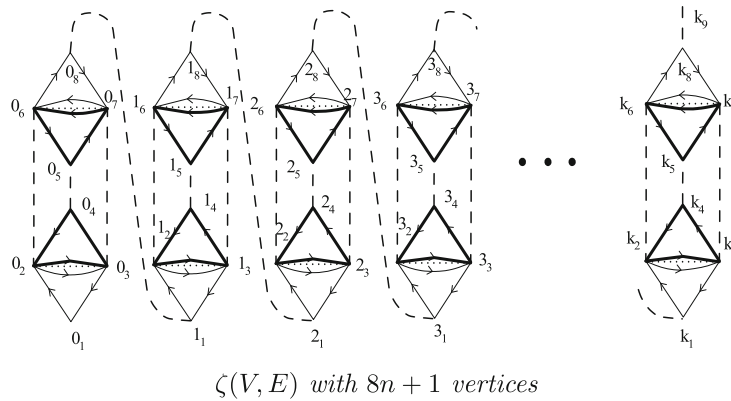


Figure 10. $\zeta(V, E)$ with $8n + 1$ vertices

$$A^3 = B^2 = C^3 = D^2 = (AC)^2 = (AD)^2 = (BC)^2 = (BD)^2 = 1$$

of the Picard group are still satisfied. There exists a word

$$w^t = (c^{-1}ada^{-1}cd)^t, \quad \text{where } t \in \mathbb{N}$$

that passes through each vertex of the diagram, that is, the vertex b_1 is connected through the word w^t , with the vertex a_t , where $t \in \mathbb{N}$ (see figure 9). Hence when we connect t copies of B_3 , yielding a transitive coset graph with $|V(\zeta)| = 8t + 1$, where $t \geq 1$, which represents quotient group of the Picard group $PSL(2, \mathbb{Z}[i])$. \square

Theorem 3. Action of $PSL(2, \mathbb{Z}[i])$ on n vertices of a coset diagram $\zeta(V, E)$, where $n \equiv 1 \pmod{8}$ is 2-transitive.

Proof. Let $\zeta(V, E)$ be coset diagrams representing the composition of k number of copies of B_3 and G be the permutation group associated with $\zeta(V, E)$ such that

$$\zeta(V, E) = B_{3_1} \circ B_{3_2} \circ \cdots \circ B_{3_k}.$$

Let Ω be the set of all vertices of $\zeta(V, E)$, that is,

$$\Omega = \{0_1, 0_2, \dots, 0_8, 1_1, 1_2, \dots, 1_8, \dots, k_1, k_2, \dots, k_8, k_9\}$$

such that $|\Omega| = 8n + 1$, where $n \geq 1$.

The composition of k number of copies of B_3 which are connected through stitching of the coset diagrams for the Picard group is shown in figure 10.

So, the coset diagram $\zeta(V, E)$ represents the transitive action of $PSL(2, \mathbb{Z}[i])$ on $8n + 1$ points.

Now, let $y_1 = c^{-1}ada^{-1}c^{-1}adc^{-1}d$ be the element of the permutation group G and y_1 fixes α , that is, $y_1 \in G_\alpha$, where $\alpha = 0_1 \in \Omega$. Now first we show that G_α acts transitively on $\Omega \setminus \{\alpha\}$. Let us suppose that $y_1 = c^{-1}ada^{-1}c^{-1}adc^{-1}d$ act on arbitrary element t_i

of $\Omega \setminus \{\alpha\}$. Then there are three possible cases, when y_1 acts on t_i and it maps to another element of $\Omega \setminus \{\alpha\}$, which are basically the vertices of $\zeta(V, E)$.

Case 1. For every $t_i \in \Omega \setminus \{\alpha\}$, $t \neq 0, k$. If the vertex t_i is not in B_{3_1} (first copy of B_3) and B_{3_k} (k -th copy of B_3) fragments, then the element y_1 of the permutation group G map each t_i as follows:

$$y_1(t_i) \mapsto \begin{cases} t - 1_8 \\ t - 1_5 \\ t_6 \\ t + 1_7 \\ t - 1_2 \\ t_3 \\ t + 1_4 \\ t + 1_1 \end{cases} \quad \text{for } i=1, \dots, 8 \quad (1)$$

Case 2. For each $t_i \in \Omega \setminus \{\alpha\}$, $t = 0$. If the vertices t_i are in the same fragment that is B_{3_1} whose first (that is, 0_1) element (say) α is fixed by y_1 , then y_1 acts on the remaining elements of B_{3_1} fragment as follows:

$$y_1(t_i) \mapsto \begin{cases} t_4 \\ t_6 \\ t + 1_7 \\ t_7 \\ t_3 \\ t + 1_4 \\ t + 1_1 \end{cases} \quad \text{for } i=2, \dots, 8 \quad (2)$$

Case 3. For each $t_i \in \Omega \setminus \{\alpha\}$, t is the k -th element. If the vertices t_i are in the k -th fragment, that is, B_{3_k} , where $k \in \mathbb{N}$, then y_1 maps vertices t_i as follows:

$$y_1(t_i) \mapsto \begin{cases} t - 1_8 \\ t - 1_5 \\ t_6 \\ t_8 \\ t - 1_2 \\ t_3 \\ t_2 \\ t_9 \end{cases} \quad \text{for } i=1, \dots, 8 \quad (3)$$

Let $H = G_\alpha = \langle y_1, A, B, D \rangle$. We need to show that for all $t_i, t_j \in \Omega \setminus \{\alpha\}$, there exists $h \in H$ such that $t_i^h = t_j$. Let $K = \langle A, B, D \rangle \leq H$. In other words, C -edges are removed. Since the subgroup K is disconnected and contains three types of orbits, namely I_t , J_t and k :

$$I_t = \{t_i, \text{ where } i = 2, 3, 4, 5, 6 \text{ and } t = 0, 1, \dots, k\},$$

thus $|I_t| = k$.

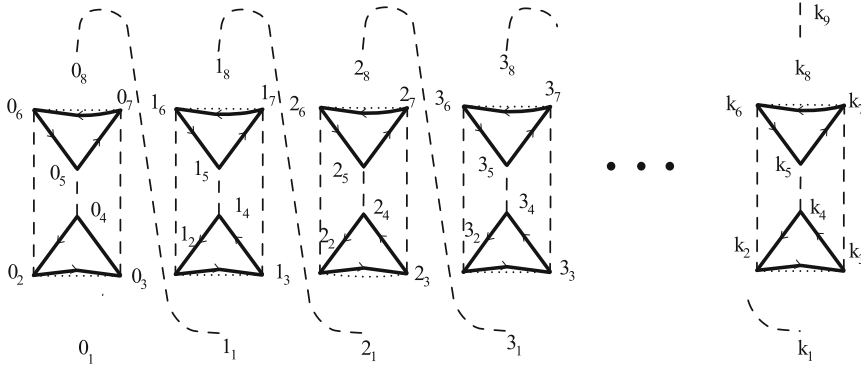


Figure 11. $\Omega \setminus \{\alpha\} = I_t \cup J_t \cup K$.

Also,

$$J_t = \{t_8, t + 1_1, \text{ where } t = 0, 1, \dots, k - 1\},$$

where $|J_t| = k - 1$ and

$$K = \{k_8, k_9\}$$

such that $\Omega \setminus \{\alpha\} = I_t \cup J_t \cup K$ and $I_t \cap J_t \cap K = \phi$, as shown in figure 11.

Consider vertex $t_7 \in I_t$ and $(t + 1)_4 \in I_{t+1}$. Then by (1), there exists $y_1 \in H$ such that

$$(t_7)^{y_1} = (t + 1)_4, \quad \text{for all } t = 0, 1, 2, \dots, k.$$

It means that y_1 acts transitively on the orbits I_t of coset diagram $\zeta(V, E)$.

Now let $\hat{y}_1 = (c^{-1}ada^{-1}cd)^t c(dc^{-1}ada^{-1}c)^t$ and \hat{y}_1 fixes α , that is, $\hat{y}_1 \in G_\alpha$. Since $c^{-1}ada^{-1}cd$ maps every t_1 to $t + 1_1$, and if we apply t time then it takes t_1 to k_1 and then c maps it to the element of I_t and by applying t times $dc^{-1}ada^{-1}c$ takes elements of orbits I_t to another distinct element of orbits I_t . Now if we apply $c^{-1}ada^{-1}cd$ on the orbit K then it will map k_8 to $k - 1_8$. If we apply t times, then it maps k_8 onto 0_8 and then c maps it to the element of I_t . By applying t times $dc^{-1}ada^{-1}c$, it takes elements of the orbits I_t to other distinct elements of the orbits I_t .

Hence G_α acts transitively over $\Omega \setminus \{\alpha\}$. Again, let y_2 be an element of G such that it fixes another vertex of the diagram $\beta \in \Omega$, such that $\beta \neq \alpha$. Let $r, s \in \Omega \setminus \{\beta\}$. Since G is transitive over Ω (by Theorem 3), therefore, there exists $y \in G$ with $y(\beta) = \alpha$. So, $y(r), y(s) \in \Omega \setminus \{\alpha\}$ and then there exists $y_1 \in G_\alpha$ such that $y_1(y(r)) = y(s)$. Now $y_2 = y^{-1}y_1y$ imply that $y_2 \in G_\beta$ with $y_2(r) = s$, where $r, s \in \Omega \setminus \{\beta\}$. \square

COROLLARY 4

Action of $PSL(2, \mathbb{Z}[i])$ on n vertices of a coset diagram $\zeta(V, E)$, where $n \equiv 1 \pmod{8}$ is primitive.

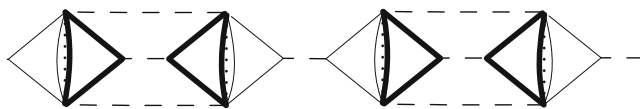


Figure 12. Composition of two copies of A_9 .

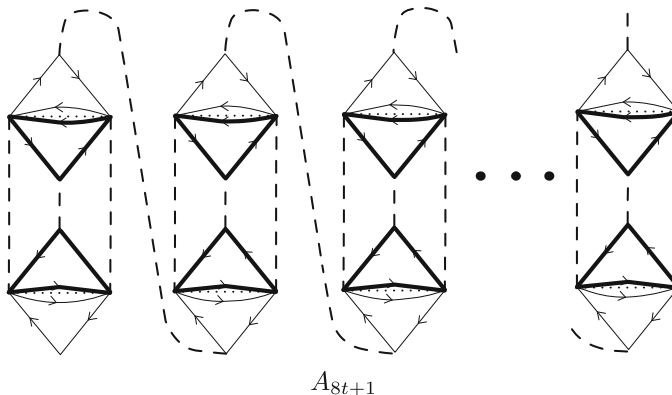


Figure 13. A_{8t+1} .

According to section 3, stitching together coset diagrams which contains a connector Λ and an adopter ∇ , merge the D -edges in this way, such that vertex a_5 omits and D connects vertex a_1 with vertex b_1 . Hence we have.

Remark 5. If we join two coset diagrams A_m and A_n , then the resulting diagram depicts the transitive alternating group A_{m+n-1} .

Theorem 6. For all $n \equiv 1, 5, 6 \pmod{8}$, the alternating group A_n is a quotient of the Picard group $PSL(2, \mathbb{Z}[i])$.

Proof. We discuss three cases separately:

Case 1. $n \equiv 1 \pmod{8}$. Consider the two diagrams of A_9 (see figure 7) containing a connector Λ as well as an adopter ∇ . So, both diagrams can be stitched with each other by placing on a common axis of symmetry. By Remark 5, as a result, A_{17} appears (see figure 12). Hence it forms a composition of two copies of A_9 . These are connected as follows: where $a_1 \stackrel{D}{\sim} b_2$.

The resulting coset diagram has a permutation representation on 17 points. Let G be the group generated by A, B, C and D . Since these are even permutations, therefore, either G is a subgroup of A_{17} or A_{17} itself. According to GAP the order of G is $\frac{17!}{2}$, so, it is A_{17} and it still satisfies all defining relations of the Picard group. By Theorem 2, $w^2 = (c^{-1}ada^{-1}cd)^2$ is the word which passes through each vertex of the coset diagram of A_{17} because of its transitivity.

Now let $t \in \mathbb{N}$. We consider t copies of A_9 containing Λ and ∇ . We put these copies of A_9 adjacent to each other along a common axis of symmetry and then we add t copies of D -edges connecting Λ_{a_m} and $\nabla_{b_{m+1}}$ so that $a_m \stackrel{D}{\sim} b_{m+1}$, where $m = 1, 2, \dots, t$.

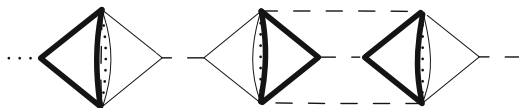


Figure 14. Composition of A_6 and A_9 .

Hence there exists a word $w^t = (c^{-1}ada^{-1}cd)^t$, such that w^t traces the path from vertex 1 to vertex a_t , where $t \in \mathbb{N}$. We obtain a new coset diagram containing $n = 8t + 1$ vertices (see figure 13). It gives permutations A, B, C and D on $8t + 1$ points which are even and generate group G isomorphic to the alternating group of degree $8t + 1$, that is A_n , where $n \equiv 1 \pmod{8}$.

Case 2. $n \equiv 6 \pmod{8}$. We already have drawn coset diagrams of A_6 and A_9 . As discussed above, A_6 contains a connector Λ and A_9 contains an adopter ∇ . So, we can stitch both by placing on a common axis of symmetry. By Remark 5, A_{14} appears. Hence it forms a composition of A_6 and A_9 , such that $6 \stackrel{D}{\sim} b_1$. We stitch these diagrams as follows:

The resulting diagram has permutation representation on 14 points. Let G be the group generated by A, B, C and D . Since these are even permutations, therefore, either G is a subgroup of A_{14} or A_{14} itself. According to GAP, the order of G is $\frac{14!}{2}$, so, it is A_{14} and still satisfies all defining relations of the Picard group. Also, (say) $u = ba^{-1}cd$ is the word that passes through each vertex of the coset diagram A_6 as shown in figure 5 and $w = c^{-1}ada^{-1}cd$ is the word connecting b_1 with a_5 passed through each vertex of A_9 (see figure 7). Both are put together to give

$$uw = (ba^{-1}cd)(c^{-1}ada^{-1}cd).$$

Thus there exists a path from vertex (say) 1 to vertex 14. Similarly, we can connect any two vertices of this coset diagram by a word. Hence, the word uw acts transitively on figure 14. Now, if we consider t number of copies of A_9 with A_6 , then there is a path from vertex 6 to vertex a_t , that is,

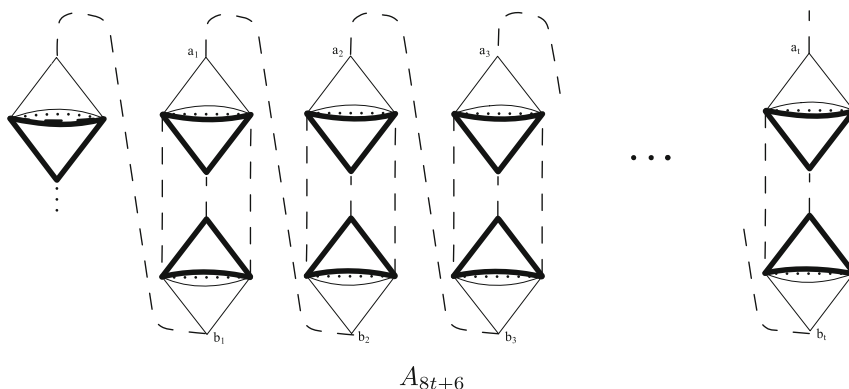
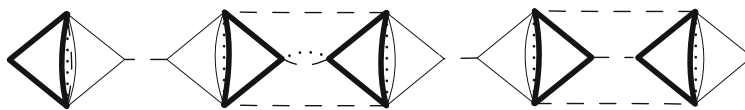
$$u[ww \dots w(t\text{-times})] = (ba^{-1}cd)[(c^{-1}ada^{-1}cd) \dots (c^{-1}ada^{-1}cd)].$$

We compose the diagrams as in figure 15.

Hence there exists a word $uw^t = (ba^{-1}cd)(c^{-1}ada^{-1}cd)^t$, such that uw^t traces the path from vertex 1 to vertex a_t , where $t \in \mathbb{N}$. Thus we obtain a new coset diagram containing $n = 8t + 6$ vertices. It gives permutations A, B, C and D on $8t + 6$ points which are even and generate the group G isomorphic to the alternating group of degree $8t + 6$, that is, A_n , where $n \equiv 6 \pmod{8}$.

Case 3. $n \equiv 5 \pmod{8}$. In this case, we join the coset diagrams of A_{13} and A_9 . Since A_{13} has a connector Λ and A_9 contains both, a connector Λ and an adopter ∇ , these diagrams are stitched together by placing on a common axis of symmetry. Hence it forms a composition of A_{13} and A_9 such that $a_1 \stackrel{D}{\sim} 1$. By Remark 5, as a result, A_{21} appears. These diagrams can be stitched together as in figure 16.

This resulting diagram has a permutation representation on 21 points. Let G be the group generated by A, B, C and D . Since these are even permutations, therefore, either G is a subgroup of A_{21} or A_{21} itself. According to GAP, the order of G is $\frac{21!}{2}$, so it is A_{21} .

**Figure 15.** A_{8t+6} .**Figure 16.** Composition of A_{13} and A_9 .

and still satisfies all defining relations of the Picard group. As discussed above, let (say) $v = a^{-1}cdc^{-1}ada^{-1}cd$ be the word that traces the path by passing through each vertex of the coset diagram A_{13} and $w = c^{-1}ada^{-1}cd$ is a word connecting b_1 with a_5 and passing through each vertex of A_9 . After composing the diagrams, the words v and w become

$$vw = (a^{-1}cdc^{-1}ada^{-1}cd)(c^{-1}ada^{-1}cd).$$

If we begin from (say) 1, then the word vw takes the vertex 1 to vertex 21, as displayed above and obtain a transitive coset diagram of A_{21} . Now we stitch together t copies of A_9 with A_{13} . Then by Theorem 3, there exists a path from vertex 13 to vertex a_t , that is,

$$\begin{aligned} u[ww \dots w(t\text{-times})] \\ = (a^{-1}cdc^{-1}ada^{-1}cd)[(c^{-1}ada^{-1}cd) \dots (c^{-1}ada^{-1}cd)]. \end{aligned}$$

We compose the diagrams as shown in figure 17.

This shows that there exists a path vw^t , such that vw^t traces the path from vertex 1 to vertex a_t , where $t \in \mathbb{N}$. By Remark 5, we obtain a new coset diagram containing $n = 8t + 13$ vertices. It gives permutation representation of A , B , C and D on $8t + 13$ points which is even and generate the group G isomorphic to the alternating group of degree $n = 8t + 13$, $t > 0$, that is, A_n , where $n \equiv 5 \pmod{8}$.

Hence, by combining all three cases it is proved that for all $n \equiv 1, 5, 6 \pmod{8}$, the alternating groups A_n are the quotients of the Picard group. \square

Remark 7. The possible finite presentation of the alternating groups A_n , where $n = 8t + 1$ and $t > 0$, such that these alternating groups are quotients of the Picard group $PSL(2, \mathbb{Z}[i])$ is

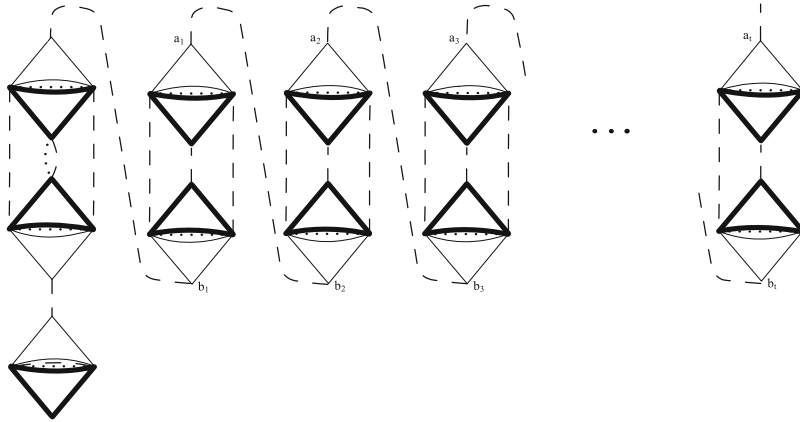

 A_{8t+13}

Figure 17. A_{8t+13} .

$$A_{8t+1} \cong \left\langle \begin{array}{l} a, b, c, d \mid a^3 = b^2 = c^3 = d^2 = (a^{-1}b)^2 = (a^{-1}c^{-1})^2 = (bc^{-1})^2 = (a^{-1}d)^2 \\ = (bd)^2 = (cdc^{-1}d)^6 = (c^{-1}d)^{8t+2} = c^{-1}a^{-1}dc^{-1}(adcdc^{-1})^{4(t-1)+2} \\ da^{-1}cdc^{-1}da^{-1}cd = c^{-1}bdcd(c^{-1}dc^{-1}dcdcd)^{3t} = 1 \end{array} \right\rangle.$$

Appendix

The code is developed in GAP for the implication of Algorithm 1.

```
G:=?
cc:=ConjugacyClasses(G);
List(cc,Representative);
cc:=Filtered(cc,e->Order(e)=3);
len1:=Length(cc);
Print("Classes of order 3=",len1,"\n");
for i in [1..len1] do
a:=cc[i];
T:=Orbit(G,a);
T:=Filtered(T,e->Order(a*(e^-1))=2);
TT:=Orbits(Centralizer(G,a),T);
len2:=Length(TT);
Print("Total number of A_4=",len2,"\n");
N:=Normalizer(G,Subgroup(G,[a]));
N:=Filtered(N,n->Order(n)=2 and Order(a*n)=2);
for j in [1..len2] do
c:=TT[j][1]^-1;
M:=Normalizer(G,Subgroup(G,[c]));
M:=Filtered(M,m->Order(m)=2 and Order(c*m)=2);
Length(M);
list:=[];
```

```

    for d in N do
        for b in M do
            if Order(b*d)=2 then
                Add(list, [a,b,c,d]);
            fi;
        od;
    od;
nn:=Length(list);
Print(" (", i, ", ", j, ") ", "\n\n"); GG:=(Order(G))/2;
    for z in [1..nn]
        do
            if
                GG=Order(Subgroup(G, [list[z][1], list[z][2],
list[z][3], list[z][4]]));
            then
                Print("\ n", StructureDescription(Subgroup(G,
[list[z][1], list[z][2],
list[z][3], list[z][4]])), "-> ", Subgroup(G, [
list[z][1], list[z][2], list[z][3], list[z][4]]));
            fi;
        od;
    od;
Print("\n", "##### ", "\n\n");
od;

```

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