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Analytical results for periodically-driven two-level models in relation to Heun functions

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Abstract. We introduce three different types of periodically-driven multiparametric two-level models whose analytical solutions are given in terms of Heun functions. These results are applied to obtain exact analytical results for certain types of periodic potentials and asymmetric double-well potentials. In particular, it is shown that under special parameter conditions, an experimentally realised periodic potential supports the exact in-gap solutions. In the asymmetric double-well potentials, some exact results of the bound-state wave functions and associated energies are found in explicit form.

Keywords. Two-level models; Heun equation; periodic potential; exact in-gap solutions.

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1. Introduction

In recent years, Heun equation (HE) and its confluent forms have attracted extensive interest of research [1–3]. This is partly due to their applications in many physical systems, including lattice systems [4–6], few-body systems [7–10], quantum spin chains [11], and quasiexactly solvable quantum potentials [12–17]. The list provided here is far from exhaustive. In particular, it has been shown that the energy spectrum of the well-known quantum Rabi model can be obtained with the help of the confluent HE [18–20]. The semiclassical form of the quantum Rabi model describes the interaction of a two-level model (TLM) with a classical monochromatic field. Such a monochromatically-driven TLM has long been an important paradigm for understanding many fundamental phenomena in diverse branches of physics [21,22]. Interestingly, the confluent HE also appears in the semiclassical form of the quantum Rabi model [23].

In recent series of works, these Heun-type equations have been used to construct exactly solvable time-dependent TLMs [24–29]. By using a proper variable transformation, the Schrödinger equation of these models can be transformed into the HE and its confluent forms. Most of the obtained TLMs describe the interaction of the TLMs and the time-dependent pulsed laser fields. The exactly solvable periodically-driven TLMs remain rare [27].

In this paper, we introduce three different types of periodically-driven multiparametric TLMs in relation to the HE. An analytical solution for these systems is given in terms of Heun function (HF). As a simple application of our results, we found certain analytical exact results for in-gap solutions in a type of periodic potential under certain special parameter conditions. In addition, it is found that the hyperbolic version of the periodic potential shows an asymmetric double-well structure where some of the bound-state wave functions and associated energies are found in an explicit form.

2. Periodically-driven two-level systems

We begin with the time-dependent TLM described by the following Hamiltonian:

$$H = \frac{f(t)}{2}\sigma_z + \frac{\nu(t)}{2}\sigma_x, \tag{1}$$

where $\sigma_{x,z}$ are the usual Pauli matrices for the two-level system. $f(t)$ and $\nu(t)$ are the time-dependent energy difference and coupling between the two levels, respectively. In our study, we focus on the case where $f(t)$ and $\nu(t)$ are periodic with respects to t. The probability amplitudes $a_1(t)$ and $a_2(t)$ in the two levels satisfy the coupled first-order differential equations

$$i\frac{\mathrm{d}a_1}{\mathrm{d}t} = \frac{\nu(t)}{2}a_2 + \frac{f(t)}{2}a_1, \tag{2}$$
$$i\frac{\mathrm{d}a_2}{\mathrm{d}t} = \frac{\nu(t)}{2}a_1 - \frac{f(t)}{2}a_2. \tag{3}$$

By eliminating $a_2(t)$ and $a_1(t)$, we get two second-order differential equations for $a_1(t)$ and $a_2(t)$

$$\frac{\mathrm{d}^2a_1}{\mathrm{d}\tau^2} - \frac{\dot{\nu}}{\nu}\frac{\mathrm{d}a_1}{\mathrm{d}\tau} + \left(\frac{f^2}{4\omega^2} + \frac{\nu^2}{4\omega^2} + i\frac{\dot{f}}{2\omega} - i\frac{f}{2\omega}\frac{\dot{\nu}}{\nu}\right)a_1 = 0, \tag{4}$$
$$\frac{\mathrm{d}^2a_2}{\mathrm{d}\tau^2} - \frac{\dot{\nu}}{\nu}\frac{\mathrm{d}a_2}{\mathrm{d}\tau} + \left(\frac{f^2}{4\omega^2} + \frac{\nu^2}{4\omega^2} - i\frac{\dot{f}}{2\omega} + i\frac{f}{2\omega}\frac{\dot{\nu}}{\nu}\right)a_2 = 0, \tag{5}$$

where we have used the variable transformation $\tau = \omega t$, the dot denotes the derivative with respect to τ, $\dot{f} = \mathrm{d}f/\mathrm{d}\tau$ and $\dot{\nu} = \mathrm{d}\nu/\mathrm{d}\tau$. Equations (4) and (5) have a property that if $\psi(\tau)$ is a solution of eq. (4), $\psi^*(\tau)$ is a solution of eq. (5). Therefore, we may only discuss the solutions of either eq. (4) or eq. (5). Here, we focus on the solutions of eq. (4).

For a given set of $f(t)$ and $\nu(t)$, a general idea for solving eq. (4) is to find appropriate variable transformations

$$z = z(\tau), \quad a_1(z) = \varphi(z)\phi(z), \tag{6}$$

so that eq. (4) can be reduced to a well-known equation for $\phi(z)$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}z^2} + P(z)\frac{\mathrm{d}\phi}{\mathrm{d}z} + Q(z)\phi = 0. \tag{7}$$

As a result, the solutions of the driven systems can be expressed by known functions. Most of the previously obtained solvable TLMs are related to the hypergeometric and confluent hypergeometric equations [30–33]. If eq. (7) takes the form of the HE, we have [34–36]

$$P(z) = \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a}, \tag{8}$$
$$Q(z) = \frac{\alpha\beta z - q}{z(z-1)(z-a)}, \tag{9}$$

where γ, δ, α, β, q, a, and $\varepsilon = \alpha + \beta + 1 - \gamma - \delta$ are parameters for the HE. The analytical solutions of the HE are given in terms of the HF, a special function [34–36]. In ref. [27], Ishkhanyan *et al* have found three different forms of $f(t)$ and $\nu(t)$,

$$f(t) = f_0 + \frac{f_1}{1+g\cos(\omega t)}, \quad \nu(t) = \nu_0 \tag{10}$$

and

$$f(t) = f_0, \quad \nu(t) = \frac{\nu_1}{(1+g\cos(\omega t))^k}, \quad k = 1/2, 1, \tag{11}$$

whose solutions are related to the HE. Here $-1 < g < 1$ is a parameter.

In this work, we found three additional cases where the TLMs are in relation to the HE, as shown in the following.

2.1 *Case I*

In the first case, $f_{\mathrm{I}}(t)$ and $\nu_{\mathrm{I}}(t)$ have the form

$$f_{\mathrm{I}}(t) = f_0 + \frac{f_1\sin(\omega t)}{1+g\cos(\omega t)}, \quad \nu_{\mathrm{I}}(t) = \nu_0, \tag{12}$$

where $f_{\mathrm{I}}(t)$ is periodic and $\nu_{\mathrm{I}}(t)$ is a constant. This model is characterised by five parameters $\{f_0, f_1, \omega, g, \nu_0\}$. When $g = 0$, the resulting model corresponds to the well-known monochromatically-driven TLM. Due to the introduction of the additional parameter g, $f_{\mathrm{I}}(t)$ represents an oscillating bias with multiple frequencies [37]. This can clearly be seen by the Taylor expansion of $f_{\mathrm{I}}(t)$ in the limit of small g,

$$f_{\mathrm{I}}(t) = f_0 + f_1\sin(\omega t)\sum_{n=0}^{\infty}(-g)^n\cos^n(\omega t). \tag{13}$$

It is found that if we apply the following transformations

$$z = \frac{\mathrm{e}^{i\tau}}{r_1}, \quad a_1(z) = z^{\lambda_1}(z-1)^{\lambda_2}\left(z - \frac{r_2}{r_1}\right)^{\lambda_3}\phi(z), \tag{14}$$

eq. (4) can be reduced to the HE for $\phi(z)$. Here, $r_1 = -(1+\sqrt{1-g^2})/g$ and $r_2 = -(1-\sqrt{1-g^2})/g$ are the two roots of $r^2 + gr/2 + 1 = 0$. The three parameters $\lambda_{1,2,3}$ are given as

$$\lambda_1 = \frac{1}{2}\sqrt{\left(\frac{f_0}{\omega} + i\frac{1}{g}\frac{f_1}{\omega}\right)^2 + \frac{\nu_0^2}{\omega^2}},$$
$$\lambda_2 = 1 - i\frac{1}{2g}\frac{f_1}{\omega}, \quad \lambda_3 = i\frac{1}{2g}\frac{f_1}{\omega}. \tag{15}$$

The relevant parameters in the HE are given as $a = r_2/r_1$, $q = r_2(1+2\lambda_1)(1-2\lambda_3)/r_1 - 4\lambda_3(f_0/2\omega + \lambda_1 + \lambda_3)/gr_1$, $\alpha = \lambda_1+\lambda_2+\lambda_3+\sqrt{(f_0/\omega - if_1/\omega g)^2 + \nu_0^2/\omega^2}/2$, $\beta = \lambda_1 + \lambda_2 + \lambda_3 - \sqrt{(f_0/\omega - if_1/\omega g)^2 + \nu_0^2/\omega^2}/2$, $\gamma = 1 + 2\lambda_1$, and $\delta = 2\lambda_2$.

In ref. [27], Ishkhanyan *et al* have found all the classes of the exactly solvable TLMs in terms of the HFs. In the following, we show that the first five-parametric model belongs to these classes. To do this, we first apply the transformations

$$a_1(\tau) = c_1(\tau)\mathrm{e}^{-i\int_0^\tau f(s)\mathrm{d}s/2\omega},$$
$$a_2(\tau) = c_2(\tau)\mathrm{e}^{i\int_0^\tau f(s)\mathrm{d}s/2\omega}, \tag{16}$$

and then get

$$i\frac{\mathrm{d}c_1}{\mathrm{d}\tau} = \frac{\nu(\tau)}{2\omega}\mathrm{e}^{-i\delta(\tau)}c_2, \tag{17}$$
$$i\frac{\mathrm{d}c_2}{\mathrm{d}\tau} = \frac{\nu(\tau)}{2\omega}\mathrm{e}^{i\delta(\tau)}c_1, \tag{18}$$

where

$$\delta(\tau) = -\int_0^\tau f(s)\mathrm{d}s/\omega.$$

These equations describe the interaction between the two-level system and the amplitude- and frequency-modulated laser field. The amplitude and frequency modulations of the laser field can be designed with the quantum optimal control theory [38,39]. Here $\nu(t)/2\omega$ corresponds to $U(t)$ in ref. [27] in different notation. To obtain solvable TLMs, it is found that $\varphi(z)$, $\nu(\tau)$, and $\delta(\tau)$ have the forms [27]

$$\varphi(z) = z^{\lambda_1}(z-1)^{\lambda_2}(z-a)^{\lambda_3}, \tag{19}$$
$$\frac{\nu(\tau)}{2\omega} = \nu^* z^{k_1}(z-1)^{k}_2(z-a)^{k_3}\frac{\mathrm{d}z}{\mathrm{d}\tau}, \tag{20}$$
$$\frac{\mathrm{d}\delta(\tau)}{\mathrm{d}\tau} = -\frac{f(\tau)}{\omega} = \left(\frac{\delta_1}{z} + \frac{\delta_2}{z-1} + \frac{\delta_3}{z-a}\right)\frac{\mathrm{d}z}{\mathrm{d}\tau}, \tag{21}$$

where the parameters $\lambda_{1,2,3}$, ν^*, $k_{1,2,3}$, and $\delta_{1,2,3}$ are constants to be determined. An exactly solvable TLM is defined by triads of parameters $\{k_1, k_2, k_3\}$, which run over a set of 35 possible choices [27]. It is found that the first five-parametric model belongs to the class $\{k_1, k_2, k_3\} = \{-1, 0, 0\}$, and can be achieved with the transformation $z = \sqrt{d}\exp(i\tau)$ and the specific parameters

$$\nu^* = -i\frac{\nu_0}{2\omega}, \quad \delta_1 = \frac{1}{2\omega}\left(2if_0 + \frac{(1+d)f_1}{\sqrt{d}}\right),$$
$$\delta_2 = \delta_3 = -\frac{(1+d)f_1}{2\sqrt{d}\omega}, \quad g = -\frac{2\sqrt{d}}{1+d}. \tag{22}$$

2.2 *Case II*

In the second case, we have

$$f_{\mathrm{II}}(t) = f_0 + \frac{f_1\sin(\omega t)}{1+g\cos(\omega t)},$$
$$\nu_{\mathrm{II}}(t) = \frac{\nu_1}{1+g\cos(\omega t)}, \tag{23}$$

where both $f_{\mathrm{II}}(t)$ and $\nu_{\mathrm{II}}(t)$ are time-dependent. It is found that the second five-parametric model belongs to the class $\{k_1, k_2, k_3\} = \{0, -1, -1\}$ [27], and can be achieved by the same transformation and the same specific parameters (22) but with $\nu^* = i\nu_1/2\omega$. In this case, the parameters $\lambda_{1,2,3}$ are given as

$$\lambda_1 = -\frac{1}{2}\left(\frac{f_0}{\omega} + i\frac{f_1}{g\omega}\right),$$
$$\lambda_{2,3} = -\frac{1}{2g}\sqrt{\frac{g^2}{1-g^2}\frac{\nu_1^2}{\omega^2} - \frac{f_1^2}{\omega^2}}. \tag{24}$$

The parameters in the HE read as, $a = r_2/r_1$, $q = -(if_1 + gf_0)(if_1 - 2g\omega\lambda_2)/r_1g^3\omega^2$, $\alpha = 1 - f_0/\omega + \lambda_2 + \lambda_3$, $\beta = -if_1/g\omega + \lambda_2 + \lambda_3$, $\gamma = 2\lambda_1$, and $\delta = 1 + 2\lambda_2$.

2.3 *Case III*

In the third case, we have

$$f_{\mathrm{III}}(t) = f_1\sin(\omega t), \quad \nu_{\mathrm{III}}(t) = \nu_1\cos(\omega t). \tag{25}$$

In this case, it is found that under the condition of $f_1 = \nu_1$, we may apply different transformations of the form

$$z = \sin^2(\tau/2), \quad \varphi(z) = (z-a)^2. \tag{26}$$

The parameters in the HE are given as, $a = 1/2$, $q = 1 - if_1/4\omega - f_1^2/8\omega^2$, $\alpha = 3/2 + \sqrt{1 + f_1^2/\omega^2}/2$, $\beta = 3/2 - \sqrt{1 + f_1^2/\omega^2}/2$, and $\gamma = \delta = 1/2$. The third two-parametric model does not belong to any class given in ref. [27]. However, if one takes a general form of $\varphi(z)$, $\nu(\tau)$, and $\delta(\tau)$

$$\varphi(z) = \mathrm{e}^{\lambda_0 z}z^{\lambda_1}(z-1)^{\lambda_2}(z-a)^{\lambda_3}, \tag{27}$$
$$\frac{\nu(\tau)}{2\omega} = \nu^* z^{k_1}(z-1)^{k}_2(z-a)^{k_3}\frac{\mathrm{d}z}{\mathrm{d}\tau}, \tag{28}$$
$$\frac{\mathrm{d}\delta(\tau)}{\mathrm{d}\tau} = -\frac{f(\tau)}{\omega} = \left(\delta_0 + \frac{\delta_1}{z} + \frac{\delta_2}{z-1} + \frac{\delta_3}{z-a}\right)\frac{\mathrm{d}z}{\mathrm{d}\tau}, \tag{29}$$

with $\lambda_0 = -i\delta_0/2$, the third model may belong to the class $\{k_1, k_2, k_3\} = \{-1/2, -1/2, 1\}$, and can be achieved by the transform $z = \sin^2(\tau/2)$ and the specific parameters $\delta_0 = -2f_1/\omega$, $\delta_{1,2,3} = 0$, $\nu^* = if_1/\omega$, $\lambda_{1,2} = 0$, and $\lambda_3 = 2$.

3. Analytical solutions in terms of the HFs

In this section, we present an analytical solution for these periodically-driven TLMs in terms of the HF.

HE has four regular singularities at $z = 0, 1, a, \infty$. If γ is not zero and a negative integer, HE has two linearly independent local Frobenius solution $\phi_{1,2}(x)$ around the regular singular point $z = 0$ [34–36]

$$\phi_1(z) = Hl(a, q; \alpha, \beta, \gamma, \delta; z) = \sum_{n=0}^{\infty} h_n z^n, \tag{30}$$

$$\phi_2(z) = z^{1-\gamma} Hl(a, q + (\varepsilon + \delta a)(1 - \gamma); \alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, \delta; z). \tag{31}$$

Here, $Hl(a, q; \alpha, \beta, \gamma, \delta; z) = \sum_{n=0}^{\infty} h_n z^n$ is usually known as the HF [36]. In such an infinite series solution, the coefficients h_n are obtained by using the three-term recurrence relation: $A(n)h_{n-1} + B(n)h_n + C(n)h_{n+1} = 0$ with the initial conditions $h_0 = 1$ and $h_{-1} = 0$. Here, $A(n) = (n - 1 + \alpha)(n - 1 + \beta)$, $B(n) = -q - n(n - 1 + \gamma)(1 + a) - n(a\delta + \varepsilon)$, and $C(n) = a(n + 1)(n + \gamma)$.

In the three cases of f_{I}, f_{II}, and f_{III}, γ is not zero and a negative integer. Therefore, $\phi_{1,2}(t)$ are two linearly independent solutions, and thus we have two linearly independent solutions for $a_1(\tau)$ near $z(\tau) = 0$

$$\psi_1(\tau) = \varphi(z) Hl(a, q; \alpha, \beta, \gamma, \delta; z), \tag{32}$$

$$\psi_2(\tau) = \varphi(z) Hl(a, q + (\varepsilon + \delta a)(1 - \gamma); \alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, \delta; z). \tag{33}$$

In terms of $\psi_{1,2}(\tau)$, the general solutions for $a_1(\tau)$ and $a_2(\tau)$ take the form

$$a_1(\tau) = c_1\psi_1(\tau) + c_2\psi_2(\tau), \tag{34}$$

$$a_2(\tau) = d_1\psi_1^*(\tau) + d_2\psi_2^*(\tau), \tag{35}$$

where the constants $c_{1,2}$ and $d_{1,2}$ are determined by the initial conditions $a_1(\tau_0)$, $\dot{a}_1(\tau_0)$, $a_2(\tau_0)$, and $\dot{a}_2(\tau_0)$. After a straight calculation, we obtain the time-evolution operator from τ_0 to τ

$$U(\tau, \tau_0) = \begin{pmatrix} U_{11}(\tau, \tau_0) & U_{12}(\tau, \tau_0) \\ -U_{12}^*(\tau, \tau_0) & U_{11}^*(\tau, \tau_0) \end{pmatrix}, \tag{36}$$

where $U_{11}(\tau, \tau_0) = \triangle_{1,2}(\tau, \tau_0)/\triangle_{1,2}(\tau_0, \tau_0) + if(\tau_0)\tilde{\triangle}_{1,2}(\tau, \tau_0)/2\omega\triangle_{1,2}(\tau_0, \tau_0)$, $U_{12}(\tau, \tau_0) = i\nu(\tau_0)\tilde{\triangle}_{1,2}(\tau, \tau_0)/2\omega\triangle_{1,2}(\tau_0, \tau_0)$, with $\triangle_{1,2}(\tau, \tau_0) = \psi_1(\tau)\dot{\psi}_2(\tau_0) - \psi_2(\tau)\dot{\psi}_1(\tau_0)$ and $\tilde{\triangle}_{1,2}(\tau, \tau_0) = \psi_1(\tau)\psi_2(\tau_0) - \psi_2(\tau)\psi_1(\tau_0)$.

In principle, the HF $Hl(a, q; \alpha, \beta, \gamma, \delta; z)$ is analytic under the condition of $|z| < \min\{1, a\}$ [34–36]. In the case of $f_{\mathrm{I,II}}$, we have $a = (1 - \sqrt{1 - g^2})/(1 + \sqrt{1 - g^2}) \leq 1$ from the condition of $-1 < g < 1$, and thus $Hl(a, q; \alpha, \beta, \gamma, \delta; z)$ is analytic in the range of $|z| < |a|$. Since $|a| < |z| = 1/(1 + \sqrt{1 - g^2}) < 1$, the time-evolution operator $U(\tau, \tau_0)$ is invalid in principle. In the case of f_{III}, we have $a = 1/2$, and thus $Hl(a, q; \alpha, \beta, \gamma, \delta; z)$ is analytic in the range of $|z| < 1/2$. As $z = \sin^2(\tau/2) \leq 1$, the time-evolution operator $U(\tau, \tau_0)$ is also invalid for a certain instant, for example, $\tau = \pi/3$. This problem can be solved in terms of a central two-point connection problem for the HE [40,41], and thus the time-evolution operator valid for any later instant can be constructed [23].

4. Applications

When $f(t)$ is periodic and $\nu(t)$ is constant, eqs. (4) and (5) have a similar form with the Schrödinger equation for certain periodic potentials. This motivates us to consider a type of the periodic potentials

$$V(x) = \frac{V_0 + V_1 \sin(x)}{1 + g\cos(x)} + \frac{V_2 + V_3 \sin(x)}{(1 + g\cos(x))^2}. \tag{37}$$

We note that the periodic potential with $V_2 = V_3 = 0$ has been realised in recent experiment [42,43]. The Schrödinger equation for such a periodic potential is given as ($2m = \hbar = 1$)

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + V(x)\psi(x) = E\phi(x). \tag{38}$$

By making the following transformations

$$z = -\frac{g\mathrm{e}^{ix}}{1 + \sqrt{1 - g^2}},$$

$$\psi(z) = z^{\lambda_1}(z - 1)^{\lambda_2}\left(z - \frac{1 - \sqrt{1 - g^2}}{1 + \sqrt{1 - g^2}}\right)^{\lambda_3}\phi(z) \tag{39}$$

with

$$\lambda_1 = -\sqrt{E - i\frac{V_1}{g}},$$

$$\lambda_2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4V_2}{1 - g^2} - i\frac{4V_3}{g\sqrt{1 - g^2}}},$$

$$\lambda_3 = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4V_2}{1 - g^2} + i\frac{4V_3}{g\sqrt{1 - g^2}}}, \tag{40}$$

the Schrödinger equation is reduced to the HE. The relevant parameters in the HE are given as, $a = (1 - \sqrt{1 - g^2})/(1 + \sqrt{1 - g^2})$, $q = -[2i(V_1 - V_3) - g(2V_0 + (1 + 2\lambda_1)((1 - \sqrt{1 - g^2})\lambda_2 + (1 + \sqrt{1 - g^2})\lambda_3)]/g(1 + \sqrt{1 - g^2})$, $\alpha = \lambda_1 + \lambda_2 + \lambda_3 - \sqrt{E + iV_1/g}$, $\beta = \lambda_1 + \lambda_2 + \lambda_3 + \sqrt{E + iV_1/g}$, $\gamma = 1 + 2\lambda_1$, and $\delta = 2\lambda_2$. If $\gamma = 1 + 2\lambda_1$ is not zero and a negative integer, we have two linearly independent solutions of the form

$$\psi_1(x) = z^{\lambda_1}(z-1)^{\lambda_2}(z-a)^{\lambda_3} \times Hl\left(a, q; \alpha, \beta, \gamma, \delta; -\frac{ge^{ix}}{1+\sqrt{1-g^2}}\right), \tag{41}$$

$$\psi_2(x) = z^{-\lambda_1}(z-1)^{\lambda_2}(z-a)^{\lambda_3} \times Hl\Big(a, q+(\varepsilon+\delta a)(1-\gamma); \alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, \delta; -\frac{ge^{ix}}{1+\sqrt{1-g^2}}\Big). \tag{42}$$

It is evident that the solutions $\psi_{1,2}(x)$ can represent the Bloch-wave solution with the wave vectors $k = \lambda_1 = -\sqrt{E - iV_1/g}$. From the expression of λ_1, we know that the wave vectors $k = \lambda_1$ may be complex. It is well-known that the Bloch waves of real wave numbers are amplitude-bounded oscillatory solutions, and the Bloch waves of complex wave numbers show unbounded exponential behaviour [45]. The energy spectrum for the infinitely periodic potential $V(x)$ consists of bands in which there exist only amplitude-bounded oscillatory solutions and gaps in which there exist unbounded oscillatory solutions. In the infinitely periodic potential $V(x)$, the solutions with the complex wave numbers are non-physical. However, in a finite potential $V(x)$, they may represent the physical solutions [46]. In the following, we shall show that under certain conditions, one can obtain certain exact in-gap solutions.

In general, there exists a necessary condition [34–36]

$$\alpha = -N, \quad N = 0, 1, 2, \ldots, \tag{43}$$

$$h_{N+1} = 0, \tag{44}$$

under which the HF $Hl(a, q; \alpha, \beta, \gamma, \delta; z)$ can be reduced to a polynomial in z. It follows that as α is related to E, eq. (43) gives the special values of E, and eq. (44) gives the relation between $V_{0,1,2,3}$ and g. In the following, we present certain exact results in the case of $V_2 = V_3 = 0$ where the resulting periodic potential has been realised in recent experiments [42,43].

In $\psi_1(x)$, from condition (43) with $N = 2$, we have

$$E = 1 - \frac{V_1^2}{4g^2}. \tag{45}$$

For an arbitrary value of V_1, we get from $h_3 = 0$

$$V_0 = 1. \tag{46}$$

The corresponding solution is given as

$$\psi(x) = \mathrm{e}^{-(V_1/2g)x}(1 + g\cos(x)). \tag{47}$$

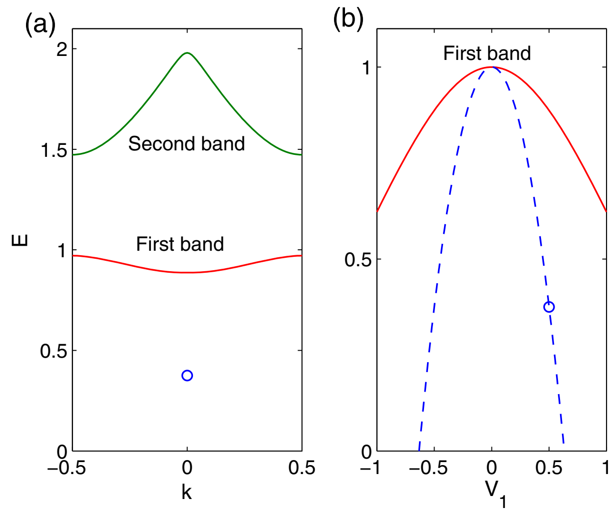

Figure 1. Band-gap structures for $V(x)$. (**a**) The first two bands for $V_0 = 1$, $V_1 = 1/2$, $V_2 = V_3 = 0$, and $g = 1/10$. The blue circle in (**a**) corresponds to the exact result $E = 1 - V_1^2/4g^2$. (**b**) The first band at $k = 0$ and the in-gap E as a function of V_1 with $V_0 = 1$, $V_2 = V_3 = 0$, and $g = 1/10$. The blue circle in (**b**) corresponds to the one in (**a**).

In comparison with the band-gap structure, if $V_1 \neq 0$, we find that $E = 1 - V_1^2/(4g^2)$ falls into the semi-infinite gap below the lowest band (see figure 1). This means that the solutions are a kind of in-gap states. If $V_1 = 0$, the in-gap solutions become stable Bloch-wave solutions, as the wave numbers become real. As the in-gap states grow without bound, they are non-physical states for the infinite periodic system. However, the in-gap states are important for the surface waves for finite periodic potential.

In addition, it is found that the hyperbolic version of the periodic potential $V(x)$ is

$$V(x) = \frac{V_0 + V_1\sinh(x)}{1 + g\cosh(x)} + \frac{V_2 + V_3\sinh(x)}{(1 + g\cosh(x))^2} \tag{48}$$

shows an asymmetric double-well structure under certain parameter conditions, as shown in figure 2. The case $V_1 = V_3 = 0$ corresponds to the symmetric double-well potential for certain specific parameters, and has been discussed in our previous work [14]. In this asymmetric double-well potential, we can use the following transforms:

$$z = -\frac{ge^x}{1+\sqrt{1-g^2}},$$

$$\psi(z) = z^{\lambda_1}(z-1)^{\lambda_2}\left(z - \frac{1-\sqrt{1-g^2}}{1+\sqrt{1-g^2}}\right)^{\lambda_3}\phi(z), \tag{49}$$

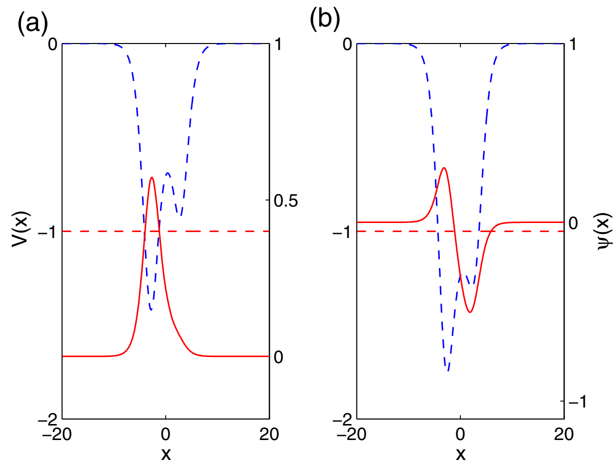

Figure 2. Profiles of the exact results $\psi(x)$ and the double-well potentials with (**a**) $V_0 = (-18g+\sqrt{V_3^2+36g^4})/4g$, $V_1 = 0$, and $V_2 = 15(1-g^2)/4 + V_3^2/16g^2$ and (**b**) $V_0 = (-18g - \sqrt{V_3^2+36g^4})/4g$, $V_1 = 0$, and $V_2 = 15(1-g^2)/4 + V_3^2/16g^2$. In (**a**) and (**b**), we have $g = 1/10$ and $V_3 = 1/10$. The dashed lines represent the analytical energies $E = -1$.

with

$$\lambda_1 = \sqrt{-E - \frac{V_1}{g}},$$
$$\lambda_2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{4V_2}{1-g^2} - \frac{4V_3}{g\sqrt{1-g^2}}},$$
$$\lambda_3 = \frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{4V_2}{1-g^2} + \frac{4V_3}{g\sqrt{1-g^2}}}, \tag{50}$$

to reduce the Schrödinger equation into the HE. The other relevant parameters in the HE are given as, $a = (1-\sqrt{1-g^2})/(1+\sqrt{1-g^2})$, $q = -[2(V_1 - V_3) + g(2V_0 - (1+2\lambda_1)(1-\sqrt{1-g^2})\lambda_2 - (1+2\lambda_1)(1+\sqrt{1-g^2})\lambda_3)]/g(1+\sqrt{1-g^2})$, $\alpha = \lambda_1 + \lambda_2 + \lambda_3 + \sqrt{-E}$, $\beta = \lambda_1 + \lambda_2 + \lambda_3 - \sqrt{-E}$, $\gamma = 1 + 2\lambda_1$, and $\delta = 2\lambda_2$. For this asymmetric double-well potential, under certain conditions, one can also obtain certain exact solutions. For example, if we take $V_2 = 15(1-g^2)/4 + V_3^2/16g^2$ and $V_1 = 0$, we have $E = -1$ and $V_0 = (-18g \pm \sqrt{V_3^2+36g^4})/4g$ from conditions (43) and (44) with $N = 1$. In the case of $V_0 = (-18g + \sqrt{V_3^2+36g^4})/4g$, the corresponding solution is given as

$$\psi(x) = N_1 e^{\lambda_1 x}(e^x - r_1)^{\lambda_2}(e^x - r_2)^{\lambda_3} \times \left(1 - \frac{\sqrt{36g^4 + V_3^2} - V_3}{6g^2} e^x\right). \tag{51}$$

In the case of $V_0 = (-18g - \sqrt{V_2^2+36g^4})/4g$, the corresponding solution is given as

$$\psi(x) = N_2 e^{\lambda_1 x}(e^x - r_1)^{\lambda_2}(e^x - r_2)^{\lambda_3} \times \left(1 - \frac{\sqrt{36g^4 + V_3^2} + V_3}{6g^2} e^x\right). \tag{52}$$

Here $N_{1,2}$ are the normalisation constants. Under certain parameter conditions, the exact solutions represent the bound-state wave functions. For example, when $g = 1/10$ and $V_3 = 1/10$, $\psi(x)$ in eq. (51) corresponds to the ground state and $\psi(x)$ in eq. (52) corresponds to the first excited state, as shown in figure 2.

In ref. [44], the potentials solvable in terms of the HFs have been fully classified. It is found that the periodic potential (37) and its hyperbolic version (48) belong to the ninth family with $\{m_1, m_2, m_3\} = \{1, 0, 0\}$. In addition, series expansions of the solutions of the HE in terms of special functions other than simple powers have been discussed [47–50]. For example, the expansion of Gauss hypergeometric functions takes the form

$$\phi(z) = \sum_{n=0}^{\infty} a_n \, {}_2F_1(\alpha, \beta, \gamma_0 - n, z), \tag{53}$$

where the coefficients a_ns obey the following three-term recurrence relation:

$$R_n a_n + Q_{n-1} a_{n-1} + P_{n-2} a_{n-2} = 0 \tag{54}$$

with

$$R_n = (1-a)(\varepsilon + \gamma + n - 1), \tag{55}$$
$$Q_n = -R_n + a(1 + n - \delta)(\varepsilon + n) + a\alpha\beta - q, \tag{56}$$
$$P_n = -\frac{a}{n+\varepsilon+\gamma}(n+\varepsilon) \times (n+\varepsilon+\gamma-\alpha)(n+\varepsilon+\gamma-\beta). \tag{57}$$

Similarly, there also exists a necessary condition

$$a_{N+1} = 0, \tag{58}$$
$$\varepsilon, \varepsilon+\gamma-\alpha \quad \text{or} \quad \varepsilon+\gamma-\beta = -N, \tag{59}$$

under which this series expansion is terminated at some $n = N$. Such a termination may result in more particular exact results. For example, in the simplest case of $N = 0$ with the parameters $V_0 = -6, V_1 = 0, V_2 = 6(1-g^2)$, and $V_3 = -6\sqrt{1-g^2}$, we have the energy $E = -9/4$ and the solution

$$\psi(x) = N_3 \frac{e^{3x/2}}{(e^x - r_1)^3}. \tag{60}$$

Here N_3 is the normalisation constant. It is evident that the exact solution describes the bound state.

5. Conclusion

In conclusion, we have presented three different sets of exactly solvable periodically-driven multiparametric TLMs whose solutions are given in terms of the HE. Our results are applied to construct exact analytical solutions for certain periodic potentials and asymmetric double-well potentials in relation to the HF. Under special parameter conditions, the HF can be terminated as a polynomial. This allows us to obtain some exact analytical results for the two kinds of potentials. In particular, it is shown that under special parameter conditions, an experimentally realised periodic potential allows the existence of the exact in-gap solutions.

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