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Moduli space of parabolic vector bundles over hyperelliptic curves

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Abstract. Let X be a smooth projective hyperelliptic curve of arbitrary genus g. In this article, we will classify the rank 2 stable vector bundles with parabolic structure along a reduced divisor of degree 4.

Keywords. Parabolic structure; vector bundles; quadrics.

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1. Introduction

It has been an interesting object in algebraic geometry to study parabolic moduli over algebraic varieties. The notion of parabolic bundle over algebraic curves has been introduced by Mehta and Seshadri [6]. They have constructed the moduli space of parabolic vector bundles over a smooth projective curve and shown that the moduli space is a projective variety. This has been generalized for higher dimensional varieties by Maruyama and Yokugawa [7]. On the other hand, Desale and Ramanan [3] gave a nice geometric description of the usual moduli space of rank 2 vector bundles over a hyperelliptic curve in terms of variety of isotropic linear subspaces of smooth intersection of two quadrics in a certain projective space.

In this paper, our aim is to give an analogous geometric description of a certain moduli space of rank 2 parabolic vector bundles over an arbitrary hyperelliptic curve of genus ≥ 2. We state the main result of this article:

Let $\omega_1, \omega_2, \ldots, \omega_{2g+2}$ be $(2g+2)$ points $(g \geq 2)$ on the affine line over the field of complex numbers and X be the irreducible nonsingular projective curve obtained as two sheeted covering of the projective line ramified precisely at $\omega_1, \omega_2, \ldots, \omega_{2g+2}$. Then X has exactly $2g+2$ Weierstrass points. We also denote them as $\omega_1, \omega_2, \ldots, \omega_{2g+2}$. Our aim in this paper is essentially to give an explicit description of the moduli space of parabolic vector bundles of rank 2 on X with parabolic structure along a reduced divisor D of degree 4.

Let Y be the two-sheeted cover of X ramified exactly at D and $\omega_1^{(1)}, \ldots, \omega_{2g+2}^{(1)}, \omega_1^{(2)}, \ldots, \omega_{2g+2}^{(2)}$ are the fibers over $\omega_1, \ldots, \omega_{2g+2}$. Then Y is also hyperelliptic with Weierstrass points $\omega_1^{(1)}, \ldots, \omega_{2g+2}^{(1)}, \omega_1^{(2)}, \ldots, \omega_{2g+2}^{(2)}$. We also denote the points in the projective line corresponding to the Weierstrass points by $\omega_1^{(1)}, \ldots, \omega_{2g+2}^{(1)}, \omega_1^{(2)}, \ldots, \omega_{2g+2}^{(2)}$. Let Q_1 and Q_2 denote the following quadrics in $(4g+3)$-dimensional projective space P,

$$Q_1 \equiv \sum_{i=1}^{2g+2} X_i^2 + \sum_{i=1}^{2g+2} Y_i^2 = 0,\ Q_2 \equiv \sum_{i=1}^{2g+2} \omega_i^{(1)} X_i^2 + \sum_{i=1}^{2g+2} \omega_i^{(2)} Y_i^2$$

and V_1, V_2 are the linear subspaces of P defined by $X_i = Y_i$ and $X_i = -Y_i$ respectively. Then we have

Theorem 1.1. *The moduli space $\mathcal{P}_\eta$ of isomorphism classes of rank* 2 *stable parabolic vector bundles with parabolic structures at x_2, x_3 and x_4 of weights* $(0, \frac{1}{2})$ *and trivial parabolic structure at x_1 with weights* $(0, 0)$, *where $D = x_1 + x_2 + x_3 + x_4$ and fixed determinant η, where η is defined in section 4 is isomorphic to the variety of* $(2g-1)$-*dimensional linear subspaces Λ of the projective space P, contained in the quadrics Q_1 and Q_2 such that* $\dim(V_1 \cap \Lambda) + \dim(V_2 \cap \Lambda) = 2g - 2$.

The idea of the proof of our theorem is very simple. If a finite group Γ acts on a smooth irreducible projective curve Y such that the quotient $X := Y/\Gamma$ is also smooth and irreducible with natural projection $p : Y \longrightarrow X$, then there is an equivalence of categories between the category of Γ-bundles on Y and the category of parabolic bundles on X with parabolic structure along the divisor of ramification points. On the other hand, there is a close relation with the moduli of Γ-bundles and the moduli space of Γ fixed points of the natural Γ action on the usual moduli space. In fact there is a forgetful map from the moduli space of Γ-bundles to the moduli space of Γ-fixed points.

If $\mathcal{M}(n, \delta)$ denote the usual moduli space of rank n stable vector bundles with fixed determinant δ, where δ is a Γ-invariant line bundle of odd degree, then the Γ-fixed locus of the natural action of Γ, namely $E \to \alpha^*(E)$, where $\alpha \in \Gamma$ and $E \in \mathcal{M}(n, \delta)$, is not in general irreducible (see [1]).

However, we will show in the case when $\Gamma = \mathbb{Z}_2$ and $n = 2$, and δ is of the form $\mathcal{O}(\sum_{i=1}^m n_i x_i)$, where all n_i's are odd integers except one and $x_1, x_2, \ldots, x_m$ are the fixed points of Γ-action on Y, it is in fact irreducible and the above forgetful map is an isomorphism. In this case, the Γ-structures on a Γ-fixed bundle (existence of Γ-structure on a Γ-fixed bundle is shown in Lemma 2.3) have non-trivial local type at all points except at one point.

We will use this fact by constructing a hyperelliptic curve Y from the given hyperelliptic curve X as a 2-sheeted cover ramified along a divisor of degree 4. By Riemann–Hurwitz formula, the genus of Y is $2g+1$. On the other hand, it is known [3] that the moduli space of rank 2 stable vector bundles over hyperelliptic curves of genus g is isomorphic to the variety of $(g-2)$-dimensional linear subspaces of a projective space of dimension $2g+1$, contained in a smooth intersection of two quadrics. Therefore, by the above discussion, the moduli space of parabolic vector bundles over X is isomorphic to the $\mathbb{Z}_2$-invariant linear subspace of dimension $2g - 1$ of a projective space of dimension $4g + 3$, contained in a smooth intersection of two quadrics.

2. Γ-bundles and parabolic bundles

2.1 Γ-*bundles*

Let Y be an irreducible smooth projective curve over $\mathbb{C}$. The group of algebraic automorphisms of Y is denoted by Aut(Y). It is known that Aut(Y) is finite. Let Γ be a subgroup of Aut(Y) and $p : Y \longrightarrow X = Y/\Gamma$ be the projection. Assume X is also smooth.

A Γ-bundle on Y is a vector bundle E together with a lift of the action of Γ to E. Let Γ_x denote the isotropy group at $x \in Y$. Then Γ_x is trivial for all but finitely many $x \in Y$. In fact Γ_x is non-trivial if $y = p(x)$ is a ramification point of X. It is also known [8] that locally at x, Γ-bundles are classified by equivalence class of representations of Γ_x into $GL(n)$.

We say that a Γ-bundle E of rank r on Y is of type τ, where τ represents representations $\rho_i : \Gamma_{x_i} \longrightarrow GL(r)$, x_i being a point of Y chosen over every ramification point $y_i \in X$ of $p : Y \longrightarrow X$, if at x_i, E is locally Γ_{x_i} isomorphic to the Γ_{x_i}-bundle defined by ρ_i.

Remark 2.1. Let $x \in Y$. Then an equivalence class of representation ρ of Γ_x into $GL(r)$ can be identified with a diagonal representation of the form

$$\rho(\alpha) = \begin{bmatrix} \zeta^{d_1} & 0 & \dots & 0 \\ 0 & \zeta^{d_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \zeta^{d_r} \end{bmatrix},$$

where α is generator of Γ_x, ζ is the primitive n_x-th root of unity and the local action of α on Y is defined by $\alpha \cdot z = \zeta \cdot z$ (z a local coordinate at x) and $0 \leq d_1 \leq \cdots \leq d_r < n_x - 1$, where n_x is the order of Γ_x.

DEFINITION 2.2

A Γ-bundle E is said to be of type $\alpha_x = (\alpha_1, \dots, \alpha_p)$, $p \leq r$ at $x \in Y$ if at x, E is locally Γ_x isomorphic to the Γ_x-bundle defined by diagonal representation $(\underbrace{\alpha_1 n_x, \dots, \alpha_1 n_x}_{k_1 \text{ times}}, \dots \underbrace{\alpha_p n_x, \dots, \alpha_p n_x}_{k_p \text{ times}})$, $k_1 + \cdots + k_p = r$.

A Γ-bundle E is Γ-semistable if the underlying bundle is semistable. A Γ-bundle E is stable if it is Γ-semistable and for every proper Γ-subbundle F of E, we have $\mu(F) < \mu(E)$, where $\mu(E) = \frac{\text{degree} E}{\text{rank} E}$.

Let $p_*^{\Gamma}(E)$ denote the Γ-invariant part of the bundle $p_* E$. Choose a point x_i over each ramification point $y_i \in X$. We define Γ-degree of E as follows:

$$\Gamma - \text{degree } E = \text{degree } p_*^{\Gamma}(E) + \sum \frac{d_1 + \cdots + d_r}{n_{x_i}},$$

where the sum has been taken over x_i.

2.2 *Parabolic bundles*

Let E be a vector bundle of rank r on X. Recall that a parabolic structure of length $p(\leq r)$ at a point $x \in X$ is a filtration

$$E_x = F^1 E_x \supsetneqq F^2 E_x \supsetneqq \cdots \supsetneqq F^p E_x,$$

where E_x denotes the fiber of E at x and weights α_i's attached to $F^i E_x'$s with $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_p < 1, i = 1, \ldots, p$. Set $k_i = \dim F^i E_x - \dim F^{i+1} E_x$. Then the parabolic degree of E is defined as

$$\text{par deg } E = \deg E + \sum k_i \alpha_i.$$

We write $\text{par } \mu(E) = \frac{\text{par deg } E}{\text{rank } E}$.

If W is a subbundle of E, it acquires, in an obvious way a quasi parabolic structure by taking the induced distinct flags. To make it a parabolic subbundle, attach weights as follows: Given i_0, $F^{i_0} W \subset F^j E$ for some j. Let j_0 be such that $F^{i_0} W \subset F^{j_0} E$ and $F^{i_0} W \not\subseteq F^{j_0+1} E$. Then weight of $F^{j_0} E$ = weight of $F^{i_0} W$. Define E to be parabolic stable (respectively semistable) if for every proper parabolic subbundle W of E, one has $\text{par } \mu(W) < \text{par } \mu(E)$ (respectively $\leq$).

Let D be a reduced divisor in X and E be a parabolic vector bundle on X, with weights at a point $x \in D$ given by $\alpha_1, \alpha_2, \ldots, \alpha_p$ whose multiplicities are $k_1, k_2, \ldots, k_p$. Then the parabolic degree of E is defined by

$$\text{par deg } E = \deg E + \sum_{x \in D} \left(\sum k_i \alpha_i \right).$$

2.3 *Comparison of* Γ*-bundles and parabolic bundles*

Let X, Y, Γ be as in section 2.1 and D be the ramification divisor. Let $D' = p^{-1}(D)$. Then for any Γ-bundle E of a given type, the bundle $p_*^\Gamma E$ has a natural parabolic structure along the divisor D [6]. If E is of type α_{x_i} at x_i, where x_i is a point lying over a ramification point y_i, $y_i \in D$, we fix weights α_{x_i} at y_i for the parabolic structure of $p_*^\Gamma E$ at y_i. Then Γ-degree of E is same as parabolic degree of $p_*^\Gamma E$.

Conversely, given a parabolic bundle on X with parabolic structure along D, one can associate a Γ-bundle on Y [2], where the author called them as orbifold bundle. In fact, this is an equivalence of categories [2], [6]. In this correspondence, a Γ-bundle is semistable if and only if the corresponding parabolic bundle is semistable ([2], Lemma 3.16). Thus the study of the moduli space of parabolic bundles of fixed degree and rank on X with parabolic structure along a divisor D is same as studying the moduli space of Γ-bundles on Y of a given type.

Let $\mathcal{M}$ denote the moduli space of stable bundles of rank 2 and fixed determinant δ, where δ is Γ-equivariant line bundle of odd degree. Then Γ has a natural action on $\mathcal{M}$. Let $\mathcal{M}^\Gamma$ denote the locus of Γ-fixed points of $\mathcal{M}$. From now on, we will assume $\Gamma = \mathbb{Z}_2$.

Lemma 2.3. *For each $E \in \mathcal{M}^\Gamma$, there exist exactly two $\mathbb{Z}_2$-structures on E.*

Proof. Let $E \in \mathcal{M}^\Gamma$. In other words, $\sigma^* E \simeq E$ where σ is a generator of Γ. We fix an isomorphism $f_1 : \sigma^* E \to E$. Now we claim that there is a Γ equivariant structure on E. It is known that giving a Γ equivariant structure on E is the same as giving an isomorphism $\phi : \sigma^* E \to E$ such that $\sigma^*\phi \circ \phi = Id_E$ as $(\sigma^*)^2 = Id,\ \sigma \in \mathbb{Z}_2$. Now since E is stable, $\sigma^* f_1 \circ f_1 = c \cdot Id_E$ for some non-zero $c \in \mathbb{C}^*$. Thus replacing the isomorphism f_1 by $\frac{1}{\sqrt{c}} f_1$, by the above observation, we will get an Γ equivariant structure on E. Our next claim is there are exactly two non isomorphic Γ equivariant structure on E. Let f_1, f_2 be two Γ-structure on E. Then we have $\sigma^* f_1 \circ f_1 = Id_E$ and $f_2 = c \cdot f_1$. But since f_2 also gives an Γ-structure, we have $\sigma^* f_2 \circ f_2 = Id_E$, which implies that $c\sigma^* f_1 \circ c f_1 = Id_E$. Thus we have $c_2^2 = 1$. Therefore, $f_2 = f_1$ or $-f_1$ which proves our claim. □

Since δ is an $\mathbb{Z}_2$-equivariant bundle, $\mathbb{Z}_2$ acts on each fiber corresponding to the points in D' and since the degree of δ is odd, the number of points in $x \in D'$ such that $\mathbb{Z}_2$ acts on the fiber δ_x as identity is odd.

From now on, we will assume the line bundle δ is of the form $\mathcal{O}(\sum_{i=1}^{r} n_i x_i)$, where $(x_1 + x_2 + \cdots + x_r) = D'$ and all n_i's are odd integers except one. Let $x_1 \in D'$ and $D_1 = D' - x_1$. Since Y is a 2:1 cover, D_1 has odd degree. For simplicity, we then take δ to be the line bundle associated to the divisor $D_1 + d x_1$, where d is an even integer. Then $\mathbb{Z}_2$ acts on the fiber δ_{x_1} by *Id* and by $-$*Id* on all other fibers associated to the points in $D' - x_1$.

Let $\rho_{x_1} : \mathbb{Z}_2 \to \mathrm{GL}(2, \mathbb{C})$ be the representation given by

$$\rho_{x_1}(\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where α is the nontrivial element of $\mathbb{Z}_2$ and $\rho_{x_i} : \mathbb{Z}_2 \to \mathrm{GL}(2, \mathbb{C})$ be the representation given by

$$\rho_{x_i}(\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

for all other points $x_i \in D', i \neq 1$. Let τ represent the representations ρ_{x_i}.

Lemma 2.4. *Each bundle $E \in \mathcal{M}^\Gamma$ admits a unique Γ-structure of type τ.*

Proof. Let $E \in \mathcal{M}^\Gamma$. Then by Lemma 2.3, E admits exactly two $\mathbb{Z}_2$-structures. More specifically, if an isomorphism $f : \alpha^* E \simeq E$ gives a Γ-structure, then the other one will be given by $-f$. These Γ-structures induce $\mathbb{Z}_2$ representations on each fibers E_{x_i} and these representations induce (taking determinant) representations on δ_{x_i}, which is *Id* on x_1 and $-$*Id* on other fibers. Thus the representation of $\mathbb{Z}_2$ on E_{x_1} is given by ρ_{x_1} or $-\rho_{x_1}$ and on the other fibers it is either ρ_{x_i} or $-\rho_{x_i}$, $i \neq 1$ and these two are equivalent. If the Γ-structure f induces the representation ρ_{x_1} at x_i, then the Γ-structure $-f$ induces the representation $-\rho_{x_1}$ at x_1. Thus on E, there exists a unique Γ-structure of type τ. □

Remark 2.5. Note that the fact $\mathrm{par}\deg(p_*^\Gamma(E)) = \Gamma\deg(E) = \frac{1}{\mathrm{ord}\,\Gamma}\deg E$ (see [8, page 18]) implies that the Γ-degree of two different Γ-structures on E coming from f and $-f$ are the same and hence the parabolic degree of the associated parabolic bundles are also the same, but the parabolic weights are different, namely $(0, \frac{1}{2})$ at $x_i, i \neq 1$ and $(0, 0)$ at x_1 and $(0, \frac{1}{2})$ at $x_i, i \neq 1$ and $(\frac{1}{2}, \frac{1}{2})$ at x_1 respectively.

Let $\mathcal{M}^\Gamma(\tau)$ denote the moduli space of Γ-stable bundles of rank 2 of fixed local type τ and fixed determinant δ . Then we have the following proposition:

PROPOSITION 2.6

$\mathcal{M}^{\Gamma}(\tau) \simeq \mathcal{M}^{\Gamma}$.

Proof. We observe that if a bundle E is Γ-stable, then the underlying bundle is semistable. But since we assume that the degree and rank are coprime, the underlying bundle is stable. Therefore, we have a natural forgetful morphism $g : \mathcal{M}^{\Gamma}(\tau) \to \mathcal{M}$ and clearly $\mathrm{Im}(g) \subseteq \mathcal{M}^{\Gamma}$.

On the other hand, by Lemma 2.4, on each $E \in \mathcal{M}^{\Gamma}$, there exists a unique Γ-structure of type τ. So the forgetful morphism $g : \mathcal{M}^{\Gamma}(\tau) \to \mathcal{M}^{\Gamma}$ is an isomorphism. □

Since the moduli space of Γ-bundles of a given type is irreducible ([8], Theorem 5), we have the obvious following corollary.

COROLLARY 2.7

$\mathcal{M}^{\Gamma}$ *is irreducible.*

Let $\mathcal{M}_P$ denote the moduli space of parabolic bundles of rank 2 and fixed determinant $p_*^{\Gamma}\delta$ with parabolic divisor D with trivial weight $(0, 0)$ at x_1 and $(0, \frac{1}{2})$ at all other points of D. Then by Proposition 2.6, we have the following obvious corollary.

COROLLARY 2.8

$\mathcal{M}_P \simeq \mathcal{M}^{\Gamma}$.

3. Construction of 2-1 cover

In this section, for a given hyperelliptic curve X, we will construct another hyperelliptic curve as a 2-sheeted cover of X. To get a 2-sheeted cover of X, we use the general method, namely, given a line bundle L and a section s of L^2 such that zero locus of s is reduced, consider the natural projection $p : \mathbb{P}(\mathcal{O} \oplus L) \longrightarrow X$ and $\mathcal{O}(1)$ the relatively ample line bundle.

Then $p_*(\mathcal{O}(1)) \simeq \mathcal{O} \oplus L^*$ which has a canonical section namely the constant section 1 of $\mathcal{O}$. This gives a section of $\mathcal{O}(1)$ over $\mathbb{P}(\mathcal{O} \oplus L)$ (as $H^0(\mathcal{O}(1)) = H^0(p_*(\mathcal{O}(1))))$ which we will denote by y. On the other hand $p_*(p^*L \otimes \mathcal{O}(1))$ is by projection formula isomorphic to $L \otimes p_*(\mathcal{O}(1)) \simeq L \otimes (\mathcal{O} \oplus L^*) \simeq L \oplus \mathcal{O}$. Hence it also has a canonical section and we denote the corresponding section of $p^*L \otimes \mathcal{O}(1)$ by x.

Now consider the section $x^2 + p^*sy^2$ of $p^*L^2 \otimes \mathcal{O}(2)$.

Let X_s denote its zero scheme. It is then clear that the restriction π of p to X_s is finite and that at any point v of X the fiber over v is the subscheme of $\mathbb{P}^1$ given by $x^2 + ay^2 = 0$, where (x, y) is a homogeneous co-ordinate system and a is the value of s identifying the fiber of L at $\mathbb{P}(\mathcal{O} \oplus L)$ with the residue field at v.

The genus of X_s is $2g + 1$, by the Riemann–Hurwitz formula. The curve X_s, in general, is not hyperelliptic. However, if we choose L to be the natural hyperelliptic line bundle h on X which is $\pi^*\mathcal{O}(1)$, where $\pi : X \longrightarrow \mathbb{P}^1$ is the natural projection map, then one can show that X_s is again hyperelliptic.

To see this, consider the hyperelliptic line bundle h on X. Then p^*h is a line bundle of degree 4 on X_s and by projection formula $p_*(p^*h) \simeq h \oplus \mathcal{O}$, which has exactly 3 sections. But by Clifford's theorem, a line bundle of degree 4 admits at most 3 sections and if it admits exactly 3 sections, then the underlying curve is hyperelliptic. Thus X_s is a hyperelliptic curve of genus $2g+1$ and p^*h is a multiple of the hyperelliptic line bundle.

Thus we have the following proposition.

PROPOSITION 3.1

Let X be a smooth projective hyperelliptic curve of genus $g$and $h$be the hyperelliptic line bundle. Let $s$be a general section of h^2. Then there exists a unique hyperelliptic curve Y of genus $2g+1$ which is a 2-sheeted cover of X ramified exactly along the zero locus of s.

Let us understand this picture geometrically. Let $\omega_1, \ldots, \omega_{2g+2}$ be the Weierstrass points of X. We denote the complex numbers $p(\omega_i)$ also by ω_i. Let $V = \sum_{i=1}^{2g+2} \mathbb{C}\omega_i$ be a vector space of dimension $2g+2$. Consider the projective space $\mathbb{P}(V) = \mathbb{P}^{2g+1}$. Let q_1 and q_2 be two non-degenerate quadratic forms in V associated to the quadrics $Q_1 = \sum_{i=1}^{2g+2} X_i^2$ and $Q_2 = \sum_{i=1}^{2g+2} \omega_i X_i^2$ and let $\mathbb{P}^1_\Phi$ be the pencil consisting the quadratic forms $\{q_\lambda\}_{\lambda\in\mathbb{P}^1}$ of the form $\mu_1 q_1 + \mu_2 q_2$ for $\mu = (\mu_1, \mu_2) \in \mathbb{P}^1$.

The family of g planes fit together in the following way. Let

$$\mathrm{Gen}(\Phi) \subset \mathbb{P}^1 \times \mathrm{Gr},$$

where $\mathrm{Gr} = \mathrm{Gr}(g+1, V)$ is the usual Grassmanian, and let it be defined by

$$\mathrm{Gen}(\Phi) = \{(\mu, E) : q_\mu |_E = 0\}.$$

It is obvious that the first projection $\mathrm{Gen}(\Phi) \longrightarrow \mathbb{P}^1$ has as fiber over μ, the variety of g planes isotropic to q_λ, which has two irreducible components whenever q_λ is smooth ([4], p. 735). Then we have as follows.

Theorem 3.2. $\mathrm{Gen}(\Phi)$ *is nonsingular and the morphism* $p_1 : \mathrm{Gen}(\Phi) \longrightarrow \mathbb{P}^1$ *has the Stein factorization*

$$\begin{tikzcd} \mathrm{Gen}(\Phi) \arrow[r, "p"] \arrow[d, "p_1"'] & X \arrow[dl, "q"] \\ \mathbb{P}^1 & \end{tikzcd},$$

where X is nonsingular, q is a double covering ramified precisely in $\mathrm{Sing}(\Phi)$, *where* $\mathrm{Sing}(\Phi)$*denotes the singular quadrics in the pencil, and p is smooth.*

Proof. See [5, Theorem, 1.10].

By the above theorem, the underlying set of the curve X can be identified with

$$\{(Q_\mu, \Sigma_\mu), \mu \in \mathbb{P}^1, \Sigma \text{ is an irreducible component of} g\text{-planes in } Q_\mu\}.$$

Remark 3.3. Here the ramification points are those $\mu \in \mathbb{P}^1$ for which the corresponding quadric Q_μ in the pencil is singular.

Let s be the section which vanishes at the points $(Q_1, \Sigma_1), (Q_1, \Sigma_2), (Q_2, \Sigma_1'), (Q_2, \Sigma_2')$. Then the image of π of these points in $\mathbb{P}^1$ are $(1, 0)$ and $(0, 1)$. Let $\pi_1 : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be the two sheeted cover ramified exactly at $(1, 0), (0, 1)$. Then the inverse image of ω_i are $\sqrt{\omega_i}, -\sqrt{\omega_i}$. Consider the vector space $V' = \sum_{i=1}^{2g+2} \mathbb{C}\sqrt{\omega_i} + \sum_{i=1}^{2g+2} \mathbb{C}(-\sqrt{\omega_i})$ and the quadrics $Q_1' = \sum_{i=1}^{2g+2} X_i^2 + \sum_{i=1}^{2g+2} Y_i^2$ and $Q_2' = \sum_{i=1}^{2g+2} \sqrt{\omega_i} X_i^2 - \sum_{i=1}^{2g+2} \sqrt{\omega_i} Y_i^2$. Then the curve obtained using Q_1', Q_2' as above can be identified with Y and the map $Y \longrightarrow X$ is just takes components of the system of $2g+1$-plane of $(\mu_1 Q_1' + \mu_2 Q_2')$ to the components of the system g-planes of $(\mu_1^2 Q_1 + \mu_2^2 Q_2)$.

4. Main theorem

Let X be a hyperelliptic curve of genus g with Weierstrass points $\omega_1, \omega_2, \dots, \omega_{2g+2}$ and h denotes the hyperelliptic line bundle on X. Choose a general section $s \in H^0(X, h^2)$. Let $p : Y \longrightarrow X$ be the 2-sheeted cover of X, ramified along the zero locus of s. By Proposition 3.1, such a curve exists and it is also hyperelliptic. Let $\omega_i^{(1)}, \omega_i^{(2)}$ be the fiber over ω_i. Then by the discussion in the previous section, the Weierstrass points of Y are precisely $\omega_1^{(1)}, \omega_1^{(2)}, \dots, \omega_{2g+2}^{(1)}, \omega_{2g+2}^{(2)}$. There is a natural $\mathbb{Z}_2$ action on Y namely, interchanging the sheets of p. Let $D = x_1 + x_2 + x_3 + x_4$ be the divisor of zeros of s and δ be a $\mathbb{Z}_2$-line bundle of the form $\mathcal{O}(4gx_1 + x_2 + x_3 + x_4)$ of degree $4g + 3$.

Consider the vector space $V := \sum_1^{2g+2} \delta_{\omega_i^{(1)}} + \sum_1^{2g+2} \delta_{\omega_i^{(2)}}$. Let Q_1 and Q_2 denote the following quadrics in $P := \mathbb{P}(V)$:

$$Q_1 = \sum_{i=1}^{2g+2} X_i^2 + \sum_{i=1}^{2g+2} Y_i^2,\ Q_2 = \sum_{i=1}^{2g+2} \omega_i^{(1)} X_i^2 + \sum_{i=1}^{2g+2} \omega_i^2 Y_i^{(2)}.$$

Then by Theorem 1 of [3], the moduli space U_δ of isomorphism classes of stable vector bundles on Y of rank 2 and fixed determinant δ is isomorphic to the variety of $2g - 1$ dimensional linear subspaces of the projective space $\mathbb{P}(V)$, contained in the quadrics Q_1 and Q_2.

This isomorphism is compatible with the $\mathbb{Z}_2$ action ([3], Lemma 5.7). Here the action of the non-trivial element of $\mathbb{Z}_2$ given by $E \to \iota^* E \otimes \beta \otimes \delta$, where β is a line bundle with $\beta^2 \cong h^{-(4g+3)}$ and ι is the hyperelliptic involution.

Let α be the generator of $\mathbb{Z}_2$. Then since $\delta \simeq \alpha^*\delta$, $\mathbb{Z}_2$ acts naturally on U_δ, simply by taking E to $\alpha^* E$. One can choose β in such a way that $\iota^* E \otimes \beta \otimes \delta \cong \alpha^* E$. Thus the isomorphism mentioned in the previous paragraph is compatible with the action of α. We denote the $\mathbb{Z}_2$-fixed points of U_δ by $U_\delta^{\mathbb{Z}_2}$. Then by Proposition 2.6, the moduli space of rank 2 stable $\mathbb{Z}_2$-bundles with determinant isomorphic to δ of type τ as in Lemma 2.4 is isomorphic to $U_\delta^{\mathbb{Z}_2}$.

PROPOSITION 4.1

Let V_1, V_2 are the linear subspaces of P defined by $X_i = Y_i$ and $X_1 = -Y_i$ respectively. Then $U_\delta^{\mathbb{Z}_2}$ is isomorphic to the variety of $(2g - 1)$-dimensional linear subspaces Λ of the projective space P, contained in the quadrics Q_1 and Q_2 such that $\dim(V_1 \cap \Lambda) + \dim(V_2 \cap \Lambda) = 2g - 2$.

Proof. The natural action of $\mathbb{Z}_2$ on Y fixes the divisor $\sum_1^{2g+2}\omega_i^{(1)} + \sum_1^{2g+2}\omega_i^{(2)}$, which induces a natural action on V, namely, $\alpha \cdot (X_1, \dots, X_{2g+2}, Y_1, \dots, Y_{2g+2}) = (Y_1, \dots, Y_{2g+2}, X_1, \dots, X_{2g+2})$. Now since U_δ of isomorphism classes of stable vector bundles on Y of rank 2 and fixed determinant δ is isomorphic to the variety of $2g-1$ dimensional linear subspaces of the projective space $\mathbb{P}(V)$, contained in the quadrics Q_1 and Q_2, $U_\delta^{\mathbb{Z}_2}$ is just the invariant $2g-1$ dimensional linear subspaces contained in $Q_1 \cap Q_2$ under this action. Since α is of order two, V is decomposed as the direct sum of 1 and -1 eigenspaces. It is clear from the action of α on V, that V_1 is 1-eigenspace and V_2 is -1-eigenspace. Now any invariant subspace will be decomposed in 1 and -1 eigenspaces and therefore each summands of this decomposition will be contained either in V_1 or in V_2. In other words, if Λ is a $2g-1$ dimensional invariant linear subspace,

$$\dim(V_1 \cap \Lambda) + \dim(V_2 \cap \Lambda) = 2g - 2,$$

which concludes the proposition. □

Let $\mathcal{P}_\eta$ denote the moduli of rank 2 parabolic bundles with fixed determinant $\eta := p_*^{\mathbb{Z}_2}\delta$ and parabolic structure along the divisor D with weights $(0, \frac{1}{2})$ at x_i, $i \neq 1$ and $(0, 0)$ at x_1. Then by Corollary 2.8 and Proposition 4.1, we have the following theorem.

Theorem 4.2. *$\mathcal{P}_\eta$ is isomorphic to the variety of $(2g-1)$-dimensional linear subspaces Λ of the projective space P, contained in the quadrics Q_1 and Q_2 such that* $\dim(V_1 \cap \Lambda) + \dim(V_2 \cap \Lambda) = 2g - 2$.

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