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SUMS OF ORDER BOUNDED DISJOINTNESS PRESERVING LINEAR OPERATORS

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Abstract: Necessary and sufficient conditions are found under which the sum of N order bounded disjointness preserving operators is n-disjoint with n and N naturals. It is shown that the decomposition of an order bounded n-disjoint operator into a sum of disjointness preserving operators is unique up to "Boolean permutation," the meaning of which is clarified in the course of the presentation.

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§ 1. Introduction

The problem of characterization of the operators between Banach lattices or vector lattices which may be written as finite sums of lattice homomorphisms was raised independently by Carothers and Feldman [1] and Huijsmans and de Pagter [2, Remark 2.3]. Bernau, Huijsmans, and de Pagter found the key concept of n-disjointness [3, Definition 1] and proved that a positive operator from a vector lattice to a Dedekind complete vector lattice decomposes into the sum of n lattice homomorphisms if and only if it is n-disjoint; see [3, Theorem 5]. Moreover, a similar result is true for order bounded disjointness preserving operators [3, Theorem 6], which yields a description of the ideal of order bounded operators generated by lattice homomorphisms [3, Theorem 8]. Radnaev [4, 5] managed to obtain the indicated decomposition result with the additional condition that the decomposition terms are pairwise disjoint. Obviously, even with this additional condition, the decomposition of a n-disjoint operator is not unique, so the question remains: In what exact sense is uniqueness be understood?

Another interesting problem by de Pagter and Schep in [6, Proposition 2.13] asks under what conditions the sum of two order bounded disjointness preserving operators is disjointness preserving as well. Kusraev and Kutateladze [7, Section 3.8] examined this question for arbitrary finite sums of disjointness preserving operators in the more general setting of n-disjoint operators. In this paper, the answers are given to both questions formulated above.

The paper is organized as follows. In Section 2 we briefly sketch the needed information concerning order bounded disjointness preserving operators. In Section 3 we introduce the new concepts of a purely n-disjoint operator and prove that each order bounded n-disjoint operator decomposes into a sum of purely k_i-disjoint components with $k_i \leq n$. Then it is verified that the sum $S_1 + \dots + S_n$ of order bounded disjointness preserving operators is purely n-disjoint if and only if S_i and S_j are purely disjoint for all $i \neq j$. Section 4 contains two main results of the paper. The first one gives necessary and sufficient conditions on a collection $S_1, \dots, S_N$ of order bounded disjointness preserving operators for $|S_1|+\dots+|S_N|$ to be purely n-disjoint with $n \leq N$. The second one tells us that purely k_i-disjoint components of an order bounded n-disjoint operator are representable (uniquely up to "Boolean permutation") as the sum of k_i pairwise purely disjoint operators that preserve disjointness.

There is a vast literature devoted to various aspects of disjointness preserving linear operators; we refer to the memoir [8] by Abramovich and Kitover and a few surveys: Boulabiar [9], Boulabiar, Buskes, and Triki [10], Gutman [11], and Huijsmans [12]. Further references can be found in the literature therein.

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We use the standard notation and terminology of Aliprantis and Burkinshaw [13] for the theory of vector lattices and positive operators. All vector lattices are assumed to be real and Archimedean. Throughout the text, an operator means a linear operator. Two elements $x, y \in E$ are called *disjoint* (denoted by $x \perp y$) whenever $|x| \wedge |y| = 0$. The *disjoint complement* $A^{\perp}$ of a nonempty set $A \subset E$ is defined by $A^{\perp} := \{x \in E : x \perp a \text{ for all } a \in A\}$ and $A^{\perp\perp}$ stands for $(A^{\perp})^{\perp}$. Denote by $\mathbb{P}(E)$ the Boolean algebra of band projections in a vector lattice E. A partition of unity in $\mathbb{P}(E)$ is a pairwise disjoin family of band projection with the sum equal to the identity operator on E. We let $:=$ denote the assignment by definition, while $\mathbb{N}$ and $\mathbb{R}$ symbolize the naturals and the reals. The notation $\mathrm{Hom}(E, F)$ is used for the set of all lattice homomorphisms from E to F.

§ 2. Preliminaries

In this section, we briefly recall some definitions and facts about lattice homomorphisms and disjointness preserving operators.

Definition 2.1. A linear operator $T : E \to F$ is said to be *disjointness preserving* if T sends disjoin elements in E to disjoint elements in F, i.e., $|x| \wedge |y| = 0$ implies $|T(x)| \wedge |T(y)| = 0$ (or equivalently $x \wedge y = 0$ implies $|T(x)| \wedge |T(y)| = 0$) for all $x, y \in E$.

It can be easily seen that a linear operator is disjointness preserving if and only if $|T(x)| = |T(|x|)|$ for all $x \in E$, while T is a lattice homomorphism if and only if $|T(x)| = T(|x|)$ for all $x \in E$. It follows that a linear operator is a lattice homomorphism if and only if it is positive and disjointness preserving. A nice description of order bounded disjointness preserving operators was found by Meyer in [14].

Theorem 2.2 (Meyer). *Let E and F be vector lattices and let T be an order bounded disjointness preserving linear operator from E to F. Then*

(1) *There exist lattice homomorphisms $T^+, T^- : E \to F$ such that $T = T^+ - T^-$ and*

$$T^+x = (Tx)^+, \quad T^-x = (Tx)^- \quad (x \in E_+);$$

in particular, T is regular and $\mathrm{Im}(T^+) \perp \mathrm{Im}(T^-)$.

(2) *The modulus $|T|$ exists, $|T|$ is a lattice homomorphism, $|T| = T^+ + T^-$ and*

$$|T(x)| = |T(|x|)| = ||T|(x)| = |T|(|x|) \quad (x \in E).$$

Proof. See Aliprantis and Burkishaw [13, Lemma 2.39 and Theorem 2.40], and Huijsmans and de Pagter [15, Theorem 1.1]. □

Corollary 2.3. *Let F be a vector lattice with the projection property. For an order bounded disjointness preserving linear operator $T : E \to F$ there exists a band projection $\pi \in \mathbb{P}(F)$ such that $T^+ = \pi|T|$ and $T^- = \pi^{\perp}|T|$. In particular, $T = (\pi - \pi^{\perp})|T|$ and $|T| = (\pi - \pi^{\perp})T$.*

The following result is a very useful characterization of lattice homomorphisms.

Theorem 2.4 (Kutateladze). *For a positive operator $T : E \to F$ between two vector lattices with F Dedekind complete the following statements are equivalent.*

(1) *T is a lattice homomorphism.*

(2) *For every operator $S : E \to F$ with $0 \le S \le T$, there exists a positive orthomorphism $R \in$* $\mathrm{Orth}(F)$ *satisfying $S = RT$.*

Proof. The proof can be found in Aliprantis and Burkinshaw [13, Theorem 2.50]; for the original proof; see Kutateladze [16]. The assumption of the Dedekind completeness of F can be weakened, but it can not be completely excluded; see Buskes and van Rooij [1] and Carothers and Feldman [17]. □

Definition 2.5. A linear operator $T : E \to F$ is said to be n*-disjoint* if, for every collection of $n+1$ pairwise disjoint elements $x_0, \dots, x_n \in E$, the meet of $\{|Tx_1|, \dots, |Tx_n|\}$ equals zero:

$$(\forall\, x_0, x_1, \dots, x_n \in E)\ x_k \perp x_l\ (k \neq l) \implies |Tx_0| \wedge \dots \wedge |Tx_n| = 0.$$

Proposition 2.6. *The sum of n order bounded disjointness preserving operators acting between vector lattices is n-disjoint.*

Proof. See Bernau, Huijsmans, and de Pagter [3, Proposition 2]. $\square$

The converse of this is also true, provided that the target space is Dedekind complete.

Theorem 2.7. *Let E and F be vector lattices with F Dedekind complete and let T be an order bounded n-disjoint operator from E to F. Then there exist n lattice homomorphisms $T_1, \dots, T_n$ from E to F such that $T = T_1 + \dots + T_n$. Moreover, $T_1, \dots, T_n$ can be chosen pairwise disjoint.*

Proof. The first assertion is obtained in Bernau, Huijsmans, and de Pagter [3, Theorem 6]; the second one in Radnaev [4, 5]. $\square$

It is easy to see that the representation of the n-disjoint operator as a sum of n disjointness preserving operators is not unique, see [3, Example 7]. Radnaev [5] observed that the decomposition of an n-disjoint operator is unique in a sense.

Proposition 2.8 (Radnaev). *Consider two disjoint collections of order bounded disjointness preserving operators $T_1, \dots, T_n$ and $S_1, \dots, S_m$ $(m, n \in \mathbb{N})$ from E to F. If $T_1 + \dots + T_n = S_1 + \dots + S_m$, then for each $i := 1, \dots, m$ there exists a disjoint collection of band projections $(\pi_{i,1}, \dots, \pi_{i,n})$ in $\mathbb{P}(F)$ such that $S_i = \sum_{j=1}^{n} \pi_{i,j} T_j$.*

§ 3. Purely n-Disjoint Operators

If an operator T is n-disjoint for some $n \in \mathbb{N}$, then T is m-disjoint for any $n \leq m \in \mathbb{N}$, but always there is the least $n \in \mathbb{N}$ for which T is still n-disjoint. At the same time, one can increase the length of the decomposition in Theorem 2.7 and Proposition 2.8 by splitting any of the operators T_k into the sum of the components $\pi_1 T_k, \dots, \pi_m T_k$ with a partition of unity $\pi_1, \dots, \pi_m$ consisting of non-zero band projections. These considerations motivate the following concept.

Definition 3.1. A linear operator $T : E \to F$ is said to be *purely n-disjoint* if n is the least natural for which πT is n-disjoint for all nonzero $\pi \in \mathbb{P}\big(T(E)^{\perp\perp}\big)$.

Clearly, if an operator is purely n-disjoint and purely m-disjoint then $n = m$. Moreover, the following result states that an order bounded n-disjoint operator uniquely decomposes into a sum of purely k-disjoint components, $k \leq n$.

Theorem 3.2. *Let E and F be vector lattices with F having the projection property, $n \in \mathbb{N}$, and T an n-disjoint linear operator from E to F with $F = T(E)^{\perp\perp}$. Then there exist a unique collection $(k_1, \pi_1), \dots, (k_l, \pi_l)$ with naturals $1 \leq k_1 < \dots < k_l \leq n$ and a partition of unity $\{\pi_1, \dots, \pi_l\}$ in $\mathbb{P}(F)$ with nonzero terms such that $\pi_i T$ is purely k_i-disjoint for all $i := 1, \dots, l$.*

Proof. Given $0 \neq \pi \in \mathbb{P}(F)$, denote by $\varkappa(\pi) = \varkappa_T(\pi)$ the least natural $k \in \mathbb{N}$ such that πT is k-disjoint. Clearly, $\varkappa(\pi) \leq n$ as πT is n-disjoint for every nonzero $\pi \in \mathbb{P}(F)$. Choose a band projection π_0 with $\varkappa(\pi_0) = \min\{\varkappa(\pi') : 0 \neq \pi' \leq \pi\}$. Then $0 \neq \pi_0 \leq \pi$ and π_0 is homogeneous with respect to $\varkappa$ in the sense that $\varkappa(\pi_0) = \varkappa(\varrho)$ for all $0 \neq \varrho \leq \pi_0$. By Definition 3.1 this means that $\pi_0 T$ is purely k-disjoint with $k = \varkappa(\pi_0)$. So, we come to the conclusion that for each nonzero $\pi \in \mathbb{P}(F)$ there exists a nonzero $\pi_0 \in \mathbb{P}(F)$ such that $\pi_0 \leq \pi$ and $\pi_0 T$ is purely $\varkappa(\pi_0)$-disjoin. It follows that there exists a partition of unity (π_ξ) in $\mathbb{P}(F)$ such that $\pi_\xi T$ is purely $\varkappa(\pi_\xi)$-disjoint for all ξ. The set of naturals $N := \{\varkappa(\pi) : 0 \neq \pi \in \mathbb{P}(F)\}$ is contained in $\{1, \dots, n\}$ and we can assume $N = \{k_1, \dots, k_l\}$ for some naturals $1 \leq k_1 < \dots < k_l \leq n$. Put $\pi_{k_i} := \bigvee\{\pi_\xi : \varkappa(\pi_\xi) = k_i\}$ and note that $\{\pi_{k_1}, \dots, \pi_{k_l}\}$ is a partition of unity in $\mathbb{P}(F)$. It remains to observe that $\varkappa(\pi_{k_i}) = k_i$ for all $i = 1, \dots, l$. Indeed, $\varkappa(\pi_{k_i}) \geq k_i$ by the definition of $\varkappa$. If $k = \varkappa(\pi_{k_i}) > k_i$ then there exists a collection of pairwise disjoin elements $x_0, \dots, x_{k_i}$ such that $\pi_{k_i}|Tx_0| \wedge \dots \wedge |Tx_{k_i}| \neq 0$. It follows that $\pi_\xi |Tx_0| \wedge \dots \wedge |Tx_{k_i}| \neq 0$ for some π_ξ with $\varkappa(\pi_\xi) = k_i$, i.e., $\varkappa(\pi_\xi) > k_i$; a contradiction.

Suppose now that there is another collection $(k_1', \pi_1'), \dots, (k_r', \pi_r')$ with the same properties as (k_1, π_1), $\dots, (k_l, \pi_l)$. If $\pi_0 := \pi_i \wedge \pi_j' \neq 0$ with $i \leq l$ and $j \leq r$, then $\pi_0 T$ is purely k_i-disjoint and purely k_j'-disjoint simultaneously and hence $k_i = k'_j$. At the same time $k_i \neq k'_s$ whenever $1 \leq s \leq r$, $s \neq j$, so that $\pi_i \wedge \pi'_s = 0$. This implies that $\pi_i \leq \pi'_j$ and similarly $\pi'_j \leq \pi_i$. Thus, we have shown that, given $i \leq l$ and $j \leq r$, either $\pi_i = \pi'_j$ and then $k_i = k'_j$, or $\pi_i \wedge \pi'_j = 0$ and then $k_i \neq k'_j$. It follows that the collections $(k_1, \pi_1), \dots (k_l, \pi_l)$ and $(k'_1, \pi'_1), \dots (k'_r, \pi'_r)$ coincide. $\square$

Lemma 3.3. *An order bounded linear operator T is n-disjoint (purely n-disjoint) if and only if so is the modulus $|T|$ of T.*

Proof. The first assertion was observed in the paper by Bernau, Huijsmans, and de Pagter [3] (see the proof of Theorem 6), and the second is easily verified by similar arguments. $\square$

Definition 3.4. Say that two lattice homomorphisms $T_1, T_2 : E \to F$ are *proportional over* $\pi \in \mathbb{P}(F)$ whenever $\pi(T_1 + T_2)$ is a lattice homomorphism; *purely disjoint over* π if $\pi T_1 \perp \pi T_2$ and there is no nonzero band projection $\rho \leq \pi$ such that T_1 and T_2 are proportional over ρ. If T_1 and T_2 are purely disjoint over any $\pi \in \mathbb{P}(F)$ with $\pi T_1 \neq 0$ and $\pi T_2 \neq 0$ then T_1 and T_2 are said to be *purely disjoint*.

Lemma 3.5. *Let E and F be vector lattices with F Dedekind complete. For a pair of lattice homomorphisms T_1 and T_2 from E to F there exist a unique band projection $\pi \in \mathbb{P}(F)$ such that T_1 and T_2 are proportional over π and purely disjoint over $\pi^\perp := I_F - \pi$. Moreover, T_1 and T_2 are disjoint if and only if $\mathrm{Im}(\pi T_1)$ and $\mathrm{Im}(\pi T_2)$ are disjoint.*

Proof. Denote $T_0 := T_1 + T_2$ and $\pi := \bigvee \Pi(T_0)$, where $\Pi(T_0) := \{\pi' \in \mathbb{P}(F) : \ \pi' T_0 \in \mathrm{Hom}(E, F)\}$. Clearly, for any $\pi' \in \Pi(T_0)$ and $\pi'' \in \mathbb{P}(F)$ the relation $\pi'' \leq \pi'$ implies $\pi'' \in \Pi(T_0)$; therefore, there exists a disjoint family $(\pi_\xi)_{\xi \in \Xi}$ in $\Pi(T_0)$ such that $\pi = \bigvee_{\xi \in \Xi} \pi_\xi$. It follows that $\pi \in \Pi(T_0)$ or, equivalently, $\pi T_0 \in \mathrm{Hom}(E, F)$, as $\pi T_0 = o\text{-}\sum_{\xi \in \Xi} \pi_\xi T_0$ and $\mathrm{Im}(\pi_\xi T_0) \perp \mathrm{Im}(\pi_\eta T_0)$ for all $\xi \neq \eta$. Now define $T := \pi T_0$ and observe that, in view of the equality $T = \pi T_1 + \pi T_2$ and the Kutateladze Theorem, there exist positive orthomorphisms $\alpha_1, \alpha_2 \in \mathrm{Orth}(F)_+$ such that $\pi T_1 = \alpha_1 T$, $\pi T_2 = \alpha_2 T$, and the equation $\alpha_1 + \alpha_2 = \pi$ holds on $\mathrm{Im}(T)$. The latter implies $\alpha_1 + \alpha_2 = \pi$, so that T_1 and T_2 are proportional over π.

Prove the second part of our claim dealing with $\pi^\perp$. Assume that $\mathrm{Im}(\pi^\perp T_1)^{\perp\perp} \neq \pi^\perp F$. Then there exists a nonzero band projection $\rho \leq \pi^\perp$ such that $\rho T_1 = 0$. Thus, $(\pi + \rho) T_0 = \pi T_0 + \rho T_2 \in \mathrm{Hom}(E, F)$, since $\pi T_0, \rho T_2 \in \mathrm{Hom}(E, F)$ and $\pi \perp \rho$. So, get the inclusion $\pi + \rho \in \Pi(T_0)$, which contradicts the definition of π, and therefore, $\mathrm{Im}(\pi^\perp T_1)^{\perp\perp} = \pi^\perp F$. The same reasoning works for T_2.

We now put $S := (\pi^\perp T_1) \wedge (\pi^\perp T_2) = \pi^\perp (T_1 \wedge T_2)$ and let ρ stand for the band projection onto the band in F generated by $S(E)$. By Kutateladze's Theorem there are orthomorphisms $\beta_1, \beta_2 \in \mathrm{Orth}(F)$ such that $0 \leq \beta_1, \beta_2 \leq \rho$ and $S = \beta_1 T_1 = \beta_2 T_2$. Moreover, $\ker(\beta_1) \cap \rho F = \ker(\beta_2) \cap \rho F = \{0\}$, since $\ker(\beta_k) = \mathrm{Im}(\beta_k)^\perp$; see [13, Theorem 2.52]. It follows that one can pick an invertible positive orthomorphism $\hat{\beta}_1 \in \mathrm{Orth}(F^u)$ such that $\hat{\beta}_1|_F = \beta_1$ and $\hat{\beta}_1^{-1} \beta = \rho$, where F^u is the universal completion of F (see, for instance, [18, Corollaries 3.8, 3.10, and 4.6]). Denote $\beta := \hat{\beta}_1 \beta_2$ and observe that $\rho T_0 = (\beta + \rho) T_2 \in \mathrm{Hom}(E, F)$; therefore, $(\pi + \rho) T_0 \in \mathrm{Hom}(E, F)$ as $\pi \perp \rho$. By the definition of π we have $\rho = 0$ and hence $S = 0$. $\square$

Remark 3.6. It follows from the above that for the same E and F, Lemma 3.5 can be stated in more detail in the following way. Given a pair of lattice homomorphisms T_1 and T_2 from E to F, there exist a unique band projection $\pi \in \mathbb{P}(F)$, a lattice homomorphism $T \in \mathrm{Hom}(E, \pi F)$, and positive orthomorphisms $\alpha_1, \alpha_2 \in \mathrm{Orth}(F)_+$ such that

$$\alpha_1 + \alpha_2 = \pi, \quad \pi T_1 = \alpha_1 T, \quad \pi T_2 = \alpha_2 T, \quad \mathrm{Im}(T)^{\perp\perp} = \pi F,$$
$$\mathrm{Im}(\pi^\perp T_1)^{\perp\perp} = \mathrm{Im}(\pi^\perp T_2)^{\perp\perp} = \pi^\perp F, \quad \pi^\perp T_1 \perp \pi^\perp T_2.$$

Moreover, T_1 and T_2 are disjoint if and only if we can choose α_1 and α_2 disjoint.

For completeness of the presentation, we also reproduce the following auxiliary fact, which can be found in [7, Proposition 3.12.B.4].

Lemma 3.7. *Given n pairwise disjoint nonzero real-valued lattice homomorphisms $h_1, \dots, h_n$ on a vector lattice E, there exist pairwise disjoint elements $x_1, \dots, x_n \in E$ such that $h_i(x_j) = \delta_{ij}$ for all $i, j := 1, \dots, n$ (with $\delta_{i,j}$ the Kronecker delta).*

PROOF. Pick $u_i \in E_+$ with $h_i(u_i) > 0$ and put $u := u_1 + \dots + u_n$. By the Kakutani–Kreins Representation Theorem the order ideal E_u in E generated by u can be identified with a norm dense vector sublattice of $C(Q)$ containing constants and separating points, where $C(Q)$ is the Banach lattice of continuous functions on a Hausdorff compact topological space Q. Moreover, u corresponds under this identification to the identically-one function $1_Q \in C(Q)$. Then the restrictions $h_1|_{E_u}, \dots, h_n|_{E_u}$ are pairwise disjoint nonzero lattice homomorphisms. Let $\hat{h}_i$ stand for the extension of $h_i|_{E_u}$ to $C(Q)$ by norm continuity. Clearly, $\hat{h}_1, \dots, \hat{h}_n$ are also pairwise disjoint nonzero lattice homomorphisms and so there exist distinct points $q_1, \dots, q_n \in Q$ such that $\hat{h}_i$ coincides with the Dirac measure $\delta_{q_i} : x \mapsto x(q_i)$ $(x \in C(Q))$. By the Tietze–Urysohn Theorem we can find pairwise disjoint continuous functions $y_1, \dots, y_n \in C(Q)$ such that $y_i(q_i) = 1$ and $0 \leq y_i(q) \leq 1$ for all $q \in Q$ and $i := 1, \dots, n$. Take $\bar{y}_i \in E_u$ so that $\|y_i - \bar{y}_i\| < \varepsilon < 1/2$ and note that $h_i(\bar{y}_i) - \varepsilon > 1 - 2\varepsilon > 0$ and $\bar{y}_i - \varepsilon 1_Q \leq y_i$. Put $x_i := (h_i(\bar{y}_i) - \varepsilon)^{-1}(\bar{y}_i - \varepsilon 1_Q)^+$ and observe that $\{x_1, \dots, x_n\} \subset E$ is the required collection. $\square$

Theorem 3.8. *Let E and F be vector lattices with F Dedekind complete. For a finite collection $S_1, \dots, S_n$ of order bounded disjointness preserving operators from E to F the sum $S := S_1 + \dots + S_n$ is purely n-disjoint if and only if $S_1(E)^{\perp\perp} = \dots = S_n(E)^{\perp\perp}$ and $S_k \perp S_l$ for all $1 \leq k, l \leq n$, $k \neq l$.*

PROOF. The "only if" part is immediate from Lemma 3.5. Indeed, applying Remark 3.6 to the lattice homomorphisms $T_1 := |S_k|$ and $T_2 := |S_l|$ yields $\pi = 0$, since otherwise we can replace $\pi|S_k| + \pi|S_l|$ by $(\alpha_1 + \alpha_2)T$ for some lattice homomorphism $T : E \to F$ and arrive at the inequality $\pi|S| \leq \sum_{j \neq k,l} \pi|S_j| + \pi T$ with $n-1$ terms on the right-hand side, whence $\pi|S|$ is $(n-1)$-disjoint, which contradicts Lemma 3.3. Thereby $|S_k| \perp |S_l|$ and $\mathrm{Im}(|S_k|)^{\perp\perp} = \mathrm{Im}(|S_l|)^{\perp\perp}$, or equivalently, $S_k \perp S_l$ and $\mathrm{Im}(S_k)^{\perp\perp} = \mathrm{Im}(S_l)^{\perp\perp}$.

To verify the "if" part, suppose that $S_1, \dots, S_n$ are pairwise disjoint and $S_k(E)^{\perp\perp} = F$ for all $k = 1, \dots, n$. We can assume without loss of generality that $S_1, \dots, S_n$ are lattice homomorphisms, since $|S| = |S_1| + \dots + |S_n|$ and $\mathrm{Im}(S_k)^{\perp\perp} = \mathrm{Im}(|S_k|)^{\perp\perp}$ for all $k := 1, \dots, n$. Fix an arbitrary nonzero band projection $\rho \in \mathbb{P}(F)$ and verify that ρS is not $(n-1)$-disjoint. Since $S_k(E)^{\perp\perp} = \bigvee\{S_k(u)^{\perp\perp} : u \in E_+\}$, we can choose a nonzero band projection $\sigma \leq \rho$ and $u_1, \dots, u_n \in E_+$ such that $\sigma S_k(u_k)^{\perp\perp} = \sigma(F)$. Put $u = u_1 + \dots + u_n$, $v = S_1(u_1) + \dots + S_n(u_n)$ and denote by E_u and F_v the order ideals in E and F generated by u and v, respectively. Let $\bar{S}$ and $\bar{S}_k$ stand for the respective restrictions of σS and σS_k onto E_u. Then $\bar{S}_1, \dots, \bar{S}_n$ are pairwise disjoint lattice homomorphisms from E_u into σF_v and $\bar{S}_k(E_u)^{\perp\perp} = \sigma F_v$ for all $k = 1, \dots, n$.

Identify σF_v with $C(Q)$ for some totally disconnected compact Hausdorff space Q and put $H := \bigcup_{k \neq l} H_{kl}$, where H_{kl} is defined as

$$H_{kl} := \{q \in Q : \ |\delta_q(\bar{S}_k + \bar{S}_l)x| = \delta_q(\bar{S}_k + \bar{S}_l)(|x|) \text{ for all } x \in E_u\}$$

with δ_q denoting the Dirac measure at $q \in Q$. Then H is a closed subset of Q, since such are H_{kl}, its interior $U := \mathrm{int}(H)$ is clopen, and let π stands for the corresponding band projection in $C(Q)$, i.e., $\pi : f \mapsto \chi_U f$ $(f \in C(Q))$. Clearly, $\pi(\bar{S}_k + \bar{S}_l)$ is a lattice homomorphism by the definition of H_{kl} and, by Lemma 3.5, we get the relation $\mathrm{Im}(\pi\bar{S}_k) \perp \mathrm{Im}(\pi\bar{S}_l)$ which is impossible by virtue of the equalities $\bar{S}_k(E_u)^{\perp\perp} = \bar{S}_l(E_u)^{\perp\perp} = \sigma F_v$. It follows that $\pi = 0$ or, equivalently, H is nowhere dense. Thus, $Q \setminus U$ is dense in Q, so there exists a real-valued lattice homomorphism h on F_u (we can take $h = \delta_q$ with $q \in Q \setminus U$) such that $h \circ \bar{S}_1, \dots, h \circ \bar{S}_n$ are pairwise disjoint nonzero lattice homomorphisms on E_u. By Lemma 3.7 there exist pairwise disjoint elements $0 \leq x_1, \dots, x_n \in E_u$ such that $(h \circ \bar{S}_k)(x_l) = \delta_{kl}$ for all $k, l := 1, \dots, n$ (with the Kronecker delta δ_{kl}). It follows that

$$1 = \bigwedge_{k=1}^{n} (h \circ \bar{S}_k)(x_k) = h\Big(\bigwedge_{k=1}^{n} \bar{S}(x_k)\Big) = h\Big(\bigwedge_{k=1}^{n} \sigma S(x_k)\Big),$$

and we arrive at the relation $0 < \sigma S(x_1) \wedge \dots \wedge \sigma S(x_n) \leq \rho S(x_1) \wedge \dots \wedge \rho S(x_n)$. Consequently, S is not $(n-1)$-disjoint and the proof is complete. $\square$

Corollary 3.9. *Let E and F be vector lattices with F Dedekind complete. Order bounded disjointness preserving operators $S_1, S_2 : E \to F$ are purely disjoint if and only if $S_1 \perp S_2$ and $S_1(E)^{\perp\perp} = S_2(E)^{\perp\perp}$.*

PROOF. Evidently, if $S_1 \perp S_2$ and $S_1(E)^{\perp\perp} = S_2(E)^{\perp\perp}$ then S_1 and S_2 are purely disjoint. Therefore, it only needs to be noted that, in accordance with Definitions 3.1 and 3.4, if S_1 and S_2 are purely disjoint then $S_1 + S_2$ is purely 2-disjoint and the required result follows from Theorem 3.8. $\square$

REMARK 3.10. It was shown in Kusraev and Kutateladze [7, Theorem 3.8.7] that Lemma 3.5 is a Boolean valued interpretation of the following simple fact: *Every two real-valued lattice homomorphisms on a vector lattice are either disjoint, or proportional.* Similarly, we can show that Theorem 3.8 is a Boolean valued interpretation of the assertion: *The sum of a finite collection of real-valued lattice homomorphisms is n-disjoint if and only if they are pairwise disjoint and the number of nonzero terms is n.*

§ 4. The Main Results

In this section, we answer the following two questions: Under what conditions is the finite sum of disjointness preserving operators n-disjoint and in which sense the decomposition of a n-disjoint operator into the sum of disjointness preserving operators is unique? We start with the two simple auxiliary facts.

Lemma 4.1. *Let $T : E \to F$ be a lattice homomorphism and $F = T(E)^{\perp\perp}$. Then there exists a Boolean isomorphism from the Boolean algebra $\mathbb{C}(T)$ of components of T onto the Boolean algebra of band projections $\mathbb{P}(F)$ such that $\varphi(S) \circ T = S$ for all $S \in \mathbb{C}(T)$.*

PROOF. See Kusraev [19, Proposition 3.3.4(1)]. $\square$

Lemma 4.2. *Let $S_1, \dots, S_N, T$ be lattice homomorphisms from a vector lattices E to a Dedekind complete vector lattice F and $S_j \in T^{\perp\perp}$ for all $j \le N$. Then there exists a partition of unity $\pi_1, \dots, \pi_N$ in $\mathbb{P}(F)$ such that*

$$\pi_1 S_1 + \dots + \pi_N S_N = S_1 \vee \dots \vee S_N.$$

PROOF. The operator $S := S_1 \vee \dots \vee S_N$ lies in the band $T^{\perp\perp}$, so that S is a lattice homomorphism. Since $0 \le S_j \le S$ by the Kutateladze Theorem, we can find $\alpha_j \in \mathrm{Orth}(F)$ with $S_j = \alpha_j T$ and $0 \le \alpha_j \le I_F$ ($j = 1, \dots, N$). There exists a partition of unity $\pi_1, \dots, \pi_N$ in $\mathbb{P}(F)$ such that $\pi_1\alpha_1 + \dots + \pi_N\alpha_N = \alpha_1 \vee \dots \vee \alpha_N$ and so $\pi_1 S_1 + \dots + \pi_N S_N = (\alpha_1 \vee \dots \vee \alpha_N)S$. Using the Meyer Theorem, for every $x \in E_+$, we deduce

$$(\alpha_1 \vee \dots \vee \alpha_N)Sx = (\alpha_1 Sx) \vee \dots \vee (\alpha_N Sx) = S_1x \vee \dots \vee S_Nx = (S_1 \vee \dots \vee S_N)x$$

and the proof is done. $\square$

DEFINITION 4.3. Given two collections $\mathscr{T} := (T_1, \dots, T_N)$ and $\mathscr{S} := (S_1, \dots, S_N)$ of linear operators from E to F, say that $\mathscr{T}$ is a $\mathbb{P}(F)$-*permutation* of $\mathscr{S}$ whenever there exists an $N \times N$ matrix $(\pi_{i,j})$ with entries from $\mathbb{P}(F)$, whose rows and columns are partitions of unity in $\mathbb{P}(F)$, such that $T_i = \sum_{j=1}^N \pi_{i,j} S_j$ for all $i := 1, \dots, N$ (and so $S_j = \sum_{i=1}^N \pi_{i,j} T_i$ for all $j := 1, \dots, N$).

We now present our first main result on the collections of order bounded disjointness preserving operators $S_1, \dots, S_N$ with purely n-disjoint sum $|S_1| + \dots + |S_N|$.

Theorem 4.4. *Let E and F be vector lattices with F Dedekind complete and $n, N \in \mathbb{N}$ with $n \le N$. For a collection of order bounded disjointness preserving operator $S_1, \dots, S_N$ from E to F the operator $|S_1| + \dots + |S_N|$ is purely n-disjoint if and only if there exists a $\mathbb{P}(F)$-permutation $T_1, \dots, T_N$ of $S_1, \dots, S_N$ such that $T_1, \dots, T_n$ are pairwise purely disjoint and, whenever $n < N$, each of $T_{n+1}, \dots, T_N$ is representable as $T_j = \sum_{k=1}^n \alpha_{j,k} T_k$ for some pairwise disjoint $0 \le \alpha_{j,1}, \dots, \alpha_{j,n} \in \mathscr{Z}(F)$ ($j := n+1, \dots, N$).*

PROOF. We can assume without loss of generality that $S(E)^{\perp\perp} = F$, where $S = |S_1| + \dots + |S_N|$.

Sufficiency. Assume that a $\mathbb{P}(F)$-permutation $(T_1, \dots, T_N)$ of $(S_1, \dots, S_N)$ satisfies the hypothesis of the theorem. Then $S = \beta_1 T_1 + \dots + \beta_n T_n$ with $\beta_k := I_F + \sum_{j=n+1}^{N} \alpha_{j,k}$ for all $k = 1, \dots, n$. Note that $\beta_k \geq I_F$ and hence β_k is invertible in $\mathrm{Orth}(F)$, so that $\mathrm{Im}(\beta_k T_k)^{\perp\perp} = \mathrm{Im}(T_k)^{\perp\perp}$. It follows $\mathrm{Im}(\beta_1 T_1)^{\perp\perp} = \dots = \mathrm{Im}(\beta_n T_n)^{\perp\perp}$ and S is purely n-disjoint by Theorem 3.8.

Necessity. Suppose S is purely n-disjoint. We will proceed in four steps.

Step 1. We verify first that there is no loss of generality in assuming that $S_1, \dots, S_N$ are lattice homomorphisms. Indeed, if the claim is true for lattice homomorphisms then there exists a $\mathbb{P}(F)$-permutation $R_1, \dots, R_N$ of $|S_1|, \dots, |S_N|$ such that $R_1, \dots, R_n$ are pairwise purely disjoint and, whenever $n < N$, each of $R_{n+1}, \dots, R_N$ is representable as $R_j = \sum_{k=1}^{n} \bar{\alpha}_{j,k} R_k$ for some pairwise disjoint $0 \leq \bar{\alpha}_{j,1}, \dots, \bar{\alpha}_{j,n} \in \mathscr{Z}(F)$ $(j := n+1, \dots, N)$. Let $(\pi_{i,j})$ be the $N \times N$ matrix of $\mathbb{P}(F)$-permutation as in Definition 4.3 and put $T_i = \sum_{j=1}^{N} \pi_{i,j} S_j$ for all $i := 1, \dots, N$. Then $|T_i| = R_i$ and hence $T_1 \dots, T_n$ are pairwise purely disjoint. By Corollary 2.3 there exists an orthomorphism τ_i of the form $\rho_i - \rho_i^{\perp}$, $\rho_i \in \mathbb{P}(F)$, such that $T_i = \tau_i R_i$ for all $i = 1, \dots, N$. Now, if $n < N$ and $n < j \leq N$ then, putting $\alpha_{j,k} := \tau_j \bar{\alpha}_{j,k} \tau_k$ we deduce

$$T_j = \tau_j R_j = \sum_{k=1}^{n} \tau_j \bar{\alpha}_{j,k} R_k = \sum_{k=1}^{n} \tau_j \bar{\alpha}_{j,k} \tau_k T_k = \sum_{k=1}^{n} \alpha_{j,k} T_k.$$

Step 2. By Theorems 2.7 and 3.8 there exist pairwise disjoint lattice homomorphisms $R_1, \dots, R_n$ with $S = R_1 + \dots + R_n$ and $R_1(E)^{\perp\perp} = \dots = R_n(E)^{\perp\perp}$. Denote by $[R_k]$ the band projection onto the band $R_k^{\perp\perp}$ in $L^r(E, F)$. Define lattice homomorphisms $T_1, \dots, T_n$ from E to F by putting

$$T_k := [R_k](S_1 \vee \dots \vee S_N) = [R_k](S_1) \vee \dots \vee [R_k](S_N) \quad (k = 1, \dots, n). \tag{1}$$

It follows from $N^{-1} S \leq S_1 \vee \dots \vee S_N \leq S$ and $S = R_1 + \dots + R_n$ that $N^{-1} R_k \leq T_k \leq R_k$ so that $T_k^{\perp\perp} = R_k^{\perp\perp}$. In particular, $T_1, \dots, T_n$ are pairwise disjoint. Moreover, $T_0 := T_1 + \dots + T_n$ is purely n-disjoint, since so are $S_1 + \dots + S_N$ and $T_0 = S_1 \vee \dots \vee S_N$. By Lemma 4.2 we can find a partition of unity $\pi_{k,1}, \dots, \pi_{k,N}$ in $\mathbb{P}(F)$ such that

$$T_k := \pi_{k,1} [R_k] S_1 + \dots + \pi_{k,N} [R_k] S_N = [R_k](S_1 \vee \dots \vee S_N) \quad (k := 1, \dots, n).$$

Ensure that $\pi_{k,j} [R_k] S_j = \pi_{k,j} S_j$ for all $k \leq n$ and $j \leq N$. Since every component of a lattice homomorphism $H : E \to F$ is of the form τH with $\tau \in \mathbb{P}(F)$, we can pick a band projection $\tau_{k,j}$ in F such that $[R_k] S_j = \tau_{k,j} S_j$ for all $j = 1, \dots, N$. If $\pi_0 \in \mathbb{P}(F)$ satisfy $\pi_0 \leq \pi_{k,j}$ and $\pi_0 \tau_{k,j} = 0$, then $0 = \pi_0 \tau_{k,j} S_j = \pi_0 [R_k] S_j = \pi_0 [R_k] \pi_{k,j} S_j = \pi_0 T_k$. It follows that $\pi_0 = 0$ as $T_k(E)^{\perp\perp} = F$ by Lemma 4.1, so that $\pi_{k,j} \leq \tau_{k,j}$. Thus $\pi_{k,j} [R_k] S_j = \pi_{k,j} \tau_{k,j} S_j = \pi_{k,j} S_j$ and we arrive at the representation

$$T_k := \pi_{k,1} S_1 + \dots + \pi_{k,N} S_N = [R_k](S_1 \vee \dots \vee S_N) \quad (j := 1, \dots, N).$$

Observe also that the band projections $\pi_{1,j}, \dots \pi_{n,j}$ are pairwise disjoint for every $j \leq N$. Indeed, if $\pi_0 := \pi_{k,j} \wedge \pi_{l,j}$ for some $k \neq l$, then the lattice homomorphisms $\pi_0 T_k = \pi_0 S_j$ and $\pi_0 T_l = \pi_0 S_j$ are disjoint as $\pi_{k,j} T_k = \pi_{k,j} S_j$, $\pi_{l,j} T_l = \pi_{l,j} S_j$ and $T_k \perp T_l$. This implies that $0 = \pi_0 S_j = \pi_0 T_k$ and, applying Lemma 4.1 to T_k, we obtain $\pi_0 = 0$ which is equivalent to $\pi_{k,j} \perp \pi_{l,j}$.

Step 3. Put $\rho_j := \sum_{k=1}^{n} \pi_{k,j}$. In order to construct $T_{n+1}, \dots, T_N$, we need to find an $(N-n) \times N$ matrix $(\pi_{k,j})_{n+1 \leq k \leq N, j \leq N}$ whose rows are partitions of unity in $\mathbb{P}(F)$ and the equality $\sum_{k=n+1}^{N} \pi_{k,j} = \rho_j^{\perp}$ holds for every $j := 1, \dots, N$. In this event we define $T_k := \sum_{j=1}^{N} \pi_{k,j} S_j$ for all $n < k \leq N$. We apply an algorithm similar to the so-called *northwest-corner method* of allocation which appears in virtually every text-book chapter on the transportation problem. We have in mind the following interpretation: the amounts $\sigma_1 := \rho_1^{\perp}, \dots, \sigma_N := \rho_N^{\perp}$ of N supply variables need to be allocated over the values $\delta_{n+1} = I_F, \dots, \delta_N = I_F$ of $N - n$ demand variables with total supply being equal to total demand:

$$\sum_{j=1}^{N} \rho_j^{\perp} = \sum_{j=n+1}^{N} \tau_j = (N-n) I_F.$$

The balance ratio is easily verified:

$$\sum_{j=1}^{N}\rho_j^{\perp}=\sum_{j=1}^{N}\left(I_F-\sum_{k=1}^{n}\pi_{k,j}\right)=NI_F-\sum_{k=1}^{n}\sum_{j=1}^{N}\pi_{k,j}=NI_F-nI_F=(N-n)I_F.$$

According to the northwest-corner method select the north west corner $(n+1,1)$ of the desired matrix and put $\pi_{n+k,1}:=0$ for all $1\le k\le N$, whenever $\sigma_1=0$; otherwise put $\pi_{n+1,1}:=\sigma_1=\rho_1^{\perp}$ and $\pi_{n+k,1}:=0$ for all $2\le k\le N$. Adjust σ_1 and δ_1 as $\sigma_1:=\sigma_1-\pi_{n+1,1}$ and $\delta_1:=\delta_1-\pi_{n+1,1}$. If it turns out that $\delta_1=0$, then put $\pi_{n+1,j}=0$ for all $2\le j\le N$, otherwise define $\pi_{n+1,2}:=\sigma_2-\pi_{n+1,1}$. In the latter case, adjust σ_2 and δ_1 as $\sigma_2:=\sigma_2-\pi_{n+1,2}$ and $\delta_1:=\delta_1-\pi_{n+1,2}$.

Working with these new σ_2 and δ_1, put $\pi_{n+k,2}=0$ for all $1\le k\le N$ whenever $\delta_1=0$; if we find that $\sigma_2=0$, put $\pi_{n+1,j}=0$ for all $3\le j\le N$ and we set $\pi_{n+1,3}:=\sigma_2(\pi_{n+1,1}+\pi_{n+1,2})^{\perp}$ otherwise. In the latter case we again adjust σ_2 and δ_1 as $\sigma_2:=\sigma_2-\pi_{n+1,3}$ and $\delta_1:=\delta_1-\pi_{n+1,3}$. Continuing, after N steps, we obtain pairwise disjoint band projections $\pi_{n+1,1},\dots,\pi_{n+1,N}$ whose sum is equal to I_F in accordance with the balance relation.

To continue, find the least $j_0\in\mathbb{N}$ with $\pi_{n+2,j_0}\ne 0$ and repeat the above reasoning, starting with $(n+2,j_0)$ as the new north-west corner. As a result, we obtain the required $(N-n)\times N$ matrix.

Step 4. It follows from (1) that $[R_k]S_j\le T_l$ and $S_j\le T_1+\dots+T_n$ for all $j\le N$. By Kutateladze Theorem there exist $0\le\beta_{j,k}\in\mathrm{Orth}(F)$ such that $S_j=\sum_{k=1}^{n}\beta_{j,k}T_k$ for all $j\le N$. Moreover, if $\beta_j:=\beta_{j,k}\wedge\beta_{j,l}$ for some $k\ne l$, then $S_j\ge\beta_j(T_k+T_l)$, which implies that $\beta_j=0$, since otherwise $\beta_j(T_k+T_l)$ is a nonzero lattice homomorphism contradicting purely n-disjointness of $T_1+\dots+T_n$. Thus $\beta_{j,k}\perp\beta_{j,l}$ and putting $\alpha_{n+1,k}:=\sum_{j=1}^{N}\pi_{n+1,j}\beta_{j,k}$ we deduce

$$T_{n+1}=\sum_{j=1}^{N}\pi_{n+1,j}S_j=\sum_{j=1}^{N}\pi_{n+1,j}\sum_{k=1}^{n}\beta_{j,k}T_k=\sum_{k=1}^{n}\alpha_{n+1,k}T_k,$$

$$\alpha_{n+1,k}\wedge\alpha_{n+1,l}=\sum_{j=1}^{N}\sum_{i=1}^{N}\pi_{n+1,j}\pi_{n+1,i}\beta_{j,k}\beta_{i,l}=\sum_{j=1}^{N}\pi_{n+1,j}\beta_{j,k}\beta_{j,l}=0.$$

The same reasoning works for $T_{n+2},\dots,T_N$ and the proof is complete. $\square$

Remark 4.5. If $S_1,\dots,S_N$ are pairwise disjoint then the operator $|S_1|+\dots+|S_N|$ is purely n-disjoint if and only if so is $S_1+\dots+S_N$, since $|S_1+\dots+S_N|=|S_1|+\dots+|S_N|$. Therefore, in this case in Theorem 4.4 we can replace $|S_1|+\dots+|S_N|$ by $S_1+\dots+S_N$.

Lemma 4.6. *Consider two collections $\mathscr{T}:=(T_1,\dots,T_N)$ and $\mathscr{S}:=(S_1,\dots,S_N)$ of order bounded disjointness preserving operators from a vector lattice E to a Dedekind complete vector lattice F. If both $\mathscr{T}$ and $\mathscr{S}$ consists of pairwise purely disjoint operators and $T_1+\dots+T_N=S_1+\dots+S_N$, then each of $\mathscr{T}$ and $\mathscr{S}$ is a $\mathbb{P}(F)$-permutation of another.*

Proof. We may assume without loss of generality that $T_k(E)^{\perp\perp}=S_j(E)^{\perp\perp}=F$ for all $1\le j,k\le N$. By an argument similar to that used in the proof of Theorem 4.4, Step 2, we deduce the representation $T_k:=\pi_{k,1}S_1+\dots+\pi_{k,N}S_N$ with a partition of unity $\pi_{k,1},\dots,\pi_{k,N}$ in $\mathbb{P}(F)$. Moreover, $\pi_{k,j}\perp\pi_{k,l}$ whenever $k\ne l$. Since $\pi_{k,j}T_k=\pi_{k,j}S_j$, we have

$$\sum_{j=1}^{N}\rho_jS_j=\sum_{j=1}^{N}\sum_{k=1}^{N}\pi_{k,j}T_k=\sum_{k=1}^{N}S_j,\quad \rho_j:=\sum_{k=1}^{N}\pi_{k,j}.$$

It follows that $\sum_{j=1}^{N}(I_F-\rho_j)S_j=0$ and hence $\rho_1=\dots=\rho_N=I_F$, since $S_1,\dots,S_N$ are pairwise purely disjoint. Thus, $\mathscr{T}$ is a $\mathbb{P}(F)$-permutation of $\mathscr{S}$ with the $N\times N$ matrix $(\pi_{k,j})$. $\square$

Definition 4.7. A collection $(k_1,\pi_1),\dots,(k_l,\pi_l)$ is said to be a *decomposition series* in $\mathbb{P}(F)$ whenever $1\le k_1<\dots<k_l$ are naturals and $\{\pi_1,\dots,\pi_l\}$ is a partition of unity in $\mathbb{P}(F)$ with nonzero terms.

We say that an n-disjoint operator T *has a decomposition series* $(k_1, \pi_1), \dots, (k_l, \pi_l)$ in $\mathbb{P}(F)$ if, in addition to the above, $k_l \leq n$ and there exist order bounded disjointness preserving operators $T_1, \dots, T_{k_l}$ from E to F such that $T = T_1 + \dots + T_{k_l}$ and $\pi_i T_1, \dots, \pi_i T_{k_i}$ are pairwise purely disjoint for every $i = 1, \dots, l$.

Note that in this event, for every $i \leq l$, we have the representation $\pi_i T = \pi_i T_1 + \dots + \pi_i T_{k_i}$ which is unique up to $\mathbb{P}(\pi_i F)$-permutation by Lemma 4.6.

Corollary 4.8. *Let E and F be vector lattices with F Dedekind complete and $n, N \in \mathbb{N}$ with $n \leq N$. For a collection of order bounded disjointness preserving operators $S_1, \dots, S_N$ from E to F the operator $|S_1| + \dots + |S_N|$ is n-disjoint if and only if there exist a $\mathbb{P}(F)$-permutation $T_1, \dots, T_N$ of $S_1, \dots, S_N$ and a unique decomposition series $(k_1, \pi_1), \dots, (k_l, \pi_l)$ in $\mathbb{P}(F)$ such that for each $1 \leq i \leq l$ the operators $\pi_i T_1, \dots, \pi_i T_{k_i}$ are pairwise purely disjoint and, whenever $k_i < N$, each of $\pi_i T_{k_i+1}, \dots, \pi_i T_N$ is representable as $\pi_i T_j = \sum_{s=1}^{k_i} \alpha_{i,j,s} T_s$ for some pairwise disjoint $0 \leq \alpha_{i,j,1}, \dots, \alpha_{i,j,k_i} \in \mathscr{Z}(\pi_i F)$ $(j := k_i + 1, \dots, N)$.*

Proof. Assume that $T := |S_1| + \dots + |S_N|$ is n-disjoint. By Theorem 3.2 there exist a unique decomposition series $(k_1, \pi_1), \dots, (k_l, \pi_l)$ such that $k_l \leq n$ and $\pi_i T$ is purely k_i-disjoint for all $i := 1, \dots, l$. By Theorem 4.4 there exists a $\mathbb{P}(\pi_i F)$-permutation $T_{i,1}, \dots, T_{i,N}$ of $\pi_i S_1, \dots, \pi_i S_N$ such that $T_{i,1}, \dots, T_{i,k_i}$ are pairwise purely disjoint and, whenever $k_i < N$, each of $T_{i,k_i+1}, \dots, T_{i,N}$ is representable as

$$T_{i,j} = \sum_{k=1}^{k_i} \alpha_{i,j,k} T_{i,k}$$

for some pairwise disjoint $0 \leq \alpha_{i,j,1}, \dots, \alpha_{i,j,k_i} \in \mathscr{Z}(\pi_i F)$ $(j := k_i + 1, \dots, N)$. Define now T_j by putting

$$T_j := T_{1,j} + \dots + T_{l,j}.$$

It is not difficult to verify that T_j satisfy the required conditions. □

Next we state our second main result answering to the uniqueness question of the decomposition of an order bounded n-disjoint operator in terms of pairwise purely disjoint operators that preserve disjointness.

Theorem 4.9. *Each order bounded n-disjoint operator T from a vector lattice E to a Dedekind complete vector lattice F has a unique decomposition series $(k_1, \pi_1), \dots, (k_l, \pi_l)$ in $\mathbb{P}(T(E)^{\perp\perp})$.*

Proof. By Theorem 2.7 a representation $T = S_1 + \dots + S_n$ holds with disjointness preserving operators $S_1, \dots, S_n$ from E to F. According to Theorem 3.2 there exists a unique decomposition series $(k_1, \pi_1), \dots, (k_l, \pi_l)$ in $\mathbb{P}(F)$ such that $\pi_i T$ is purely k_i-disjoint for all $i := 1, \dots, l$. Apply Theorem 4.4 with $N := n$ and $n := k_i$ to the operators $\pi_i S_1, \dots, \pi_i S_n$ and find a pairwise purely disjoint $\mathbb{P}(\pi_i F)$-permutation $T_{i,1}, \dots, T_{i,n}$ of $\pi_i S_1, \dots, \pi_i S_n$. If $k_i < n$ then, just as in the proof of sufficiency in Theorem 4.4, $T_{i,j}$ is replaced by $\beta_{i,j} T_{i,j}$ with $\beta_{i,j} := I_F + \sum_{k=k_i+1}^{n} \alpha_{i,j,k}$ for some pairwise disjoint $\alpha_{i,j,k} \in \mathrm{Orth}(\pi_i F)$, keeping the same designation $T_{i,j}$ for the new operator. Clearly, $T_{i,1}, \dots, T_{i,k_i}$ are disjointness preserving operators from E to $\pi_i F$ and $T_{i,1} + \dots + T_{i,k_i} = \pi_i S_1 + \dots + \pi_i S_n$. Now, for $k_i < j \leq k_{i+1}$ and $1 \leq i < l$ define $T_j : E \to F$ by $T_j := T_{1,j} + \dots + T_{l,j}$. It can be easily seen that $T_1, \dots, T_{k_l}$ are order bounded disjointness preserving operators from E to F and

$$\sum_{j=1}^{k_l} T_j = \sum_{i=1}^{l} \sum_{j=1}^{k_i} T_j = \sum_{i=1}^{l} \sum_{j=1}^{k_i} \sum_{s=1}^{l} T_{s,j} = \sum_{i=1}^{l} \left(\sum_{s=1}^{l} \sum_{j=1}^{k_l} T_{s,j} \right) = \sum_{i=1}^{l} \sum_{j=1}^{n} \pi_i S_j = T.$$

Since $\pi_i T_j = T_{i,j}$ for $j \leq k_i$ and $\pi_i T_{k_i+j} = 0$ for $1 \leq j \leq k_l - k_i$ by construction, the operators $\pi_i T_1, \dots, \pi_i T_{k_i}$ are pairwise purely disjoint for every $i = 1, \dots, l$ and this completes the proof according to Definition 4.7. □

Remark 4.10. The problem of finding conditions for a finite sum of order bounded disjointness preserving operators to be n-disjoint was treated in the book by Kusraev and Kutateladze [7, Section 3.8] using Boolean valued analysis. Theorem 4.4 coincides essentially with Theorem 3.8.7 in [7]. However, the above proof uses the standard toolkit and does not depend on the Boolean valued approach.

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