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Traveling wave solutions of a diffusive predator–prey model with modified Leslie–Gower and Holling-type II schemes

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Abstract. We study a diffusive predator–prey model with modified Leslie–Gower and Holling-II schemes with $D = 0$. We establish the existence of traveling wave solutions connecting a positive equilibrium and a boundary equilibrium via the 'shooting method', and the non-existence by the 'eigenvalue method'. It should be emphasized that a threshold value $c^* = \sqrt{4\alpha}$ is found in our paper.

Keywords. Diffusive predator–prey model; traveling wave solution; modified Leslie–Gower; Holling-type II scheme; shooting method.

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1. Introduction

The existence and non-existence of traveling solutions are very important topics in the study of asymptotic behaviors and dynamics for nonlinear evolution equations. Because of their significant effects in applied sciences such as physics, epidemics and ecology, etc, they have gained much attention during the past forty years. Lots of relevant methods and theory are established (see [2,5,8,10,13–15,17–21] and the references therein). However, most of the research papers listed above concentrate on monotone dynamics, while few of them focus on non-monotone dynamics (see [4,5]).

An interesting non-monotone model

$$\begin{aligned}\dot{x} &= \left(a_1 - bx - \frac{c_1 y}{x + k_1}\right) x,\\ \dot{y} &= \left(a_2 - \frac{c_2 y}{x + k_2}\right) y,\end{aligned} \tag{1.1}$$

has been considered by Nindjin and Aziz-Alaoui in [11]. The two species food chain model (1.1) describes a prey population x which serves as food for a predator. The model parameters $a_1, a_2, b, c_1, c_2, k_1$ are assumed to be positive. They are defined as follows: a_1

is the growth rate of prey x, b measures the strength of competition among individuals of species x, c_1 is the maximum value of the per capita reduction rate of x due to y, k_1 (respectively, k_2) measures the extent to which environment provides protection to prey x (respectively, to the predator y), a_2 describes the growth rate of y, and c_2 has a similar meaning to c_1.

Taking the diffusion of the species into account and doing dimensionless transformation, one can obtain the following model:

$$\begin{aligned}\frac{\partial U}{\partial t} &= D\frac{\partial^2 U}{\partial x^2} + \left(1 - U - \frac{\beta_1 W}{U + k_1}\right) U,\\ \frac{\partial W}{\partial t} &= \frac{\partial^2 W}{\partial x^2} + \alpha W \left(1 - \frac{\beta_2 W}{U + k_2}\right),\end{aligned} \tag{1.2}$$

where U and W denote the prey species and the predator species respectively. More details are in [16]. It is obvious that $E_0 = (0, 0)$, $E_1 = (1, 0)$, $E_2 = (0, \frac{k_2}{\beta_2})$ are three equilibria of (1.2). A positive equilibrium can be obtained from the following proposition, which was discussed in [11].

PROPOSITION 1.1

System (1.2) *has a unique interior equilibrium* $E_* = (u^*, w^*)$ (*i.e.*, $u^* > 0, w^* > 0$) *if the following conditions hold*:

$$\frac{k_2}{\beta_2} < \frac{k_1}{\beta_1}, \tag{1.3}$$

where

$$u^* = \frac{1}{2\beta_2}[-(\beta_1 - \beta_2 + \beta_2 k_1) + \Delta^{\frac{1}{2}}], \quad w^* = \frac{u^* + k_2}{\beta_2},$$

and $\Delta = (\beta_1 - \beta_2 + \beta_2 k_1)^2 + 4\beta_2(\beta_2 k_1 - \beta_1 k_2)$.

The purpose of our paper is to establish the existence and non-existence of nonnegative traveling wave solutions connecting (u^*, w^*) and $(1, 0)$ of system (1.2). Traveling wave solutions are of the form $U(x, t) = u(x + ct)$, $W(x, t) = w(x + ct)$, where $c > 0$ is the wave speed parameter. Setting $s = x + ct$, we see that u, w can be regarded as functions of one variable s. Therefore the system (1.2) with $D = 0$ becomes

$$\begin{aligned} u' &= \frac{1}{c}\left(1 - u - \frac{\beta_1 w}{u + k_1}\right) u,\\ w'' &= cw' - \alpha w \left(1 - \frac{\beta_2 w}{u + k_2}\right).\end{aligned} \tag{1.4}$$

We point out that $D = 0$ is a special case, which is the limiting case when D is small. It corresponds to a senario in which the prey species diffuses much more slowly than the predator species. For example, a plant species being consumed by a relatively mobile herbivore.

Moreover, the wave solutions satisfy the following boundary condition

$$\begin{aligned} &u(-\infty) = 1, \ w(-\infty) = 0,\\ &u(+\infty) = u^*, \ w(+\infty) = w^*.\end{aligned} \tag{1.5}$$

The equilibrium $(1, 0)$ represents the population of the prey at the environmental carrying capacity in the absence of predators. The equilibrium (u^*, w^*) represents the time constant coexistence of both species. The existence of such wave solutions biologically indicates that if the habitat is initially uniformly saturated with prey, inductive predator at the boundary of the habitat may result in a 'wave of invasion' of predators.

Our paper is organized as follows. We consider the existence and non-existence of the solutions of the equations (1.4) and (1.5) in section 2. The 'shooting method' is used to obtain the existence of such solutions since the 'upper-lower solutions method' is not valid when $D = 0$. Our results show that there is a threshold value $c^* = \sqrt{4\alpha}$ such that the existence of traveling waves is obtained if $c \geq c^*$ and the non-existence is obtained if $c < c^*$. In section 3, some numerical simulations are presented to illustrate the analytic results and future work is discussed.

2. Traveling wave solutions with $D = 0$

In this section, we investigate the existence of traveling wave solutions of system (1.2) when $D = 0$ which is equivalent to the existence of solutions of system (1.4). A method which is called the 'shooting method' will be applied here. In order to use this method, we rewrite system (1.4) as a first order system in $\mathbb{R}$:

$$\begin{aligned} u' &= \frac{1}{c}\left(1 - u - \frac{\beta_1 w}{u + k_1}\right)u, \\ w' &= z, \\ z' &= cz - \alpha w\left(1 - \frac{\beta_2 w}{u + k_2}\right). \end{aligned} \tag{2.1}$$

Therefore, the equilibria E_0, E_1, E_2 and E_* of (1.4) are equivalent to the critical points $(0, 0, 0)$, $(1, 0, 0)$, $(0, \frac{k_2}{\beta_2}, 0)$ and $(u^*, w^*, 0)$ of system (2.1) respectively.

Obviously, the first equation in (1.4) has singularity at $u = -k_1$, and the third equation has singularity at $u = -k_2$. Let $-k = \max\{-k_1, -k_2\}$ and $\mathcal{D} := (-k, \infty) \times \mathbb{R} \times \mathbb{R}$. For any initial point $(u_0, w_0, z_0) \in \mathcal{D}$, the trajectory starts from (u_0, w_0, z_0) and will stay in $\mathcal{D}$ forever. On the other hand, the critical points $(0, 0, 0)$, $(1, 0, 0)$,$(0, \frac{k_2}{\beta_2}, 0)$ and $(u^*, w^*, 0)$ of system (2.1) lie in the domain $\mathcal{D}$. In the following discussions, we shall only consider system (2.1) in $\mathcal{D}$.

From (1.5), we consider the solutions of system (2.1) with $u(s) \geq 0$, $w(s) \geq 0$ satisfying

$$\begin{aligned} &u(-\infty) = 1,\ w(-\infty) = 0,\ z(-\infty) = 0, \\ &u(+\infty) = u^*,\ w(+\infty) = w^*,\ z(+\infty) = 0. \end{aligned} \tag{2.2}$$

2.1 *A variant of Wazewski's theorem and a Wazewski set*

The 'shooting method' combines the qualitative analysis for orbits on the phase plane, the Lyapunov function method and the invariant manifold principle together with the construction of a Wazewski set. It seems that it is very challenging for the construction of the Wazewski set and Lyapunov function. Since Dunbar's pioneer work, not much references have been published on this aspect. In view of this, the method is based on a variant of Wazewski's theorem, so we state some notations and the related theorem in the following. For more details, see [4–6].

Consider the differential equation

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}s} = f(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^n, \tag{2.3}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies a Lipschitz condition. Let $\mathbf{y}(s; \mathbf{y}_0)$ be the unique solution of system (2.3) satisfying $\mathbf{y}(0; \mathbf{y_0}) = \mathbf{y}_0$. For convenience, set $\mathbf{y}(s; \mathbf{y_0}) = \mathbf{y}_0 \cdot s$. Let $\mathbf{U} \cdot \mathbf{S}$ be the set of points $\mathbf{y}_0 \cdot \mathbf{s}$, where $\mathbf{y}_0 \in \mathbf{U},\ s \in \mathbf{S}$. For a given subset $\mathbf{W} \subseteq \mathbb{R}^n$, let

$$\mathbf{W}^- = \{\mathbf{y}_0 \in \mathbf{W} \mid \text{any } s > 0, \mathbf{y}_0 \cdot [0, s] \nsubseteq \mathbf{W}\}.$$

$\mathbf{W}^-$ is called the immediate exit set of $\mathbf{W}$. Let $\Sigma \subseteq \mathbf{W}$ be given. Define

$$\Sigma^0 = \{\mathbf{y}_0 \in \Sigma \mid \text{ there is an } s_0 = s_0(\mathbf{y}_0) \text{ such that } \mathbf{y}_0 \cdot s_0 \notin \mathbf{W}\}.$$

For $\mathbf{y}_0 \in \Sigma^0$, define $T(\mathbf{y}_0) = \sup\{s \mid \mathbf{y}_0 \cdot [0, s] \subseteq \mathbf{W}\}$. $T(\mathbf{y}_0)$ is called an exit time. Note that $\mathbf{y}_0 \cdot T(\mathbf{y}_0) \in \mathbf{W}^-$. On the other hand, $T(\mathbf{y}_0) = 0$ if and only if $\mathbf{y}_0 \in \mathbf{W}^-$. The notation cl $(\mathbf{W})$ is used for the closure of $\mathbf{W}$. Proposition 2.1 would be devoted to proving the main result in this section.

PROPOSITION 2.1

Suppose

(1) $\mathbf{y}_0 \in \Sigma$, *and* $\mathbf{y}_0 \cdot [0, s] \subseteq \mathrm{cl}\,(\mathbf{W})$, *then* $\mathbf{y}_0 \cdot [0, s] \subseteq \mathbf{W}$.
(2) $\mathbf{y}_0 \in \Sigma, \mathbf{y}_0 \cdot s \in \mathbf{W}, \mathbf{y}_0 \cdot s \notin \mathbf{W}^-$, *then there is an open set* V_s *about* $\mathbf{y}_0 \cdot s$ *disjoint from* $\mathbf{W}^-$.
(3) $\Sigma = \Sigma^0$, Σ *is compact, and* Σ *intersects a trajectory of* (2.3) *only once.*

Then the mapping $F(\mathbf{y}_0) = \mathbf{y}_0 \cdot T(\mathbf{y}_0)$ *is a homeomorphism from* Σ *to its image on* $\mathbf{W}^-$.

A set $\mathbf{W} \subseteq \mathbb{R}^n$ that satisfies statements (1) and (2) is called a Wazewski set. With regard to system (1.4), we give some sets. Define some sets of $\mathcal{D}$:

$$\mathbb{P} = \{(u, w, z) \in \mathcal{D} \mid u < u^*, w > w^*, z > 0\},\ \mathbb{Q} = \{(u, w, z) \in \mathcal{D} \mid u > u^*, w < w^*, z < 0\},$$
$$J = \{(u, w, z) \in \mathcal{D} \mid u \geq u^*, w \leq 0, z = 0\},\ \mathbf{W} = \mathcal{D} \backslash (\mathbb{P} \cup \mathbb{Q}).$$

Now we study the structure of $\mathbf{W}^-$.

Lemma 2.1. $\mathbf{W}^-$ *has the following structure*:

$$\mathbf{W}^- = \partial \mathbf{W} \backslash (J \cup \{(u^*, w^*, 0)\}).$$

Proof. Obviously by the vector field, if $(u, w, z) = (u^*, w^*, 0)$ or $(u, w, z) \in J$, the trajectory will not leave $\mathbf{W}$ immediately. Then it is sufficient to prove that any point in $\partial \mathbf{W} \backslash (J \cup \{(u^*, w^*, 0)\})$ will leave $\mathbf{W}$ immediately.

Since $\partial \mathbf{W} \backslash (J \cup \{(u^*, w^*, 0)\}) = \{\partial \mathbb{P} \backslash \{(u^*, w^*, 0)\}\} \cup \{\partial \mathbb{Q} \backslash (J \cup \{(u^*, w^*, 0)\})\}$, then we should claim that any trajectory through the point in $\partial \mathbb{P} \backslash \{(u^*, w^*, 0)\}$ and $\partial \mathbb{Q} \backslash (J \cup \{(u^*, w^*, 0)\})$ will enter the region $\mathbb{P}$ or $\mathbb{Q}$.

The set $\partial\mathbb{P}\backslash\{(u^*, w^*, 0)\}$ is represented in the form

$$\begin{aligned}\partial\mathbb{P}\backslash\{(u^*, w^*, 0)\} = &\{u = u^*, w > w^*, z > 0\}\\ &\cup\{-k < u < u^*, w = w^*, z > 0\}\\ &\cup\{-k < u < u^*, w > w^*, z = 0\}\\ &\cup\{u = u^*, w = w^*, z > 0\}\cup\{u = u^*, w > w^*, z = 0\}\\ &\cup\{-k < u < u^*, w = w^*, z = 0\}.\end{aligned}$$

If a trajectory passes through a point in $\{u = u^*, w > w^*, z > 0\}$, it would enter the region $\mathbb{P}$ since the first equation of system (2.1) implies $u' < 0$. Similarly, the trajectory through a point in any other subset enters the region $\mathbb{P}$ too.

On the other hand, the set $\partial\mathbb{Q}\backslash(J \cup (u^*, w^*, 0))$ is represented as

$$\begin{aligned}\partial\mathbb{Q}\backslash(J \cup (u^*, w^*, 0)) = &\{u = u^*, -\infty < w < w^*, z < 0\}\\ &\cup\{u > u^*, w = w^*, z < 0\}\\ &\cup\{u > u^*, 0 < w < w^*, z = 0\}\\ &\cup\{u = u^*, w = w^*, z < 0\}\\ &\cup\{u = u^*, 0 < w < w^*, z = 0\}\\ &\cup\{u > u^*, w = w^*, z = 0\}.\end{aligned}$$

If a trajectory passes through a point in $\{u = u^*, -\infty < w < w^*, z < 0\}$, then it would enter the region $\mathbb{Q}$ since the first equation of system (2.1) implies $u' > 0$. The same conclusion is obtained when the trajectory passes through a point in any other subset. The proof is complete. $\square$

Obviously, $\mathbf{W}^-$ is not a connected set. One component of $\mathbf{W}^-$ is $\partial\mathbb{P}\backslash\{(u^*, w^*, 0)\}$, and the other one is $\partial\mathbb{Q}\backslash(J \cup \{(u^*, w^*, 0)\}$.

2.2 *The traveling wave solution*

In this section, we investigate the existence of the traveling wave solutions which satisfies (2.2). The Jacobic matrix at $(1, 0, 0)$ is

$$J(1,0,0) = \begin{pmatrix} -\frac{1}{c} & -\frac{\beta_1}{c(1+k_1)} & 0\\ 0 & 0 & 1\\ 0 & -\alpha & c\end{pmatrix}. \tag{2.4}$$

Thus the characteristic equation of system (2.1) at $(1, 0, 0)$ is

$$\left(\lambda + \frac{1}{c}\right)[\lambda(\lambda - c) + \alpha] = 0,$$

and the eigenvalues are

$$\lambda_1 = -\frac{1}{c}, \quad \lambda_2 = \frac{c - \sqrt{c^2 - 4\alpha}}{2}, \quad \lambda_3 = \frac{c + \sqrt{c^2 - 4\alpha}}{2}. \tag{2.5}$$

If $0 < c < \sqrt{4\alpha}$, then (λ_2, λ_3) is a complex conjugate pair of eigenvalues with positive part. Thus there is a two dimensional unstable manifold based at $(1, 0, 0)$. The critical point $(1, 0, 0)$ is a spiral point on the unstable manifold. Therefore a trajectory approaching $(1, 0, 0)$ must have $w(s) < 0$ for some s. This violates the requirement that $w(s)$ is nonnegative. This demonstrates the non-existence of traveling wave solutions satisfying (2.2) when $0 < c < \sqrt{4\alpha}$. Hence we have as follows.

Theorem 2.1. *If* $0 < c < \sqrt{4\alpha}$, *then the traveling wave solutions with* (2.2) *do not exist, which means that system* (1.2) *with* $D = 0$ *does not have the nonnegative traveling wave solutions connecting* $(1, 0)$ *and* (u^*, w^*).

As for the case where $c \geq \sqrt{4\alpha}$, the analysis is complicated. For better illustration, we first state our main result here.

Theorem 2.2. *Suppose* $k_1 > 1 - u^*$. *If* $c \geq \sqrt{4\alpha}$, *then the traveling wave solution with* (2.2) *exists. That is, system* (1.2) *with* $D = 0$ *has a nonnegative traveling wave solution connecting* $(1, 0)$ *and* (u^*, w^*). *Furthermore, let*

$$A_1 = -\left(1 - 2u^* - \frac{\beta_1 w^* k_1}{(u^* + k_1)^2}\right), \quad A_2 = \frac{\beta_1 u^*}{(u^* + k_1)}.$$

There exists a critical value

$$\alpha^* = \alpha^*(u^*, c) \geq \frac{A_1^2}{4\left(A_1 + \frac{A_2}{\beta_2}\right)}$$

such that

(1) *if* $\alpha \leq \alpha^*$, *the solution converges monotonically to* $(u^*, w^*, 0)$ *for large* s,
(2) *if* $\alpha > \alpha^*$, *the solution exhibits exponential oscillations around* $(u^*, w^*, 0)$ *for large* s.

2.3 *Proof of Theorem* 2.2

Under the condition $c > \sqrt{4\alpha}$, the characteristic equation has three distinct real eigenvalues: $\lambda_1 < 0 < \lambda_2 < \lambda_3$. Let χ_1, χ_2, χ_3 be the eigenvectors associated with $\lambda_1, \lambda_2, \lambda_3$ respectively. We can choose them as

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} -\dfrac{\beta_1}{(1 + k_1)(c\lambda_2 + 1)} \\ 1 \\ \lambda_2 \end{pmatrix}, \quad \chi_3 = \begin{pmatrix} -\dfrac{\beta_1}{(1 + k_1)(c\lambda_3 + 1)} \\ 1 \\ \lambda_3 \end{pmatrix}. \tag{2.6}$$

Applying Theorem 6.1 of [7, page 242], there is a one-dimensional unstable manifold corresponding to the largest eigenvalues λ_3, denoted by $\mathcal{U}_1$. There is also a two-dimensional unstable manifold $\mathcal{U}_2$ corresponding to λ_2 and λ_3. Obviously, we know that $\mathcal{U}_1 \subseteq \mathcal{U}_2$. Points on $\mathcal{U}_1$ are parametrically represented in a small neighborhood of $(1, 0, 0)$ by a function $\mathbf{u}_1 : \mathbb{R}^1 \to \mathbb{R}^3$, where

$$\mathbf{u}_1(m) = (1, 0, 0)^T + m\chi_3 + o(|m|). \tag{2.7}$$

Points on $\mathcal{U}_2$ are represented by $\mathbf{u}_2 : \mathbb{R}^2 \to \mathbb{R}^3$, where

$$\mathbf{u}_2(m, n) = (1, 0, 0)^T + m\chi_3 + n\chi_2 + o(|m|, |n|). \tag{2.8}$$

An octant near the critical point $(1, 0, 0)$ is denoted by

$$\Theta_1 = \{(u, w, z) \in \mathcal{D} \mid u \leq 1, w \geq 0, z \geq 0\}.$$

We will discuss the properties of the trajectory on the branch of the strongly unstable manifold $\mathcal{U}_1$ in the octant Θ_1.

Lemma 2.2. *If $c \geq \sqrt{4\alpha}$, then for the trajectories on the branch of the strongly unstable manifold $\mathcal{U}_1$ in the octant Θ_1, we have $w(s) > 0$ and $z(s) > \frac{c}{2}w(s)$ for all $s > 0$.*

Proof. Consider a trajectory on the branch of the strongly unstable manifold $\mathcal{U}_1$ in the octant $\{(u, w, z) \in \mathcal{D} \mid u < 1, w > 0, z > 0\}$. It is easy to see that it converges to $(1, 0, 0)$ tangent to the eigenvector χ_3. From the expressions of λ_3 and χ_3, we know that the second and third components of the tangent vector satisfies $z = \lambda_3 w \geq \frac{c}{2}w$. Thus there is a point on the trajectory whose components satisfy $u < 1, w > 0, z > \frac{c}{2}w$. Because system (2.1) is autonomous, the time variable may be reset to this point corresponding to $s = 0$. Hence it is sufficient to prove a solution of (2.1) with $u(0) < 1$, $w(0) > 0$ and $z(0) > \frac{c}{2}w(0)$.

Suppose, by contradiction, there is an $s > 0$ such that $z(s) \leq \frac{c}{2}w(s)$. Let $s_1 = \inf\{s > 0 | z(s) \leq \frac{c}{2}w(s)\}$. For $0 \leq s \leq s_1$, $w'(s) = z(s) \geq \frac{c}{2}w(s)$ and $w(0) > 0$, so $w(s_1) > 0$. Also $z(s_1) = \frac{c}{2}w(s_1)$ and $z(s) > \frac{c}{2}w(s)$ for $0 \leq s < s_1$. Therefore,

$$z'(s_1) - \frac{c}{2}w'(s_1) = z'(s_1) - \frac{c}{2}z(s_1) \leq 0.$$

Substituting from (2.1),

$$cz(s_1) - \alpha w(s_1)\left(1 - \frac{\beta_2 w(s_1)}{u(s_1) + k_2}\right) - \frac{c}{2}z(s_1) \leq 0.$$

Noticing that $z(s_1) = \frac{c}{2}w(s_1)$, it follows that

$$\frac{c^2}{4}w(s_1) - \alpha w(s_1)\left(1 - \frac{\beta_2 w(s_1)}{u(s_1) + k_2}\right) \leq 0.$$

Since $w(s_1) > 0$, we have

$$\frac{c^2}{4} - \alpha < \frac{c^2}{4} - \alpha\left(1 - \frac{\beta_2 w(s_1)}{u(s_1) + k_2}\right) \leq 0,$$

which is a contradiction to $c \geq \sqrt{4\alpha}$. So we conclude that $z(s) > \frac{c}{2}w(s)$ for all $s > 0$ and also $w(s) > 0$ for all $s > 0$. The proof is complete. □

Lemma 2.3. *Suppose $c \geq \sqrt{4\alpha}$. A trajectory on the portion of the strongly unstable manifold $\mathcal{U}_1$ in the octant Θ_1 must satisfy*

$$w(s) \geq -\left(1 + \frac{c^2}{2}\right)\frac{u + k_1}{\beta_1}(u(s) - 1) \quad \text{for all } s. \tag{2.9}$$

Proof. First we consider the case when the trajectory is in $\mathcal{U}_1 \cap \Theta_1$. Since

$$\chi_3 = \begin{pmatrix} -\dfrac{\beta_1}{(1+k_1)(c\lambda_3+1)} \\ 1 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix},$$

and for any $\mathbf{u} = (u, w, z)^T \in \mathcal{U}_1$, we obtain

$$\xi_2 = -\frac{(1+k_1)(c\lambda_3+1)}{\beta_1}\xi_1.$$

Since it is approximated from (2.7) that $u = 1 + m\xi_1$, $w = m\xi_2$, one has

$$w = -\frac{1+k_1}{\beta_1}(c\lambda_3+1)m\xi_1 = -\frac{1+k_1}{\beta_1}(c\lambda_3+1)(u-1).$$

Substituting $\lambda_3 = \dfrac{c+\sqrt{c^2-4\alpha}}{2}$ into the above expression, we obtain

$$\begin{aligned} w &= -\frac{1+k_1}{\beta_1}\left(\frac{c^2+c\sqrt{c^2-4\alpha}}{2}+1\right)(u-1) \\ &= \bar{D}(u) - \frac{1+k_1}{\beta_1}\frac{c\sqrt{c^2-4\alpha}}{2}(u-1), \end{aligned}$$

where

$$\bar{D}(u) = -\left(1+\frac{c^2}{2}\right)\frac{1+k_1}{\beta_1}(u-1).$$

Noticing that $\mathbf{u} \in \mathcal{U}_1 \cap \Theta_1$, $u \leq 1$. It follows that the inequality (2.9) is valid for $\mathbf{u} = (u, w, z)^T \in \mathcal{U}_1 \cap \Theta_1$. That means there is $s_1 > -\infty$ such that

$$w(s) \geq \bar{D}(u)(s) \quad \text{for all } -\infty < s \leq s_1.$$

Next we study the case when the trajectory is out of $\mathcal{U}_1 \cap \Theta_1$. Assume, by contradiction, that there is $s_2 \geq s_1$ such that

$$w(s_2) = \bar{D}(u)(s_2) \tag{2.10}$$

and

$$w(s) < \bar{D}(u)(s) \quad \text{for } s > s_2.$$

Furthermore, one can have

$$\dot{w}(s_2) + \frac{\bar{D}(u)}{\mathrm{d}u}\dot{u}\bigg|_{s=s_2} < 0. \tag{2.11}$$

But (2.11) is not valid if we calculate the right side of the inequality carefully. For $s = s_2$, we have

$$\begin{aligned}\dot{w}(s) - \frac{\mathrm{d}\bar{D}(u)(s)}{\mathrm{d}s} &= z - \frac{\mathrm{d}\bar{D}(u)(s)}{\mathrm{d}s} \\ &> \frac{c}{2}w - \frac{1+\frac{c^2}{2}}{\beta_1}(1-2u-k_1)\frac{1}{c}u\left(1-u-\frac{\beta_1 w}{u+k_1}\right).\end{aligned}$$

Substituting (2.10) into the right side of the above term yields

$$\begin{aligned}&\frac{c}{2}\frac{u+k_1}{\beta_1}\left(1+\frac{c^2}{2}\right)(1-u) - \left(1+\frac{c^2}{2}\right)\frac{1-2u-k_1}{\beta_1} \\ &\quad\times\frac{u}{c}\left[1-u-\left(1+\frac{c^2}{2}\right)(1-u)\right] \\ &= \frac{c}{2}\left(1+\frac{c^2}{2}\right)\frac{1-u}{\beta_1}[u+k_1+(1-2u-k_1)u] \\ &= \frac{c}{2}\left(1+\frac{c^2}{2}\right)\frac{1-u}{\beta_1}(2u+k_1)(1-u) > 0.\end{aligned}$$

Hence $\dot{w}(s) - \frac{\mathrm{d}\bar{D}(u)(s)}{\mathrm{d}s} = z - \frac{\mathrm{d}\bar{D}(u)(s)}{\mathrm{d}s} > 0$, and it implies that (2.11) is not valid. It follows that (2.9) holds true for all $s > 0$. The proof is complete. □

Lemma 2.4. *Set* $A = \frac{\alpha\beta_2(1+k_1)}{k_2\beta_1}$. *For any given* $d > \frac{c(1+\frac{A}{u^*})+\sqrt{(1+\frac{A}{u^*})^2c^2+4A}}{2}$, *if*

$$0 < w(0) < -\left(1+\frac{cd}{u^*}\right)\frac{1+k_1}{\beta_1}(u(0)-1) \ \ \textit{and} \ \ z(0) < \mathrm{d}w(0)$$

then

$$w(s) < -\left(1+\frac{cd}{u^*}\right)\frac{1+k_1}{\beta_1}(u(s)-1) \quad \textit{for } s > 0 \tag{2.12}$$

and

$$z(s) < \mathrm{d}w(s) \quad \textit{for } s > 0 \tag{2.13}$$

as long as $w(s) > 0$ *and* $u^* < u(s) < 1$ *for* $s > 0$.

Proof. We prove by contradiction. Suppose the above statement does not hold. Then only two cases can occur.

Case I. There is $s_1 > 0$ such that (2.12) and (2.13) are valid for $0 < s < s_1$, and

$$w(s_1) = -\left(1+\frac{cd}{u^*}\right)\frac{1+k_1}{\beta_1}(u(s_1)-1), \tag{2.14}$$

$$z(s_1) \le \mathrm{d}w(s_1). \tag{2.15}$$

Hence, we have

$$\dot{w}(s_1) + \left(1 + \frac{cd}{u^*}\right)\left(\frac{1+k_1}{\beta_1}\right)\dot{u}(s_1) > 0, \tag{2.16}$$

that is,

$$z(s_1) + \left(1 + \frac{cd}{u^*}\right)\left(\frac{1+k_1}{\beta_1}\right)\frac{1}{c}u(s_1)\left(1 - u(s_1) - \frac{\beta_1 w(s_1)}{u(s_1)+k_1}\right) > 0.$$

By (2.15), we have

$$\begin{aligned}&\left[d - \frac{1}{c}\left(1 + \frac{cd}{u^*}\right)\left(\frac{1+k_1}{\beta_1}\right)\frac{\beta_1 u(s_1)}{u(s_1)+k_1}\right]w(s_1)\\ &\quad + \left(1 + \frac{cd}{u^*}\right)\left(\frac{1+k_1}{\beta_1}\right)\frac{1}{c}u(s_1)(1-u(s_1)) > 0,\end{aligned}$$

which together with (2.14) leads to

$$\begin{aligned}&\left\{\left[d - \left(1 + \frac{cd}{u^*}\right)\frac{u(s_1)}{c}\right]\left(1 + \frac{cd}{u^*}\right)\left(\frac{1+k_1}{\beta_1}\right)\right.\\ &\quad \left. + \left(1 + \frac{cd}{u^*}\right)\left(\frac{1+k_1}{\beta_1}\right)\frac{1}{c}u(s_1)\right\}(1-u(s_1)) > 0.\end{aligned}$$

Hence

$$d - \left(1 + \frac{cd}{u^*}\right)\frac{u(s_1)}{c} + \frac{u(s_1)}{c} > 0,$$

which is a contradiction since $u(s_1) > u^*$. Therefore, Case I can not occur.

Case II. There exists s_1 such that (2.12) and (2.13) are valid for $0 < s < s_1$, and

$$w(s_1) \leq -\left(1 + \frac{cd}{u^*}\right)\left(\frac{1+k_1}{\beta_1}\right)(u(s_1) - 1) \tag{2.17}$$

and

$$z(s_1) = \mathrm{d}w(s_1). \tag{2.18}$$

Then we have

$$\dot{z}(s_1) - \mathrm{d}\dot{w}(s_1) > 0 \Rightarrow cz(s_1) - \alpha w(s_1)\left(1 - \frac{\beta_2 w(s_1)}{u(s_1)+k_2}\right) - \mathrm{d}z(s_1) > 0. \tag{2.19}$$

By (2.18), it is easy to see that

$$(cd - d^2)w(s_1) - \alpha w(s_1)\left(1 - \frac{\beta_2 w(s_1)}{u(s_1)+k_2}\right) > 0.$$

Since $w(s_1) > 0$,

$$(cd - d^2) + \frac{\alpha\beta_2}{u(s_1) + k_2} w(s_1) > 0.$$

By (2.17), we obtain

$$\left(1 + \frac{A}{u^*}\right) cd - d^2 + A = (cd - d^2) + \frac{\alpha\beta_2}{k_2}\left(1 + \frac{cd}{u^*}\right)\left(\frac{1+k_1}{\beta_1}\right) > 0,$$

and hence, $d \in \left(0, \frac{c(1+\frac{A}{u^*})+\sqrt{(1+\frac{A}{u^*})^2c^2+4A}}{2}\right)$, which is a contradiction. Therefore, Case II can not occur.

Summarizing the above discussion, we know that (2.12) and (2.13) are valid for all $s > 0$. The proof is complete. □

Lemma 2.5. *In a sufficiently small neighborhood of* $(1, 0, 0)$, *the two-dimensional unstable manifold* $\mathcal{U}_2$ *intersects the plane, which is defined by* $z = 0$ *in a smooth* C^1 *curve* Γ.

The proof is similar with that of [5, Lemma 5], which we omit here.

Lemma 2.6. *A solution of* (2.1) *on* $\mathcal{U}_1$ *would exit* $\mathbf{W}$ *and enter the region* $\mathbb{P}$.

Proof. Lemmas 2.2–2.4 show that a solution of system (2.1) on the strongly unstable manifold $\mathcal{U}_1$ is contained in the set

$$\begin{aligned}\mathcal{C} = \Bigg\{&(u, w, z) \in \mathbb{D} \;\Big|\; u^* < u < 1, -\left(1 + \frac{c^2}{2}\right)\frac{1+k_1}{\beta_1}(u-1)\\ &\leq w < -\left(1 + \frac{cd}{u^*}\right)\frac{1+k_1}{\beta_1}(u-1),\ \tfrac{c}{2}w < z < \mathrm{d}w\Bigg\},\end{aligned}$$

where $d > \frac{c(1+\frac{A}{u^*})+\sqrt{(1+\frac{A}{u^*})^2c^2+4A}}{2}$. Since $u(s) < 1$, we have

$$\begin{aligned}&w(s) \geq -\left(1 + \frac{c^2}{2}\right)\frac{1+k_1}{\beta_1}(u(s)-1) > \frac{u(s)+k_1}{\beta_1}(1-u(s))\\ &\Rightarrow 1 - u(s) - \frac{\beta_1 w(s)}{u(s)+k_1} < 0 \Rightarrow u'(s) < 0.\end{aligned}$$

Thus for a solution of system (2.1), $u(s)$ decreases until $u(s_1) = u^*$ for some finite s_1. The trajectory of this solution therefore hits $\partial\mathbf{W}$ on the face $u = u^*$, $w > w^*$, $z > 0$, (2.1) shows that it will leave $\mathbf{W}$ at some finite time by the common boundary of $\mathbf{W}$ and $\mathbb{P}$, and enters the region $\mathbb{P}$. □

Lemma 2.7. *A trajectory of system* (2.1) *passing through a point of* Γ *in the region where* $u < 1$ *would leave* $\mathbf{W}$ *at some finite time and then enter the region* $\mathbb{Q}$.

Proof. We know that a point (u, w, z) with $z = 0$ is on curve Γ. Thus, the third element in (2.8) can be approximated to the first order by $n = \frac{-\lambda_3 m}{\lambda_2}$. Recall that the portion of interest of the curve Γ is in the region where $u < 1$, then there is

$$-m\frac{\beta_1}{(1+k_1)(c\lambda_3+1)} - n\frac{\beta_1}{(1+k_1)(c\lambda_2+1)} < 0.$$

It follows that $m < 0$, hence $m + n > 0$, which together with (2.8) implies that the w component of points along the curve Γ will satisfy $w > 0$. From the direction of the vector field on the quarter to plane by $u^* < u, w > 0, z > 0$, any trajectory passing through a point of Γ would immediately enter the region $\mathbb{Q}$. The proof is complete. $\square$

We define $\sum$ in the following way. First, we choose a sufficiently small circle around $(1, 0, 0)$ on the two-dimensional unstable manifold $\mathcal{U}_2$, and it satisfies the following:

(i) The circle is small enough that the intersection point of the circle and the one-dimensional strongly unstable manifold $\mathcal{U}_1$ is close enough to the eigenvector χ_3 to satisfy Lemmas 2.2–2.4.
(ii) The circle is contained in the neighborhood of $(1, 0, 0)$ given in Lemma 2.5, then it intersects with the curve Γ.

Hence, $\sum$ is defined to be the arc of this circle contained on the octant Θ_1 whose endpoints are the intersections of the circle with $\mathcal{U}_1$ and curve Γ.

Now we can apply Proposition 2.1 to find a trajectory in phase space which does not enter the region $\mathbb{P}$ or $\mathbb{Q}$. Such a result is presented in Lemma 2.8.

Lemma 2.8. *There is a* $\mathbf{y}_0 \in \sum$ *such that the solution* $\mathbf{y}(s; \mathbf{y}_0)$ *of system* (2.1) *remains in* $\mathbf{W}$ *for all* s; *i.e.*, $\sum \neq \sum^0$.

Proof. We first prove set $\mathbf{W}$ is a Wazewski set. Since $\mathbf{W}$ is a closed set, then statement (1) in Proposition 2.1 is valid. Suppose $\mathbf{y}_0 \in \sum,\ s < T(\mathbf{y}_0),\ \mathbf{y}(s, \mathbf{y}_0) \in \mathbf{W}\backslash\mathbf{W}^-$. Then $\mathbf{y}(s, \mathbf{y}_0) \in \operatorname{int}\mathbf{W} \cup \mathbf{J}$. To verify statement (2) in Proposition 2.1, we only need to prove the following claim.

Claim. $\mathbf{y}(s, \mathbf{y}_0) \notin \mathbf{J} = \{(u, w, z)|u \geq u^*, w < 0, z = 0\} \cup \{(u, w, z)|u \geq u^*, w = 0, z = 0\}$.

Suppose the above claim is not true, then two cases can occur.

Case I. There is $s_1 < T(\mathbf{y}_0)$ such that $\mathbf{y}(s_1, \mathbf{y}_0) \in \{(u, w, z)|u \geq u^*, w < 0, z = 0\}$. From the calculation of the vector field, we have $u'(s_1) > 0,\ z'(s_1) > 0$. If $u(s_1) > u^*$, there exists a sufficiently small δ such that $\mathbf{y}(s_1 - \delta, \mathbf{y}_0)$ lies in region $\mathbb{Q}$, which is a contradiction to $s_1 < T(\mathbf{y}_0)$. If $u(s_1) = u^*$, then either $u(s) < u^*$ for all $s < s_1$, or there is $s_1' < s_1$ such that $u(s_1') = u^*$ and $u(s) < u^*$ for $s_1' < s < s_1$. The fact that $u(s) < u^*$ for all $s < s_1$, leads to a contradiction that $\mathbf{y}_0 \notin \Sigma$. Then s_1' must exist. But we can find δ small enough such that $\mathbf{y}(s_1' + \delta, \mathbf{y}_0)$ stays in region $\mathbb{P}$ no matter whether $z(s_1') > 0$ from the calculation of the vector field, which is also a contradiction to $s_1 < T(\mathbf{y}_0)$. That means this case cannot occur.

Case II. There is s_1 such that $\mathbf{y}(s_1, \mathbf{y}_0) \in \{(u, w, z)|u \geq u^*, w = 0, z = 0\}$. Since $\{(u, w, z)|u \geq u^*, w = 0, z = 0\}$ is an invariant set, no $\mathbf{y}_0 \in \sum$ is valid but $\mathbf{y}_0$ belongs to the stable manifold corresponding to the eigenvalue $\lambda_1 < 0$, which is a contradiction.

Thus this case cannot occur either. It follows that statement (2) in Proposition 2.1 holds true and hence, **W** is a Wazewski set.

Now we are in a position to verify statement (3) in Proposition 2.1. Obviously, the arc $\sum$ is compact and it has only one intersection with a trajectory of system (2.1). Lemmas 2.2–2.7 show that the image of one endpoint of $\sum$ lies in the component $\partial\mathbb{P}\backslash\{(u^*, w^*, 0)\}$ of $\mathbf{W}^-$, and the image of the other endpoint is in the component $\partial\mathbb{Q}\backslash\{\mathbf{J}\bigcup(u^*, w^*, 0)\}$ of $\mathbf{W}^-$. If $\sum = \sum^0$, then $\mathbf{F}$ would be a homeomorphism of the connected set $\sum$ to its image in the disconnected set $\mathbf{W}^-$. Therefore, $\sum \neq \sum^0$ and the lemma is proved. $\square$

Moreover, the next lemma shows us that $\mathbf{y}(s; \mathbf{y}_0) \in \mathfrak{D} \subset \mathbf{W}$. For convenience, let $\tilde{\mathbf{y}}(s)$ denote the solution $\mathbf{y}(s; \mathbf{y}_0)$, and the coordinate functions be $\tilde{u}(s)$, $\tilde{w}(s)$ and $\tilde{z}(s)$. Therefore, $\mathbf{y}_0 = \tilde{\mathbf{y}}(0) = (\tilde{u}(0), \tilde{w}(0), \tilde{z}(0))^T$.

Lemma 2.9. *The solution $\tilde{\mathbf{y}}(s)$ stays in the region*

$$\mathfrak{D} = \{(u, w, z) \in \mathbb{D} \mid 0 < u < 1, 0 < w < l_1(u), l_2(w) < z < l_3(w)\},$$

where

$$l_1(u) = \begin{cases} -\left(1+\dfrac{cd}{u^*}\right)\left(\dfrac{1+k_1}{\beta_1}\right)(u-1) & u^* < u \le 1, \\ -\left(1+\dfrac{cd}{u^*}\right)\left(\dfrac{1+k_1}{\beta_1}\right)(u^*-1) & 0 \le u \le u^*, \end{cases}$$

$$l_2(w) = -\frac{\bar{A}}{c} w \text{ with } \bar{A} > \frac{\alpha\beta_2}{k_2}\left(1+\frac{cd}{u^*}\right)\left(\frac{1+k_1}{\beta_1}\right),$$

$$l_3(w) = dw,$$

where d satisfies the condition in Lemma 2.4.

Proof. Since the plane defined by $u = 0$ is an invariant manifold, then the coordinate $\tilde{u}(s)$ of $\tilde{\mathbf{y}}(s)$ is strictly positive for all s since $\tilde{u}(0) > 0$.

We claim that the second coordinate of $\tilde{\mathbf{y}}$, $\tilde{w}$ is also positive. Suppose, by contradiction, $\tilde{\mathbf{y}}(s)$ enters the region $N = \{(u, w, z)|w < 0\}$. Let $s_1 = \inf\{s|\tilde{\mathbf{y}}(s) \in N\}$. Then $\tilde{w}(s_1) = 0$. The vector field on the plane defined by $w = 0$ shows that $\tilde{w}'(s_1) = \tilde{z}(s_1) \le 0$. The u-axis is an invariant manifold, so $\tilde{z}(s_1) < 0$, and we obtain $\tilde{u}(s_1) < u^*$. Otherwise, $\tilde{\mathbf{y}}(s_1) \in \mathbb{Q}$, which is a contradiction. Therefore, we have $\tilde{u}(s_1) < u^*$. Then only two cases can occur.

Case I. $\tilde{w}(s) < 0$ for all $s > s_1$. We show that $\tilde{z}(s) < 0$ for all $s > s_1$. If not, we suppose that there is $\hat{s} > s_1$ and small $\delta > 0$ such that $\tilde{z}(\hat{s}) = 0$, $\tilde{z}(\hat{s}+\delta) > 0$, hence $\tilde{z}'(s) > 0$ for $s \ge \hat{s}$. It follows that $\tilde{w}'(s) > \tilde{z}(\hat{s}) > 0$ for $s > \hat{s}$, then we have $\tilde{w}(s) > 0$ at some finite time, which is a contradiction. Thus $\tilde{z}(s) < 0$ is valid for all $s > s_1$. By the first equation of (2.1) and $\tilde{u}(s_1) < u^*$, we have $\tilde{u}'(s) \ge \frac{1}{c}\min[\tilde{u}(s_1)(1-\tilde{u}(s_1)), u^*(1-u^*)]$ for $s > s_1$. Then $\tilde{u}(s_1)$ increases to u^* at finite time, say $\tilde{u}(s_2) = u^*$. Together with $\tilde{w}(s_2) < 0$, $\tilde{z}(s_2) < 0$, the vector field shows that $\tilde{\mathbf{y}}(s)$ enters $\mathbb{Q}$. This is impossible.

Case II. There is $\bar{s} > s_1$ such that $\tilde{w}(s) < 0$ for $s_1 < s < \bar{s}$ and $\tilde{w}(\bar{s}) = 0$. Obviously, $\tilde{w}'(\bar{s}) = \tilde{z}(\bar{s}) \ge 0$, similar to the above discussion. The invariance of u-axis leads to $\tilde{w}'(\bar{s}) = \tilde{z}(\bar{s}) > 0$. Hence there is $\delta > 0$ such that $\tilde{z}(\bar{s}+\delta) > \frac{c}{2}\tilde{w}(\bar{s}+\delta) > 0$. By Lemma 2.2, we have $\tilde{z}(s) > \frac{c}{2}\tilde{w}(s)$ and $\tilde{w}(s) > 0$ for all $s > \bar{s}+\delta$. Now the vector field shows that the trajectory would enter region $\mathbb{P}$, which is not impossible either.

Summarizing Cases I and II, we conclude that $\tilde{w}(s) > 0$ at all times. Next we prove $\tilde{z}(s) > l_2(\tilde{w})(s)$ for all s. If not, suppose there is s_3 such that $\tilde{z}(s_3) \leq -\frac{\bar{A}}{c}\tilde{w}(s_3) < 0$. If there is an $s_4 > s_3$ such that $\tilde{z}(s_4) = -\frac{\bar{A}}{c}\tilde{w}(s_4)$, then $\tilde{z}'(s_4) + \frac{\bar{A}}{c}\tilde{w}'(s_4) \geq 0$. Hence for $s = s_4$, we have $0 < \tilde{w} < l_1(\tilde{u})$, $\frac{1-\tilde{u}}{\tilde{u}+k_2} \leq \frac{1}{k_2}$, and obtain

$$
\begin{aligned}
& c\tilde{z} - \alpha\tilde{w}\left(1 - \frac{\beta_2\tilde{w}}{\tilde{u}+k_2}\right) + \frac{\bar{A}}{c}\tilde{z} \geq 0 \\
& \Rightarrow -\bar{A}\tilde{w} - \alpha\tilde{w}\left(1 - \frac{\beta_2\tilde{w}}{\tilde{u}+k_2}\right) - \frac{\bar{A}^2}{c^2}\tilde{w} \geq 0 \\
& \Rightarrow -\alpha - \frac{\bar{A}^2}{c^2} \geq \bar{A} - \frac{\alpha\beta_2\tilde{w}}{\tilde{u}+k_2} > \bar{A} - \frac{\alpha\beta_2 l_1(\tilde{u})}{\tilde{u}+k_2} \\
& \qquad\qquad\qquad \geq \bar{A} - \frac{\alpha\beta_2}{k_2}\left(1 + \frac{cd}{u^*}\right)\left(\frac{1+k_1}{\beta_1}\right) > 0,
\end{aligned}
$$

which is impossible. So if $\tilde{z}(s_3) < -\frac{\bar{A}}{c}\tilde{w}(s_3)$, then $\tilde{z}(s) < -\frac{\bar{A}}{c}\tilde{w}(s)$ for all $s > s_3$. Furthermore, if

$$
\begin{aligned}
\tilde{z}'(s) &= c\tilde{z}(s) - \alpha\tilde{w}(s)\left(1 - \frac{\beta_2\tilde{w}(s)}{\tilde{u}(s)+k_2}\right) \\
&< \left(-\bar{A} + \frac{\alpha\beta_2}{k_2}\left(1 + \frac{cd}{u^*}\right)\frac{1+k_1}{\beta_1}\right)\tilde{w}(s) - \alpha\tilde{w}(s) < -\alpha\tilde{w}(s) < 0,
\end{aligned}
$$

it yields that $\tilde{z}(s) < \tilde{z}(s_3) < 0$ for $s > s_3$. It then follows that $\tilde{w}'(s) < \tilde{z}(s_3) < 0$ for $s > s_3$, so $\tilde{w}(s) < 0$ for some finite s, again a contradiction. Therefore, one has $\tilde{z}(s) > l_2(\tilde{w})(s)$ for all s.

Note that a trajectory starting on $\sum$ converges to $(1, 0, 0)$ tangent to χ_2 or χ_3. Points on χ_2 or χ_3 have $z = \lambda_2 w$ or $z = \lambda_3 w$. Since $\lambda_2, \lambda_3 < d$, Lemma 2.4 shows that $\tilde{z}(s) < \mathrm{d}\tilde{w}(s)$ for all s. The proof is complete. □

The remaining work is to construct a V function to apply the Invariance Principle to show that $\tilde{\mathbf{y}}(s) \to (u^*, w^*, 0)$. We define

$$
\begin{aligned}
V(u, w, z) = {} & \frac{c\alpha}{\beta_1}\left[u - u^* + (k_1 - k_2 - u^* + \frac{k_1 u^*}{k_2})\ln\frac{u+k_2}{u^*+k_2} - \frac{k_1 u^*}{k_2}\ln\frac{u}{u^*}\right] \\
& + \left[c(w - w^*) - z\right] + w^*\left[\frac{z}{w} - c\ln\left(\frac{w}{w^*}\right)\right].
\end{aligned}
$$

Calculating $\frac{\mathrm{d}V}{\mathrm{d}t}$ along the solution of (2.1), we have

$$
\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{c\alpha}{\beta_1}\left[\frac{(u-u^*)(u+k_1)}{u(u+k_2)}\right]\frac{\mathrm{d}u}{\mathrm{d}t} + \left(c - \frac{w^*}{w^2}z - \frac{cw^*}{w}\right)\frac{\mathrm{d}w}{\mathrm{d}t} - \left(1 - \frac{w^*}{w}\right)\frac{\mathrm{d}z}{\mathrm{d}t}.
$$

that is,

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} = {} & \frac{\alpha}{\beta_1}\left[\frac{(u-u^*)(u+k_1)}{(u+k_2)}\right]\left(1 - u - \frac{\beta_1 w}{u+k_1}\right) - \frac{w^*}{w^2}z^2 \\
& + \alpha\left(1 - \frac{\beta_2 w}{u+k_2}\right)(w - w^*).
\end{aligned}
$$

Note that

$$
\begin{aligned}
1-u-\frac{\beta_1 w}{u+k_1} &= 1-u-\frac{\beta_1 w}{u+k_1}-\left(1-u^*-\frac{\beta_1 w^*}{u^*+k_1}\right)\\
&= -(u-u^*)-\left(\frac{\beta_1 w}{u+k_1}-\frac{\beta_1 w^*}{u^*+k_1}\right)\\
&= -(u-u^*)-\left(\frac{\beta_1 w}{u+k_1}-\frac{\beta_1 w^*}{u+k_1}+\frac{\beta_1 w^*}{u+k_1}-\frac{\beta_1 w^*}{u^*+k_1}\right)\\
&= -(u-u^*)-\frac{\beta_1}{u+k_1}(w-w^*)+\frac{\beta_1 w^*(u-u^*)}{(u^*+k_1)(u+k_1)}\\
&= -(u-u^*)-\frac{\beta_1}{u+k_1}(w-w^*)+\frac{u-u^*}{u+k_1}(1-u^*)
\end{aligned}
$$

and

$$1-\frac{\beta_2 w}{u+k_2}=\frac{-\beta_2}{u+k_2}(w-w^*)+\frac{u-u^*}{u+k_2}.$$

If $k_1 > 1-u^*$, we have

$$\frac{\mathrm{d}V}{\mathrm{d}t} \le -\frac{\alpha}{\beta_1}\frac{u+k_1-1+u^*}{u+k_2}(u-u^*)^2-\frac{\alpha\beta_2}{u+k_2}(w-w^*)^2-\frac{w^*}{w}z^2 \le 0.$$

Note that $\dot{V}=0$ if and only if (u, w, z) is the equilibrium point $(u^*, w^*, 0)$. The Invariant Principle implies that $\tilde{\mathbf{y}}(s) \to (u^*, w^*, 0)$ as $s \to +\infty$. Let

$$A_1=-\left(1-2u^*-\frac{\beta_1 w^* k_1}{(u^*+k_1)^2}\right),\ A_2=\frac{\beta_1 u^*}{(u^*+k_1)}.$$

If $k_1 > 1-u^*$, then

$$
\begin{aligned}
1-2u^*-\frac{\beta_1 w^* k_1}{(u^*+k_1)^2} &= (1-u^*)\left(1-\frac{k_1}{u^*+k_1}\right)-u^*\\
&= \left(\frac{1-u^*}{u^*+k_1}-1\right)u^*=\frac{u^*}{u^*+k_1}(1-2u^*-k_1)<0,
\end{aligned}
$$

which implies

$$A_1 > 0,\ A_2 > 0. \tag{2.20}$$

Evaluating the Jacobian matrix of the right side of system (2.1) at $(u^*, w^*, 0)$, we obtain

$$
\begin{aligned}
J(u^*, w^*, 0) &= \begin{pmatrix} \frac{1}{c}\left[1-2u^*-\frac{\beta_1 w^* k_1}{(u^*+k_1)^2}\right] & -\frac{1}{c}\left(\frac{\beta_1 u^*}{u^*+k_1}\right) & 0\\ 0 & 0 & 1\\ -\frac{\alpha}{\beta_2} & \alpha & c \end{pmatrix}\\
&= \begin{pmatrix} -\frac{A_1}{c} & -\frac{A_2}{c} & 0\\ 0 & 0 & 1\\ -\frac{\alpha}{\beta_2} & \alpha & c \end{pmatrix},
\end{aligned}
$$

and the corresponding characteristic polynomial is

$$P(\lambda)=\left(\lambda+\frac{A_1}{c}\right)(c\lambda+\alpha-\lambda^2)+\frac{\alpha A_2}{c\beta_2}.$$

We are interested in the eigenvalues of $P(\lambda) = 0$, and study them by delicate analysis.

Let $\lambda \to +\infty$, then $P(\lambda) \to -\infty$. By the graph of $P(\lambda)$, we know $P(\lambda) = 0$ has a positive root λ', which satisfies

$$\lambda' \geq \frac{c + \sqrt{c^2 + 4\alpha}}{2} > c. \tag{2.21}$$

Now we can rewrite $P(\lambda)$ as

$$P(\lambda) = (\lambda - \lambda')(a + b\lambda - \lambda^2),$$

where a, b are undetermined. Since

$$(\lambda - \lambda')(a + b\lambda - \lambda^2) = (\lambda + \frac{A_1}{c})(c\lambda + \alpha - \lambda^2) + \frac{\alpha A_2}{c\beta_2},$$

it follows that

$$\begin{aligned} b + \lambda' &= -\frac{A_1}{c} + c, \\ a\lambda' &= -\frac{A_1}{c}\alpha - \frac{\alpha A_2}{c\beta_2}. \end{aligned}$$

By virtue of (2.20) and (2.21), we have $a < 0, b < 0$. Noticing that the other two roots of $P(\lambda) = 0$ are determined by

$$\lambda^2 - b\lambda - a = 0, \tag{2.22}$$

it is easy to obtain that $\operatorname{Re}\lambda = \frac{b}{2} < 0$, hence $P(\lambda) = 0$ has three eigenvalues, one is positive and the other two with negative real parts. Hence there is a two-dimensional stable manifold and a one-dimensional unstable manifold at $(u^*, w^*, 0)$. By (2.22), we can find a critical $\alpha^* = \alpha^*(u^*, c)$ such that if $\alpha > \alpha^*$, there is a complex conjugate pair of eigenvalues with negative real part; if $\alpha < \alpha^*$, there are two distinct negative eigenvalues; if $\alpha = \alpha^*$, there is a repeated negative real value. Therefore, if $\alpha \leq \alpha^*$, solutions of system (2.1) on the stable manifold approach $(u^*, w^*, 0)$ monotonously. If $\alpha > \alpha^*$, the solution on the stable manifold will spiral in towards $(u^*, w^*, 0)$ with damped oscillations. Furthermore, by direct computation, we have

$$\alpha^* = \frac{\lambda'(-\frac{A_1}{c} + c - \lambda')^2}{4\left(\frac{A_1}{c} + \frac{A_2}{c\beta_2}\right)} \geq \frac{A_1^2}{4\left(A_1 + \frac{A_2}{\beta_2}\right)}.$$

Summarizing the above discussion, Theorem 2.2 is true for $c > \sqrt{4\alpha}$. The existence of such traveling wave solutions for $c = \sqrt{4\alpha}$ can be proved in a similar way to that in [5], where the limit argument was applied. We omit it here. Thus the proof of Theorem 2.2 is complete.

3. Discussion

We consider a diffusive predator-prey model with modified Leslie–Gower and Holling-II schemes with $D = 0$. The stability of the four equilibria of system (1.2) was discussed by Tian [16]. The boundary equilibrium $(1, 0)$ is unstable and the positive equilibrium

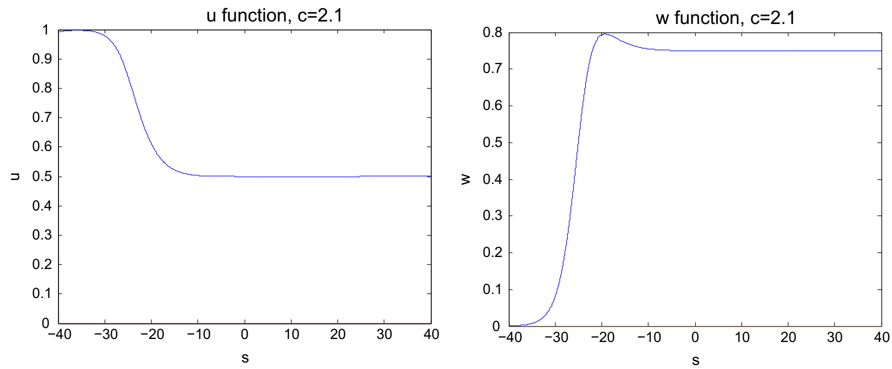

Figure 1. Traveling wave solution of system (1.2).

(u^*, w^*) is stable under some assumptions. Biologically, we concentrate on the case of such traveling wave connecting $(1, 0)$ to (u^*, w^*), which describes the invasion of the predators. We use the 'shooting method' to show the existence of a nonnegative traveling wave solution, and prove the non-existence by the 'eigenvalue method'.

Now we present some simulation results for the main results in section 2. By Theorem 2.2, system (1.2) admits a nonnegative traveling wave connecting $(1, 0)$ to (u^*, w^*). Let $k_1 = 1,\ \beta_1 = 1,\ \alpha = 1,\ k_2 = 1,\ \beta_2 = 2,\ c = 2.1 > c^*(= 2)$. It is easy to check that the conditions in Theorem 2.2 are valid under the above parameters. The evolution of the solutions is shown in figure 1. We do not show the case where $c = c^* = 2$ as the graphs are quite similiar.

A particular interesting question that is left open here but could be approached with the techniques by Dunbar [6], is to explore the case where $D > 0$. In [6], Dunbar studied diffusive Lotka–Volterra equations with $D > 0$, and found out a heteroclinic connection between $(1, 0)$ with (u^*, w^*) by using the shooting method in $\mathbf{R}^4$. However, the analysis could become more complicated in $\mathbf{R}^4$, compared with that in $\mathbf{R}^3$ shown here.

We only investigate the traveling wave connecting $(1, 0)$ to (u^*, w^*) in our present study. However, the traveling wave connecting other equilibria are also worthy to be considered because of their important insights in ecological balance. For example, such traveling wave connecting $(1, 0)$ to $(0, \frac{k_2}{\beta_2})$ describes a scenario that introducing predator into the system could drive prey to extinction. Currently, the commonly used technique to explore the traveling wave in a non-monotone system includes the shooting arguments, iterative method [8,9,15,20] and perturbation analysis [1,12]. The effective approach to solve the existence and non-existence of the traveling wave with a general connection between different equilibria of system (1.2) still remain wide open.

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