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Harmonic oscillator Wigner function extension to exceptional polynomials

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Abstract. In this paper, we construct isospectral Hamiltonians without shape invariant potentials for a harmonic oscillator Wigner function on a real line. In this case, we actually remove the ground state of the second Hamiltonian, which forms a special case, $m = 0$, of an exceptional Laguerre differential equation with solutions $\{L_n^{-2}\}_{n=2}^{\infty}$ as eigenfunctions form a complete orthogonal set in the Hilbert space.

Keywords. Bound states; exceptional polynomials; Wigner function.

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1. Introduction

The Sturm–Liouville equation is given as

$$p(x)y_n'' + q(x)y_n' + r(x)y_n = \lambda_n y_n, \tag{1}$$

where $p(x)$, $q(x)$ and $r(x)$ are polynomials of order 2, 1 and 0, respectively. It is well known in the mathematical literature that the only possible solutions to the Sturm–Liouville equation are Hermite, Laguerre and Jacobi polynomials. This famous theorem is known as Bochner theorem [1]. It should be noted that y_0 is constant in all these systems. In the last decade, Gómez-Ullate *et al* [2,3] have constructed a new orthogonal polynomial system (OPS), which starts with y_n $(n = 1, 2, \ldots)$ by a suitable modification to its weight function and forms a complete set. The new OPS was constructed by relaxing the constraint on the Sturm–Liouville problem, where $r(x)$ is not constant. It is possible to extend all three classical OPSs to exceptional Laguerre polynomials, exceptional Hermite and exceptional Jacobi polynomials. Initially, Laguerre and Jacobi polynomials were shown to allow such extensions known as exceptional polynomials. Later on, the extensions to Hermite polynomials are found to be exceptional Hermite polynomials [4].

Recently, the classification of the exceptional orthogonal polynomials is given in terms of the codimension of the exceptional family [5]. The codimension of the exceptional family is defined as the total number of missing degrees in the polynomial sequence. These missing degrees in the polynomials put constraints on the Sturm–Liouville equation and in turn on the Bochner theorem. They have also shown that every system of exceptional orthogonal polynomials is related to the respective classical orthogonal polynomials by a sequence of Darboux transformations.

Quesne was the first to construct the Darboux transformations of codimension two in [6,7]. Odake and Sasaki have constructed an exceptional OPS for arbitrary codimension [8,9]. The role of Darboux transformations was further studied and clarified in [10, 11]. Exceptional orthogonal families were generated through higher-order Darboux transformations [11–13]. Exceptional polynomials are developed using prepotential approach [14]. Exceptional polynomials are also extended to the PT symmetry Hamiltonians [15] and the well-known examples are PT-symmetric Scarf II potentials [16–18]. An exceptional polynomial for a symmetry group preserving the form of the Rayleigh–Schrödinger equation is constructed [19]. Alternative derivation of infinitely many exceptional Wilson and Askey–Wilson polynomials is presented in [20]. These exceptional polynomial systems appear as solutions to the quantum mechanical problems in exactly solvable models [21,22] or in superintegrable systems [23].

In this paper, we construct the Wigner function for harmonic potential on a real line, given in terms of the exceptional Laguerre polynomials. We give a brief description of the exceptional Laguerre polynomials below.

One can construct the exceptional X_1-Laguerre polynomials $\mathcal{L}_n^k(x),\ k>0$, using the Gram–Schmidt procedure from the sequence [2,3]

$$v_1 = x+k+1, \quad v_i = (x+k)^i, \quad i \geq 2 \tag{2}$$

and using the weight function

$$\hat{W}_k(x) = \frac{x^k \mathrm{e}^{-x}}{(x+k)^2} \tag{3}$$

as defined in the interval $x \in (0,\infty)$ and the scalar product

$$(f,g)_k = \int_0^\infty \mathrm{d}x\, \hat{W}_k(x) f(x) g(x). \tag{4}$$

The weight function of the normal Laguerre polynomial $W_k(x) = x^k \mathrm{e}^{-x}$ is multiplied by suitable factors by which one obtains a new $\hat{W}_k(x)$ such that one can construct a new OPS excluding the zero degree polynomial. The exceptional X_1-Laguerre differential equation is

$$T_k(y) = \lambda y, \tag{5}$$

where $\lambda = n-1$ with $n = 1, 2, \ldots$ and

$$T_k(y) = -xy'' + \left(\frac{x-k}{x+k}\right)[(k+x+1)y' - y]. \tag{6}$$

In general, one can construct X_m-Laguerre polynomials and X_m-Laguerre differential equation. For more details, we refer the reader to a recent review on exceptional Laguerre polynomials [24].

It is well known that the Wigner function for harmonic potential on a real line is given in terms of Laguerre polynomials. It has been shown in [25] that whenever a potential admits Laguerre/Jacobi as a solution, then one can construct an isospectral Hamiltonian with exceptional Laguerre/Jacobi as a solution and the potential is determined uniquely. In our problem, it turns out that $m = 0$ in the exceptional Laguerre differential equation.

2. Wigner function

The Wigner function is proposed as an alternative to the Schrödinger picture to solve quantum mechanical problems. The Wigner function maps the quantum wave function to a probability distribution in the phase space, where the state of the system is described by the position x and momentum p in terms of the classical Hamiltonian. The main aim of the Wigner function is to find quantum corrections to classical statistical mechanics where Boltzmann factors contain energies which in turn are expressed as functions of position and momentum. It is well known that there is no Heisenberg uncertainty relation in classical mechanics and due to this, there are constraints on the probability distribution and Wigner function. Therefore, for a given quantum state, the probability distribution in the phase space can be negative; hence, these distributions are called quasiprobability distributions [26].

In physics, the Wigner function has found many applications in the fields of statistical mechanics, quantum optics and classical optics, for details refer to [26]. In particular, in quantum optics, the Wigner function is used for the study of classical and non-classical states [26]. The Wigner function is also used to study the phase-space representation of quantum mechanics using Glauber coherent states [27]. The Wigner functions for the number states $\hat{\rho} = |n\rangle\langle n|$ in Fock basis are Laguerre polynomials [28].

The time-dependent Wigner function is given by

$$W(x,p,t) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} \psi^*(x+y)\psi(x-y)\,\mathrm{e}^{2\mathrm{i}py/\hbar}\,\mathrm{d}y, \tag{7}$$

where ψ is the quantum wave function and the Wigner function in terms of the density matrix $\hat{\rho}(t)$ is given by

$$W(x,p,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{\mathrm{i}py/\hbar} \times \left\langle x + \frac{1}{2}y|\hat{\rho}(t)|x - \frac{1}{2}y\right\rangle \tag{8}$$

and the Wigner function in terms of the Moyal function is given by

$$W_{|E''\rangle\langle E'|}(x,p,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{\mathrm{i}py/\hbar}\left\langle x + \frac{1}{2}y|E''\right\rangle\left\langle E'|x - \frac{1}{2}y\right\rangle. \tag{9}$$

The Moyal function in terms of the quantum Liouville equation is given as

$$\left[\frac{p^2}{2M} + U - \frac{\hbar^2}{8M}\frac{\partial^2}{\partial x^2} + \sum_{l=1}^{\infty}\frac{(-1)^l(\hbar/2)^{2l}}{2l!}\frac{\partial^{2l}U}{\partial x^{2l}}\frac{\partial^{2l}}{\partial p^{2l}}\right]W_{|E\rangle} = EW_{|E\rangle} \tag{10}$$

and

$$\left[\frac{p}{M}\frac{\partial}{\partial x}-\frac{\partial U}{\partial x}\frac{\partial}{\partial p} -\sum_{l=1}^{\infty}\frac{(-1)^l(\hbar/2)^{2l}}{(2l+1)!}\frac{\partial^{2l+1}U}{\partial x^{2l+1}}\frac{\partial^{2l+1}}{\partial p^{2l+1}}\right]W_{|E\rangle}=0, \tag{11}$$

where $W_{|E\rangle}=W_{|E\rangle\langle E|}$ is the diagonal Moyal function, for details refer to [26]. The harmonic oscillator potential is given by

$$U(x)=\frac{1}{2}M\omega x^2. \tag{12}$$

It has been shown in [26] that the solution to the Wigner function for the harmonic potential on a real line is the Laguerre polynomials. We have shown in [25] that any potential has solutions as the Laguerre polynomials and by adding an extra potential one can construct exceptional polynomials as its solution. Hence, by adding an extra term to the harmonic oscillator potential, one obtains

$$U(x)=\frac{1}{2}M\omega x^2+\frac{1}{2}V(x). \tag{13}$$

The Moyal Wigner equations in terms of the new potential will become

$$\left[\frac{p^2}{2M}+\frac{1}{2}M\omega x^2+\frac{1}{2}V(x)-\frac{\hbar^2}{8M}\frac{\partial^2}{\partial x^2}-\frac{\hbar^2M\omega^2}{8}\frac{\partial^2}{\partial p^2} +\frac{1}{2}\sum_{l=1}^{\infty}\frac{(-1)^l(\hbar/2)^{2l}}{2l!}\frac{\partial^{2l}V(x)}{\partial x^{2l}}\frac{\partial^{2l}}{\partial p^{2l}}\right]W_{|E\rangle} = EW_{|E\rangle} \tag{14}$$

and

$$\left[\frac{p}{M}\frac{\partial}{\partial x}-M\omega^2x\frac{\partial}{\partial p}-\frac{1}{2}\frac{\partial V}{\partial x}\frac{\partial}{\partial p} -\frac{1}{2}\sum_{l=1}^{\infty}\frac{(-1)^l(\hbar/2)^{2l}}{(2l+1)!}\frac{\partial^{2l+1}V}{\partial x^{2l+1}}\frac{\partial^{2l+1}}{\partial p^{2l+1}}\right]W_{|E\rangle} =0. \tag{15}$$

One can write the second equation as

$$\left[\frac{p}{M}\frac{\partial}{\partial x}-M\omega^2x\frac{\partial}{\partial p}-\frac{1}{2}\frac{\partial}{\partial x}\frac{\partial}{\partial p}\left(V(x) +\sum_{l=1}^{\infty}\frac{(-1)^l(\hbar/2)^{2l}}{(2l)!}\frac{\partial^{2l}V}{\partial x^{2l}}\frac{\partial^{2l}}{\partial p^{2l}}\right)\right]W_{|E\rangle}=0. \tag{16}$$

By making the following change to variable $\kappa=(M\omega/\hbar)$ and introducing the dimensionless position $\xi=\kappa x$, momentum $\zeta=p/\hbar\kappa)$ and energy $\eta=E/(\hbar\omega)$, the above equations take the form

$$\left[\frac{\partial^2}{\partial\zeta^2}+\frac{\partial^2}{\partial\xi^2}\right]W_{|E\rangle}-\left[2\eta-(\zeta^2+\xi^2)+\frac{1}{2}V(\xi)\right]W_{|E\rangle} -c_1\frac{1}{2}\left[\sum_{l=1}^{\infty}\frac{(-1)^l(\hbar/2)^{2l}}{(2l)!}\frac{\partial^{2l}V}{\partial\xi^{2l}}\frac{\partial^{2l}}{\partial\zeta^{2l}}\right]W_{|E\rangle}=0 \tag{17}$$

and

$$\left[\zeta\frac{\partial}{\partial\xi}-\xi\frac{\partial}{\partial\zeta}\right]W_{|E\rangle}-c_2\frac{1}{2}\frac{\partial}{\partial\xi}\frac{\partial}{\partial\zeta}\left[V(\xi) +c_1\sum_{l=1}^{\infty}\frac{(-1)^l(\hbar/2)^{2l}}{(2l)!}\frac{\partial^{2l}V}{\partial\xi^{2l}}\frac{\partial^{2l}}{\partial\zeta^{2l}}\right]W_{|E\rangle}=0. \tag{18}$$

We make another change of variable

$$y(\xi,\zeta)=\xi^2+\zeta^2 \tag{19}$$

which will give

$$\frac{\partial W_{|E\rangle}}{\partial\xi}=2\xi\frac{\partial W_{|E\rangle}}{\partial y},\quad \frac{\partial W_{|E\rangle}}{\partial\zeta}=2\zeta\frac{\partial W_{|E\rangle}}{\partial y}, \tag{20}$$

and then

$$\frac{\partial^2W_{|E\rangle}}{\partial\xi^2}=4\xi^2\frac{\partial^2W_{|E\rangle}}{\partial y^2}+2\frac{\partial W_{|E\rangle}}{\partial y},$$
$$\frac{\partial^2W_{|E\rangle}}{\partial\zeta^2}=4\zeta^2\frac{\partial^2W_{|E\rangle}}{\partial y^2}+2\frac{\partial W_{|E\rangle}}{\partial y}. \tag{21}$$

With the help of these equations, the first term in eq. (18) is zero

$$[2\zeta\xi-2\xi\zeta]\frac{\partial W_{|E\rangle}}{\partial y}=0 \tag{22}$$

and substituting it into the second equation, one obtains

$$V(y)+c_1D_1\sum_{l=1}^{\infty}\frac{(-1)^l(\hbar/2)^{2l}}{(2l)!}\frac{\partial^{2l}V}{\partial y^{2l}}\frac{\partial^{2l}}{\partial y^{2l}}=A. \tag{23}$$

Substituting eq. (23) into eq. (17), we obtain

$$\left[y\frac{\partial^2}{\partial y^2}+\frac{\partial}{\partial y}+2\eta-y+V(y)-A\right]W_{|E\rangle}=cW_{|E\rangle}. \tag{24}$$

We make an ansatz

$$W_{|E\rangle}=L(y)\,\mathrm{e}^{-y}, \tag{25}$$

then

$$y\frac{\partial^2}{\partial y^2}L(y)+(1-2y)\frac{\partial}{\partial y}L(y) +((2\eta+V(y))-1-A)L(y)=0. \tag{26}$$

If $V(y) = 0$ and $A = 0$, then the above differential equation is the Laguerre differential equation. Without loss of generality, we choose the constant $A = -1$. Then the differential equation reduces to

$$y\frac{\partial^2}{\partial y^2}L(y) + (1-y)\frac{\partial}{\partial y}L(y) + (\lambda + V_e(y))L(y) = 0. \tag{27}$$

We apply the following theorem:

Theorem 1. *Add an extra term $V_e(x)$ to the Laguerre/Jacobi differential equation and demand the solutions to be $g(x) = f(x)/(x+m)$ and $g(x) = f(x)/(x-b)$ for the new differential equations, where $f(x)$ are the Laguerre and Jacobi polynomials, respectively. Then $g(x)$ satisfies the X_1-exceptional differential equation for the Laguerre and Jacobi, respectively, $V_e(x,m)$ can be determined uniquely.*

Proof. Let $g(x) = L_\lambda^m(x)$ satisfy the Laguerre differential equation

$$x\frac{d^2}{dx^2}g(x) + (m+1-x)\frac{d}{dx}g(x) + \lambda g(x) = 0, \tag{28}$$

where λ is an integer. By adding an extra term $V_e(x,m)$ to the Laguerre differential equation:

$$\begin{aligned} x\frac{d^2}{dx^2}h(x) &+ (m+1-x)\frac{d}{dx}h(x) \\ &+(\lambda + V_e(x,m))h(x) = 0, \end{aligned} \tag{29}$$

and setting $h(x) = f(x)/(x+m)$ and $\lambda = n-1$, where $f(x)$ satisfies the X_1 exceptional Laguerre differential equation

$$\begin{aligned} -xf''(x) &+ \left(\frac{x-m}{x+m}\right)\left[(m+x+1)f'(x) - f(x)\right] \\ &= (n-1)f(x), \end{aligned} \tag{30}$$

one determines $V_e(x,m)$ to be

$$V_e(x,m) = \frac{2m}{(x+m)^2} - \frac{1}{(x+m)}. \tag{31}$$

If $g(x) = f(x)/(x+m)^j$ with $f(x)$ satisfying the X_j exceptional differential equation

$$\begin{aligned} -xf''(x) &+ \left(\frac{x-m}{x+m}\right)\left[\left((m+x+1) - \frac{2x(j-1)}{x-m}\right)f'(x) - jf(x)\right] = (n-j)f(x), \end{aligned} \tag{32}$$

one obtains $V_e(x,m)$ to be

$$V_e(x,m) = \frac{j(j+1)m}{(x+m)^2} - \frac{j}{(x+m)} \tag{33}$$

for the general case. □

It should be noted that our construction of exceptional polynomials is at the level of differential equations. Hence, the superpotential is constructed using the operator $\hat{\mathcal{O}}$ [2,3,7], which connects the ordinary Laguerre polynomials to the exceptional Laguerre polynomials

$$\hat{\mathcal{O}}L_\nu^{k-1}(x) = \mathcal{L}_{\nu+1}^k(x), \tag{34}$$

where $\hat{\mathcal{O}} = (x+k)((d/dx) - 1) - 1$.

We repeat the demonstration of the supersymmetry for the three-dimensional (3D) oscillator [25]. The wave function for the exceptional 3D oscillator is given by

$$|\psi_\nu^+\rangle = \frac{\xi^{l/2}e^{-(\xi/2)}}{(\xi + ((2l+1)/2))}\mathcal{L}_n^k(\xi) \tag{35}$$

and the wave function for the 3D oscillator is given by

$$|\psi_\nu^-\rangle = \xi^{l/2}e^{-(\xi/2)}L_n^k(\xi). \tag{36}$$

Substituting the exceptional Laguerre polynomial solution and Laguerre polynomial solution into eq. (34) determines the superpotential $\boldsymbol{W}(x)$:

$$\boldsymbol{W}(x) = -\frac{l}{2\xi} - \frac{1}{2} - \frac{2}{2\xi + k}, \tag{37}$$

Here we take $k = 2l+1$. We recover the results of Quesne [7]. In supersymmetry, the superpotential $\boldsymbol{W}(x)$ is defined in terms of the intertwining operators $\hat{A}$ and $\hat{A}^\dagger$ as

$$\hat{A} = \frac{d}{dx} + \boldsymbol{W}(x), \quad \hat{A}^\dagger = -\frac{d}{dx} + \boldsymbol{W}(x). \tag{38}$$

The superpotential, $\boldsymbol{W}(x)$, can be obtained by replacing d/dx in $\hat{A}$ in terms of $\hat{\mathcal{O}}$:

$$\hat{A} = \frac{d}{dx} - \frac{l}{2\xi} - \frac{1}{2} - \frac{2}{2\xi + k}. \tag{39}$$

By taking $\xi = \frac{1}{2}x^2$ one obtains

$$\boldsymbol{W}(x) = -\frac{l}{x} - \frac{x}{2} - \frac{2x}{x^2 + k}, \tag{40}$$

then one obtains

$$\boldsymbol{W}^2(x) + \boldsymbol{W}'(x) = V_l^+(x) = \frac{1}{2}x^2 + \frac{l(l+1)}{x^2} - E$$

and

$$\boldsymbol{W}^2(x) - \boldsymbol{W}'(x) = V^-(x) = V_{l-1}^+(x) + V_e(x).$$

In our case of the harmonic oscillator Wigner function $m = 0$, then the potential will be

$$V_e(x,m) = -\frac{1}{x} \tag{41}$$

and differential eq. (27) will become

$$-xf''(x)+[(x+1)f'(x)-f(x)]=(n-1)f(x). \quad (42)$$

The solution to the differential eq. (42) is given in terms of the Laguerre polynomials $\{L_n^{-2}\}_{n=2}^{\infty}$. It has been proved in [29] that the differential equation of the form (42) is an exceptional differential equation for the special case $m = 0$ and their solutions are given in terms of the Laguerre polynomials $\{L_n^{-2}\}_{n=2}^{\infty}$ as eigenfunctions form a complete orthogonal set in the Hilbert space.

3. Conclusion

In this paper, we construct isospectral Hamiltonians without shape invariant potentials for the harmonic oscillator Wigner function. In this case, we actually removed the ground state. We have also shown that the solutions of the second Hamiltonian are also the Laguerre polynomials and the Laguerre differential equation forms a special case, $m = 0$, of the exceptional Laguerre differential equation. We expect this to have application in the field of quantum optics.

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