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CONTRIBUTION TO THE GENERAL LINEAR CONJUGATION PROBLEM FOR A PIECEWISE ANALYTIC VECTOR

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Abstract: Establishing an analogy between the theories of Riemann–Hilbert vector problem and linear ODEs, for the n-dimensional homogeneous linear conjugation problem on a simple smooth closed contour Γ partitioning the complex plane into two domains D^+ and D^- we show that if we know $n-1$ particular solutions such that the determinant of the size $n-1$ matrix of their components omitting those with index k is nonvanishing on $D^+ \cup \Gamma$ and the determinant of the matrix of their components omitting those with index j is nonvanishing on $\Gamma \cup D^- \setminus \{\infty\}$, where $k, j = \overline{1, n}$, then the canonical system of solutions to the linear conjugation problem can be constructed in closed form.

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Take a simple smooth closed contour Γ partitioning the complex plane into two domains D^+ and D^-, with $0 \in D^+$ and $\infty \in D^-$, and consider a size n matrix function H-continuous on Γ

$$G(t) = \begin{pmatrix} g_{11}(t) & g_{12}(t) & \dots & g_{1n}(t) \\ g_{21}(t) & g_{22}(t) & \dots & g_{2n}(t) \\ \dots & \dots & \dots & \dots \\ g_{n1}(t) & g_{n2}(t) & \dots & g_{nn}(t) \end{pmatrix}, \quad \Delta(t) = \det G(t) \neq 0, \ t \in \Gamma. \tag{1}$$

The homogeneous linear conjugation problem for an n-dimensional vector, or the vector Riemann–Hilbert problem, consists in finding a piecewise analytic vector function $\mathbf{w}(z) = (w^1(z), w^2(z), \dots, w^n(z))$ of the specified order at infinity with the limit values $\mathbf{w}^{\pm}(t)$ H-continuous on Γ and subject to the condition

$$\mathbf{w}^+(t) = G(t)\mathbf{w}^-(t)$$

or, in scalar form, to the conditions

$$\begin{aligned} w^{1+}(t) &= g_{11}(t)w^{1-}(t) + g_{12}(t)w^{2-}(t) + \dots + g_{1n}(t)w^{n-}(t), \\ w^{2+}(t) &= g_{21}(t)w^{1-}(t) + g_{22}(t)w^{2-}(t) + \dots + g_{2n}(t)w^{n-}(t), \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ w^{n+}(t) &= g_{n1}(t)w^{1-}(t) + g_{n2}(t)w^{2-}(t) + \dots + g_{nn}(t)w^{n-}(t). \end{aligned} \tag{2}$$

The qualitative theory of problem (2) in the classes of Hölder functions of arbitrary dimension is presented in [1], and for wider classes of matrix functions, in [2]. However, there are few examples of matrix functions for which the solution to the problem can be expressed in closed form via Cauchy-type integrals and solutions to certain linear algebraic systems. One of these examples is the linear conjugation problem for meromorphic matrix functions [3]. A constructive algorithm for solving this problem is also proposed in [2]. An algorithm is presented in [4] for efficiently constructing a canonical system of solutions whenever n solutions to the linear conjugation problem are available such that the determinant consisting of their components has finitely many zeros in the corresponding domain. The author demonstrated in [5] the possibility of constructing in closed form a canonical system of solutions to the linear conjugation problem for a three-dimensional vector in the presence of just two particular solutions of dimension is less by 1 than the dimension of the problem. A constructive algorithm is presented in [6] for implementing a right Wiener–Hopf factorization on the real axis from the available $n-1$ solutions to the n-dimensional

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problem $G\Phi^+ = \Phi^-$, which is equivalent to constructing a left factorization (used here) for the transpose matrix function.

This article establishes an analogy between the theory of the n-dimensional Riemann–Hilbert problem and the theory of linear ordinary differential equations, according to which the Ostrogradskiĭ–Liouville formula, given $n-1$ linearly independent solutions to a homogeneous equation of order n, enables us to extend this collection of solutions to a fundamental system. Here we reduce the n-dimensional homogeneous Riemann–Hilbert problem in the presence of $n-1$ particular solutions satisfying certain requirements to an $(n-1)$-dimensional problem with a matrix function analytically continuable into the domain D^+, while the relations (6), (9), and (10) provide an analog of the Ostrogradskiĭ–Liouville formula; they can be combined into one formula if we ignore the function of the canonical system of solutions they are written for and suppress the particular form of the undetermined polynomials involved.

Suppose that

$$\mathbf{w}_i(z) = \left(w_i^1(z), w_i^2(z), \dots, w_i^n(z)\right), \quad i = \overline{1, n-1}, \tag{3}$$

are solutions to (2) without finite poles and of orders k_i at infinity (a positive order means the order of a pole). Denote the required canonical system of solutions by

$$\mathbf{v}_i(z) = \left(v_i^1(z), v_i^2(z), \dots, v_i^n(z)\right), \quad i = \overline{1, n}, \tag{4}$$

where $\mathbf{v}_i(z)$ is of order $-\varkappa_i$ at infinity for $i = \overline{1,n}$. Arrange the integers $\varkappa_i$, the particular indices of the matrix function (1), in decreasing order $\varkappa_1 \ge \varkappa_2 \ge \dots \ge \varkappa_n$, while

$$\varkappa_1 + \varkappa_2 + \dots + \varkappa_n = \varkappa = \operatorname{ind} \det G(t)$$

is the total index of (1).

Denote by $X(z) = \left\| v_j^i(z) \right\|$, for $i, j = \overline{1,n}$, the canonical matrix whose columns consist of the components of the vector functions (4),

$$\Delta(z) = \det X(z) \quad (G(t) = X^+(t)[X^-(t)]^{-1},\ \Delta(t) = \Delta^+(t)/\Delta^-(t),\ t \in \Gamma).$$

Denote by $\Omega^k(z)$ the matrix function of size $n-1$ obtained from the rectangular matrix function $\Omega(z) = \left\| w_j^i(z) \right\|$, where $i = \overline{1,n}$ and $j = \overline{1, n-1}$, whose columns consist of the components of the vector functions (3) with the row k omitted.

Proposition. *Assume that for the solutions* (3) *to the linear conjugation problem* (2) *for some values of the indices k and s, where $k, s = \overline{1,n}$, the determinant of $\Omega^k(z)$ is nowhere vanishing in $D^+ \cup \Gamma$ and the determinant of $\Omega^s(z)$ is nowhere vanishing in $\Gamma \cup D^- \setminus \{\infty\}$. Then the canonical system of solutions to* (2) *can be constructed in closed form.*

Proof. Suppose that the solutions (3) expand in the functions of the canonical system (4) as

$$\mathbf{w}_i(z) = \sum_{j=1}^{n} p_i^j(z) \mathbf{v}_j(z), \quad i = \overline{1, n-1}, \tag{5}$$

where $p_i^j(z)$ for $i = \overline{1, n-1}$ and $j = \overline{1,n}$ are polynomials. Denote by $\Omega_1(z)$ the size n matrix function obtained by adjoining to $\Omega(z)$ the first column consisting of the components of the vector function $\mathbf{v}_1(z) = \left(v_1^1(z), v_1^2(z), \dots, v_1^n(z)\right)$. Put $\Delta_1(z) = \det \Omega_1(z)$ and let $\Delta_k^1(z)$ be the algebraic complement of the entry $v_1^k(z)$ of $\Omega_1(z)$, i.e., $\Delta_k^1(z) = (-1)^{k+1} \det \Omega^k(z)$. Then

$$\Delta_1(z) = \sum_{k=1}^{n} v_1^k(z) \Delta_k^1(z).$$

On the other hand, (5) yields

$$\Omega_1(z)=\begin{pmatrix} v_1^1(z) & \sum\limits_{j=1}^{n} p_1^j(z)v_j^1(z) & \dots & \sum\limits_{j=1}^{n} p_{n-1}^j(z)v_j^1(z)\\ v_1^2(z) & \sum\limits_{j=1}^{n} p_1^j(z)v_j^2(z) & \dots & \sum\limits_{j=1}^{n} p_{n-1}^j(z)v_j^2(z)\\ \dots & \dots & \dots & \dots\\ v_1^n(z) & \sum\limits_{j=1}^{n} p_1^j(z)v_j^n(z) & \dots & \sum\limits_{j=1}^{n} p_{n-1}^j(z)v_j^n(z) \end{pmatrix}$$

$$=\begin{pmatrix} v_1^1(z) & v_2^1(z) & \dots & v_n^1(z)\\ v_1^2(z) & v_2^2(z) & \dots & v_n^2(z)\\ \dots & \dots & \dots & \dots\\ v_1^n(z) & v_2^n(z) & \dots & v_n^n(z) \end{pmatrix}\begin{pmatrix} 1 & p_1^1(z) & \dots & p_{n-1}^1(z)\\ 0 & p_1^2(z) & \dots & p_{n-1}^2(z)\\ \dots & \dots & \dots & \dots\\ 0 & p_1^n(z) & \dots & p_{n-1}^n(z) \end{pmatrix}=X(z)P_1(z),$$

where $P_1(z)$ stands for the polynomial matrix in this expansion. Hence, $\Delta_1(z)=\Delta(z)p_1(z)$, where $p_1(z)$ is the algebraic complement of the entry at the intersection of the first row and first column of $P_1(z)$, which enables us to express the relation between the components of the first vector function $\mathbf{v}_1(z)$ of the canonical system (4) and those of the vector functions (3):

$$p_1(z)\Delta(z)=\sum_{k=1}^{n} v_1^k(z)\Delta_k^1(z). \tag{6}$$

On Γ we have

$$p_1(t)\Delta^+(t)=\sum_{k=1}^{n} v_1^{k+}(t)\Delta_k^{1+}(t), \tag{7}$$

$$p_1(t)\Delta^-(t)=\sum_{k=1}^{n} v_1^{k-}(t)\Delta_k^{1-}(t). \tag{8}$$

Consider the matrix function $\Omega_2(z)$ obtained from $\Omega_1(z)$ by replacing its first column with the components of the second vector function $\mathbf{v}_2(z)$ of the canonical system (4). Then

$$\Delta_2(z)=\det\Omega_2(z)=\sum_{k=1}^{n} v_2^k(z)\Delta_k^1(z).$$

It is not difficult to verify that $\Omega_2(z)=X(z)P_2(z)$, where the polynomial matrix $P_2(z)$ is of the form

$$\begin{pmatrix} 0 & p_1^1(z) & \dots & p_{n-1}^1(z)\\ 1 & p_1^2(z) & \dots & p_{n-1}^2(z)\\ \dots & \dots & \dots & \dots\\ 0 & p_1^n(z) & \dots & p_{n-1}^n(z) \end{pmatrix}.$$

Denoting by $p_2(z)$ the algebraic complement of the entry at the intersection of the second row and first column of $P_2(z)$, we arrive at the relation

$$p_2(z)\Delta(z)=\sum_{k=1}^{n} v_2^k(z)\Delta_k^1(z) \tag{9}$$

between the components of the second vector function $\mathbf{v}_2(z)$ of the canonical system (4) and those of the vector functions (3).

Finally, considering the matrix function $\Omega_n(z)$ obtained from $\Omega_1(z)$ by replacing its first column with the components of the vector function $\mathbf{v}_n(z)$ of the canonical system (4) and denoting its determinant by $\Delta_n(z)$, we arrive at

$$p_n(z)\Delta(z)=\sum_{k=1}^{n} v_n^k(z)\Delta_k^1(z), \tag{10}$$

in which $p_n(z)$ stands for the algebraic complement of the entry at the intersection of the last row and first column of the matrix

$$P_n(z) = \begin{pmatrix} 0 & p_1^1(z) & \dots & p_{n-1}^1(z) \\ 0 & p_1^2(z) & \dots & p_{n-1}^2(z) \\ \dots & \dots & \dots & \dots \\ 1 & p_1^n(z) & \dots & p_{n-1}^n(z) \end{pmatrix}.$$

These expressions imply in particular that if one of the vector functions, for instance $\mathbf{v}_k(z)$, of the canonical system (4) is a linear combination of the vector functions (3) with polynomial coefficients then $\Delta_k(z) \equiv 0$ and $p_k(z) \equiv 0$.

Proceed to constructing the first vector function $\mathbf{v}_1(z)$ of the canonical system (4). Observe firstly that it suffices to consider the case of the solutions (3) satisfying

$$\Delta_n^{1+}(z) \neq 0,\ z \in D^+ \cup \Gamma, \quad \Delta_n^{1-}(z) \neq 0,\ z \in \Gamma \cup D^- \setminus \{\infty\}. \tag{11}$$

We can always reduce to this case by permuting the boundary conditions (2) and relabeling the components of the required vector function $\mathbf{w}(z)$, which amounts to multiplying the matrix function (1) on the left and/or right with size n permutation matrices.

Eliminating $v_1^{n-}(t)$ from the first $n-1$ boundary conditions (2) expressed for $\mathbf{v}_1(z)$ using (8), we arrive on Γ at the linear conjugation problem

$$v_1^{1+} = \frac{g_{11}\Delta_n^{1-} - g_{1n}\Delta_1^{1-}}{\Delta_n^{1-}} v_1^{1-} + \dots + \frac{g_{1(n-1)}\Delta_n^{1-} - g_{1n}\Delta_{n-1}^{1-}}{\Delta_n^{1-}} v_1^{(n-1)-} + \frac{g_{1n}p_1\Delta^-}{\Delta_n^{1-}}, \dots,$$

$$v_1^{(n-1)+} = \frac{g_{(n-1)1}\Delta_n^{1-} - g_{(n-1)n}\Delta_1^{1-}}{\Delta_n^{1-}} v_1^{1-} + \dots + \frac{g_{(n-1)(n-1)}\Delta_n^{1-} - g_{(n-1)n}\Delta_{n-1}^{1-}}{\Delta_n^{1-}} v_1^{(n-1)-} + \frac{g_{(n-1)n}p_1\Delta^-}{\Delta_n^{1-}}.$$

Denote by $\omega_{ij}^k(z)$ the algebraic complement of the entry of $\Omega^k(z)$ at the intersection of the row i and the column j for $i, j = \overline{1, n-1}$. Consider the difference

$$g_{11}\Delta_n^{1-} - g_{1n}\Delta_1^{1-} = (-1)^{n+1} g_{11} \sum_{j=1}^{n-1} w_j^{1-}\omega_{1j}^{n-} - (-1)^2 g_{1n} \sum_{j=1}^{n-1} w_j^{n-}\omega_{(n-1)j}^{1-}.$$

It is not difficult to verify the validity of the equalities $\omega_{(n-1)k}^{1-}(t) = (-1)^{n-2}\omega_{1k}^{n-}(t)$ for $k = \overline{1, n-1}$, according to which this difference becomes

$$g_{11}\Delta_n^{1-} - g_{1n}\Delta_1^{1-} = (-1)^{n+1} \sum_{j=1}^{n-1} \big(g_{11}w_j^{1-} + g_{1n}w_j^{n-}\big)\omega_{1j}^{n-}.$$

Accounting for the boundary conditions (2) expressed for the vector functions (3), we obtain

$$g_{11}\Delta_n^{1-} - g_{1n}\Delta_1^{1-} = (-1)^{n+1} \sum_{j=1}^{n-1} \big(w_j^{1+} - g_{12}w_j^{2-} - \dots - g_{1(n-1)}w_j^{(n-1)-}\big)\omega_{1j}^{n-}.$$

Obviously,

$$\sum_{j=1}^{n-1} w_j^{k-}(t)\omega_{1j}^{n-}(t) \equiv 0, \quad k = \overline{2, n-1};$$

here the entries in the row k of $\Omega^{n-}(t)$, for $k = \overline{2, n-1}$, are multiplied by the algebraic complements of the entries in the first row. Therefore, we finally arrive at

$$g_{11}\Delta_n^{1-} - g_{1n}\Delta_1^{1-} = (-1)^{n+1} \sum_{j=1}^{n-1} w_j^{1+}\omega_{1j}^{n-}.$$

For the differences

$$g_{jk}(t)\Delta_n^{1-}(t) - g_{jn}(t)\Delta_j^{1-}(t), \quad j,k = \overline{1,n-1}\ (j \neq 1,\ k \neq 1),$$

expanding along the row j the minor corresponding to the algebraic complement of $\Delta_n^{1-}(t)$ and using the equalities

$$\omega_{(n-1)k}^{j-}(t) = (-1)^{n-j-1}\omega_{jk}^{n-}(t), \quad j,k = \overline{1,n-1},$$

upon the transformations similar to those with $j = k = 1$ for the first $n-1$ components of the vector function $\mathbf{v}_1(z)$ of the canonical system (4), we obtain on Γ the linear conjugation problem

$$v_1^{1+} = (-1)^{n+1}\sum_{j=1}^{n-1}\frac{w_j^{1+}\omega_{1j}^{n-}}{\Delta_n^{1-}}v_1^{1-} + \cdots + (-1)^{n+1}\sum_{j=1}^{n-1}\frac{w_j^{1+}\omega_{(n-1)j}^{n-}}{\Delta_n^{1-}}v_1^{(n-1)-} + \frac{g_{1n}p_1\Delta^-}{\Delta_n^{1-}}, \ldots,$$

$$v_1^{(n-1)+} = (-1)^{n+1}\sum_{j=1}^{n-1}\frac{w_j^{(n-1)+}\omega_{1j}^{n-}}{\Delta_n^{1-}}v_1^{1-} + \cdots + (-1)^{n+1}\sum_{j=1}^{n-1}\frac{w_j^{(n-1)+}\omega_{(n-1)j}^{n-}}{\Delta_n^{1-}}v_1^{(n-1)-} + \frac{g_{(n-1)n}p_1\Delta^-}{\Delta_n^{1-}}.$$

Introduce the new unknown vector functions

$$\mathbf{W}(z) = (W^1(z), W^2(z), \ldots, W^{(n-1)}(z))$$

by using the equalities

$$\mathbf{W}^+(z) = \mathbf{V}^+(z), \quad z \in D^+, \tag{12}$$

$$\mathbf{W}^-(z) = [\Omega^{n-}(z)]^{-1}\mathbf{V}^-(z), \quad z \in D^-, \tag{13}$$

in which $\mathbf{V}(z)$ stands for

$$\mathbf{V}(z) = \big(v_1^1(z), v_1^2(z), \ldots, v_1^{(n-1)}(z)\big), \tag{14}$$

while the components of (13) are of the form

$$\begin{gathered}
W^{1-}(z) = (-1)^{n+1}\sum_{j=1}^{n-1}\frac{\omega_{j1}^{n-}(z)v_1^{j-}(z)}{\Delta_n^{1-}(z)}, \\
W^{2-}(z) = (-1)^{n+1}\sum_{j=1}^{n-1}\frac{\omega_{j2}^{n-}(z)v_1^{j-}(z)}{\Delta_n^{1-}(z)}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
W^{(n-1)-}(z) = (-1)^{n+1}\sum_{j=1}^{n-1}\frac{\omega_{j(n-1)}^{n-}(z)v_1^{j-}(z)}{\Delta_n^{1-}(z)}, \quad z \in D^-.
\end{gathered} \tag{15}$$

Then for the limit values of $\mathbf{W}(z)$ on Γ we obtain the linear conjugation problem

$$\mathbf{W}^+(t) = \Omega^{n+}(t)\mathbf{W}^-(t) + \mathbf{f}(t), \quad t \in \Gamma, \tag{16}$$

with the matrix function

$$\Omega^{n+}(t) = \begin{pmatrix} w_1^{1+}(t) & w_2^{1+}(t) & \ldots & w_{(n-1)}^{1+}(t) \\ w_1^{2+}(t) & w_2^{2+}(t) & \ldots & w_{(n-1)}^{2+}(t) \\ \ldots & \ldots & \ldots & \ldots \\ w_1^{(n-1)+}(t) & w_2^{(n-1)+}(t) & \ldots & w_{(n-1)}^{(n-1)+}(t) \end{pmatrix}$$

analytic on D^+ with the determinant

$$\det\Omega^{n+}(t) = (-1)^{n+1}\Delta_n^{1+}(t) \neq 0, \quad t \in \Gamma,$$

and the vector function

$$\mathbf{f}(t) = \left(\frac{g_{1n}p_1\Delta^-}{\Delta_n^{1-}}, \frac{g_{2n}p_1\Delta^-}{\Delta_n^{1-}}, \dots, \frac{g_{(n-1)n}p_1\Delta^-}{\Delta_n^{1-}}\right).$$

Since the vector function $\mathbf{v}_1(z)$ of the canonical system (4) must be of the least possible order at infinity, it follows that $-\varkappa_1 \le \min(k_1, k_2, \dots, k_{n-1})$. Thus, according to (15), in general we should seek a solution to (16) in the class of functions bounded at infinity. Putting on Γ

$$\mathbf{W}_1^+(t) = [\Omega^{n+}(t)]^{-1}\mathbf{W}^+(t), \quad \mathbf{W}_1^-(t) = \mathbf{W}^-(t), \tag{17}$$

by (11) we arrive at the "jump problem" for finding a piecewise analytic bounded at infinity vector function $\mathbf{W}_1(z)$:

$$\mathbf{W}_1^+(t) = \mathbf{W}_1^-(t) + \mathbf{g}(t), \quad t \in \Gamma, \tag{18}$$

with the H-continuous vector function

$$\mathbf{g}(t) = [\Omega^{n+}(t)]^{-1}\mathbf{f}(t) = (-1)^{n+1}\frac{p_1(t)\Delta^-(t)}{\Delta_n^{1+}(t)\Delta_n^{1-}(t)}\left(\sum_{j=1}^{n-1} g_{jn}(t)\omega_{j1}^{n+}(t), \dots, \sum_{j=1}^{n-1} g_{jn}(t)\omega_{j(n-1)}^{n+}(t)\right).$$

Then on Γ we have

$$\mathbf{W}_1^+(t) = P[\mathbf{g}(t)] - \mathbf{c}, \quad \mathbf{W}_1^-(t) = -Q[\mathbf{g}(t)] + \mathbf{c}, \tag{19}$$

where $P = (I + S)/2$ and $Q = (I - S)/2$ with the identity operator I and a singular operator S, while $\mathbf{c} = (c^1, \dots, c^n)$ is a constant vector. Thus, for the limit values on Γ of the first $n-1$ components of the canonical system (4) we obtain

$$\mathbf{V}^+(t) = \Omega^{n+}(t)(P[\mathbf{g}(t)] - \mathbf{c}), \quad \mathbf{V}^-(t) = \Omega^{n-}(t)(-Q[\mathbf{g}(t)] + \mathbf{c}). \tag{20}$$

Expressions for the last component $v_1^n(z)$ of (4) follow from (7) and (8).

Inserting the found expression for $\mathbf{v}_1(z)$ into the boundary conditions (2), we see that its components are solutions to the problem for every polynomial $p_1(z)$. Hence, in order to find the first vector function $\mathbf{v}_1(z)$ of the canonical system of solutions, we should pick the undetermined coefficients of the polynomial $p_1(z)$ and the components of the constant vector $\mathbf{c}$ so that the vector function with components determined from (20), (7), and (8) be of the least possible order $-\varkappa_1$ at infinity. If this order is attained for $p_1(z) \equiv 0$ then the first vector function of the canonical system (4) coincides with one of the specified solutions or their linear combination with constant coefficients.

The degree s_1 of $p_1(z)$ can be estimated from above. Indeed, comparing the orders at infinity of both sides of (8) yields

$$s_1 - \varkappa \le \sum_{i=1}^{n-1} k_i - \varkappa_1 \le \sum_{i=1}^{n-1} k_i + \min(k_1, k_2, \dots, k_{n-1}).$$

Hence,

$$s_1 \le \sum_{i=1}^{n-1} k_i + \varkappa + \min(k_1, k_2, \dots, k_{n-1}). \tag{21}$$

In order to represent the second vector function $\mathbf{v}_2(z)$ of the canonical system (4), we should apply (20), (7), and (8) with $p_1(z)$ replaced by $p_2(z)$ and the components of the constant vector $\mathbf{c}$ assumed to be polynomials of degree at most l such that the order of the vector function determined from (20), (7), and (8), equal to $\max(k_1 + l, k_2 + l, \dots, k_{n-1} + l)$, be at most

$$-\varkappa_2 = -\varkappa + \varkappa_1 + \varkappa_3 + \dots + \varkappa_n \le -\varkappa + (n-1)\varkappa_1. \tag{22}$$

For the degree s_2 of $p_2(z)$ we obtain from (9) and (22) that

$$s_2 - \varkappa \le \sum_{i=1}^{n-1} k_i - \varkappa_2 \le \sum_{i=1}^{n-1} k_i - \varkappa + (n-1)\varkappa_1$$

and so

$$s_2 \le \sum_{i=1}^{n-1} k_i + (n-1)\varkappa_1.$$

We should pick the polynomials $p_2(z)$ and $c^i(z)$, for $i = \overline{1,n}$, yet undetermined, so that $\mathbf{v}_2(z)$ be of order $-\varkappa_2 \ge -\varkappa_1$ at infinity and differ from the already constructed vector function $\mathbf{v}_1(z)$ multiplied by any polynomial.

We construct the remaining vector functions of the canonical system of solutions (4) similarly. Say, we seek the vector function $\mathbf{v}_n(z)$ from (20), (7), and (8) as of order $-\varkappa_n = -\varkappa + \varkappa_1 + \dots + \varkappa_{n-1}$ at infinity that differs from the linear combinations of the previously constructed vector functions $\mathbf{v}_1(z), \mathbf{v}_2(z), \dots, \mathbf{v}_{n-1}(z)$ with polynomial coefficients.

The proof is complete. □

Some example realizing the presented scheme for constructing a canonical system of solutions for the case $n = 3$ appears in [5].

Remark. The reason for the restrictions in (11) is mostly technical and related to the method for solving (16), (12)–(14) and obtaining simpler formulas in (20). The proposed method for constructing a canonical system of solutions can be extended to the case of vanishing $\Delta_n^{1\pm}(z)$ in the corresponding domains with the use of [3], while to the case of finitely many points on the contour, with the use of [7].

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