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# ON THE CHARACTERIZATION OF THE CORE OF A $\pi$-PREFRATTINI SUBGROUP OF A FINITE SOLUBLE GROUP

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**Abstract:** Let $\pi$ be a set of primes and let $H$ be a $\pi$-prefrattini subgroup of a finite soluble group $G$. We prove that there exist elements $x, y, z \in G$ such that $H \cap H^x \cap H^y \cap H^z = \Phi_\pi(G)$.

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## 1. Introduction

It was shown in [1] that if $\mathfrak{F}$ is a saturated formation, $H$ is a $\mathfrak{F}$-prefrattini subgroup of a finite soluble group $G$, and $\Delta_{\mathfrak{F}}(G)$ is the intersection of all $\mathfrak{F}$-abnormal maximal subgroups of $G$ then

(1) $H \cap H^x \cap H^y \cap H^z = \Delta_{\mathfrak{F}}(G)$ for some $x$, $y$, and $z$ in $G$;

(2) if either $G$ is $S_4$-free or the formation $\mathfrak{F}$ is composed of $S_3$-free groups then $H \cap H^x \cap H^y = \Delta_{\mathfrak{F}}(G)$ for some $x$ and $y$ in $G$.

Partial aspects of this result were considered in [2] ($\mathfrak{F}$ is the formation of identity groups, and $\Delta_{\mathfrak{F}}(G) = \Phi(G)$ is the Frattini subgroup of $G$) and [3] ($\mathfrak{F}$ is the formation of all nilpotent groups, $\Delta_{\mathfrak{F}}(G) = \Delta(G)$ is the Gaschütz subgroup of $G$, i.e., the intersection of all abnormal maximal subgroups of $G$).

Since $\Delta_{\mathfrak{F}}(G) = \mathrm{Core}_G(H) = \bigcap_{x \in G} H^x$ for every $\mathfrak{F}$-prefrattini subgroup $H$ of $G$, the thing is in essence about the possibility of representing of the core of a $\mathfrak{F}$-prefrattini subgroup as the intersection of a bounded number (three or four) conjugate subgroups.

Note that the statement of such a problem is initiated by the central results of [4–7]. In [4], Passman proved that, in every $p$-soluble group $G$, there exist three Sylow $p$-subgroup whose intersection is equal to $O_p(G)$. In [5], Zenkov showed that an analogous result holds in every finite group. In [6], Dolfi proved that if $2 \notin \pi$ then, in every $\pi$-soluble group $G$, there exists three Hall $\pi$-subgroups whose intersection is equal to $O_\pi(G)$. Later in [7], Dolfi established that if $H$ is a Hall $\pi$-subgroup of a $\pi$-soluble group $G$ then $H \cap H^x \cap H^y = O_\pi(G)$ for some elements $x$ and $y$ in $G$ (see also [8]).

In this article, we extend the above result on $\mathfrak{F}$-prefrattini subgroups of a finite soluble group $G$ to its $\pi$-prefrattini subgroups. Observe that, in general, the set of all $\pi$-prefrattini subgroups of a group $G$ does not coincide with the set of all its $\mathfrak{F}$-prefrattini subgroups (this is shown in Section 5).

Let $\pi$ be a set of primes and let $\Phi_\pi(G)$ be the intersection of all maximal subgroups of the group $G$ whose indices do not divide by the numbers of $\pi$. Our main aim is to prove the following

**Theorem.** *Let $\pi$ be a set of primes. If $H$ is a $\pi$-prefrattini subgroup of a finite soluble group $G$ then*

(1) *$H \cap H^x \cap H^y \cap H^z = \Phi_\pi(G)$ for some $x$, $y$, and $z$ in $G$;*

(2) *if the group $G$ is $S_4$-free then $H \cap H^x \cap H^y = \Phi_\pi(G)$ for some $x, y \in G$;*

(3) *if $2 \notin \pi$ and $3 \notin \pi$ then $H \cap H^x \cap H^y = \Phi_\pi(G)$ for some $x, y \in G$.*

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## 2. Preliminary Results

Below in the article, by a group we always mean a finite soluble group. We use the definitions and notations of [9].

The concept of prefrattini subgroup was proposed by Gaschütz in 1962 in [10]. In the original exposition, a prefrattini subgroup is defined as the intersection of the complements to the crowns of all complemented chief factors of some fixed chief series of the group. Later such an approach was widely studied and repeatedly generalized. It was most impressively developed in the article [11] by Hawkes, who, given a saturated formation $\mathfrak{F}$, introduced the notion of $\mathfrak{F}$-prefrattini subgroup by considering complements to the crowns not of all complemented chief factors but only of those that are $\mathfrak{F}$-excentral.

Note that approaches are known (see, for example, [12–14]) that do not use the notion of crown of a complemented chief factor. One of such approaches, which regards a generalized prefrattini subgroup of a group $G$ as the intersection of some of its maximal subgroups is used in our definition of $\pi$-prefrattini subgroup.

Definition 2.1. Suppose that $\pi$ is a set of primes, while $1 = A_0 \subset A_1 \subset \cdots \subset A_n = G$ is a chief series of a group $G$, and $\{A_i/A_{i-1} \mid i \in I\}$ is the set of all complemented $\pi'$-factors of this series. Let $M_i$ ($i \in I$) be a maximal subgroup of $G$ complementing the chief factor $A_i/A_{i-1}$. Then the subgroup $\bigcap_{i\in I} M_i$ is called a *$\pi$-prefrattini* subgroup of $G$ (if $G$ has no complemented chief $\pi'$-factors then the group $G$ itself is regarded as its $\pi$-prefrattini subgroup).

A check shows that the definition of $\pi$-prefrattini subgroup is correct: It does not depend on the choice of a chief series of the group. The definition also implies that a $\pi$-prefrattini subgroup exists in every group.

Further we will need information on the properties of $\Phi_\pi(G)$. The following lemma is proved by an easy check.

**Lemma 2.1.** *The following hold for every group $G$ and every set $\pi$ of primes:*
(1) $O_\pi(G) \subseteq \Phi_\pi(G)$ *and* $\Phi_\pi(G)/O_\pi(G) = \Phi(G/O_\pi(G))$;
(2) *if* $N \lhd G$ *then* $\Phi_\pi(G)N/N \subseteq \Phi_\pi(G/N)$;
(3) *if* $N \lhd G$ *and* $N \subseteq \Phi_\pi(G)$ *then* $\Phi_\pi(G)N/N = \Phi_\pi(G/N)$;
(4) $\Phi_\pi(G/\Phi_\pi(G)) = 1$.

The main properties of $\pi$-prefrattini subgroups will be given in the form of lemmas. Recall only that if $H$ is a subgroup of $G$ and $A/B$ is its normal section then it is said that
—$H$ *covers* $A/B$ if $HB \supseteq A$;
—$H$ *avoids* $A/B$ if $H \cap A \subseteq B$.

**Lemma 2.2** [15, Theorem 2]. *Let $H$ be a $\pi$-prefrattini subgroup of a group $G$. Then*
(1) *if* $N \lhd G$ *then* $HN/N$ *is a $\pi$-prefrattini subgroup of* $G/N$;
(2) $H$ *covers all chief $\pi$-factors and all Frattini chief factors of* $G$;
(3) $H$ *avoids all complemented chief $\pi'$-factors of* $G$;
(4) $\mathrm{Core}_G(H) = \Phi_\pi(G)$;
(5) *every two $\pi$-prefrattini subgroups of $G$ are conjugate.*

**Lemma 2.3.** *Let $N$ be a minimal normal $\pi'$-subgroup of $G$. If $M$ is a maximal subgroup of $G$ that complements $N$ then every $\pi$-prefrattini subgroup in $M$ is a $\pi$-prefrattini subgroup of $G$.*

Proof. Let $H$ be a $\pi$-prefrattini subgroup in $M$. Let $1 = A_0 \subset A_1 \subset \cdots \subset A_n = M$ be a chief factor of $M$ and let $\{A_i/A_{i-1} \mid i \in I\}$ be the set of all complemented chief $\pi'$-factors of this series. By Definition 2.1, $H = \bigcap_{i\in I} M_i$, where $M_i$ ($i \in I$) is a maximal subgroup in $M$ that complements the chief $\pi'$-factor of $A_i/A_{i-1}$.

Consider the normal series

$$1 \subset N = A_0N \subset A_1N \subset \cdots \subset A_nN = MN = G \tag{1}$$

of $G$. Since

$$A_jN/A_{j-1}N \cong A_j/A_j \cap A_{j-1}N = A_j/A_{j-1}(A_j \cap N) = A_j/A_{j-1}$$

for each $j \in \{1, 2, \ldots, n\}$, this series is a chief series of $G$. Since the mapping $\alpha : mN \mapsto m$ is an isomorphism of the groups $G/N$ and $M$, the factor $A_jN/A_{j-1}N$ is a $\pi'$-factor if and only if the factor $A_j/A_{j-1}$ is a $\pi'$-factor. Therefore, in series (1), the only $\pi'$-factors are the factors $N$ and $A_jN/A_{j-1}N$ for all $j \in I$; moreover, all these factors are complemented: the subgroup $N$ is complemented by the subgroup $M$ by hypothesis, and the complement to the chief factor $A_jN/A_{j-1}N$ ($j \in I$) is the subgroup $M_jN$.

This implies that $M \cap \left(\bigcap_{i\in I} M_iN\right)$ is a $\pi$-prefrattini subgroup of $G$. Obviously,

$$H = \bigcap_{i\in I} M_i \subseteq M \cap \left(\bigcap_{i\in I} M_iN\right).$$

Since by Lemma 2.2 the subgroups $H$ and $M \cap \left(\bigcap_{i\in I} M_iN\right)$ avoid all complemented chief $\pi'$-factors of series (1) and cover all its remaining chief factors, by Lemma A.1.7 in [9], we have $|H| = |M \cap (\bigcap_{i\in I} M_iN)|$. This and $H \subseteq M \cap (\bigcap_{i\in I} M_iN)$ imply that $H = M \cap (\bigcap_{i\in I} M_iN)$, i.e., $H$ is a $\pi$-prefrattini subgroup of $G$. The lemma is proved.

**Lemma 2.4.** *Let $N$ be a normal $\pi'$-subgroup of a group $G$ in* $\mathrm{Soc}(G)$. *If $H$ is a subgroup of $G$ such that $G = HN$ and $H \cap N = 1$ then $\Phi_\pi(G) = C_{\Phi_\pi(H)}(N)$.*

PROOF. Let $T$ be the intersection of all maximal subgroups of $G$ that include $N$ and whose indices do not divide by the numbers of $\pi$. Then $\Phi_\pi(G) \subseteq T$ and $T/N = \Phi_\pi(G/N)$. By the natural isomorphism $H \cong HN/N = G/N$, we have $\Phi_\pi(G/N) = \Phi_\pi(H)N/N$. Now, $\Phi_\pi(G)N/N \subseteq \Phi_\pi(G/N)$ implies the inclusion $\Phi_\pi(G) \subseteq \Phi_\pi(H)N$.

Let $W$ be a minimal normal subgroup of $G$ contained in $N$. Then $N = W \times W^*$, where $W^*$ is a normal subgroup of $G$. Since the subgroup $W$ is complemented in $G$ by the maximal subgroup $W^*H$ whose index does not divide by the numbers of $\pi$, $\Phi_\pi(G) \cap N = 1$. Consequently,

$$\Phi_\pi(G) \subseteq C_G(N) \cap \Phi_\pi(H)N = (C_G(N) \cap \Phi_\pi(H))N = C_{\Phi_\pi(H)}(N) \times N.$$

Let $K = C_{\Phi_\pi(H)}(N)$. Clearly, $N_G(K) \supseteq \langle H, N\rangle = G$, i.e., $K$ is a normal subgroup of $G$. Suppose that there exists a maximal subgroup $M$ in $G$ not including $K$ whose index does not divide by the numbers of $\pi$. In this case, $KM = G$. If $N \subseteq M$ then $M/N$ is a maximal subgroup in $G/N$, and hence $M \cap H$ is a maximal subgroup in $H$. Moreover, the index $|H : M \cap H| = |G : M|$ does not divide by the numbers of $\pi$. But then $K \subseteq \Phi_\pi(H) \subseteq M \cap H \subseteq M$, which contradicts the assumption.

Thus, $N$ does not lie in $M$. Let $D = M \cap N$. Since $N \subseteq \mathrm{Soc}(G)$, there exists a minimal normal subgroup $V$ in $G$ such that $N = V \times D$. Hence, $M$ and $DH$ are maximal subgroups of $G$ complementing $V$. The subgroups $M$ and $DH$ are not conjugate in $G$ since $DH$ has a normal subgroup $K$ and $M$ does not include $K$ by assumption. Then, by Proposition A.16.9 in [9], $M \cap DH$ is a maximal subgroup in $DH$. Since $M \cap DH = (M \cap H)D$, the subgroup $M \cap H$ is maximal in $H$. Since $MK = G$ and $K \subseteq H$, we have $MH = G$. Consequently, $|G| = |MH| = |M||H|/|M \cap H|$, and hence $|G : M| = |G|/|M| = |H : H \cap M|$ does not divide by the numbers of $\pi$; therefore, $K \subseteq \Phi_\pi(H) \subseteq M \cap H \subseteq M$. We again have a contradiction.

Thus, every maximal subgroup of $G$ whose index does not divide by the numbers of $\pi$ includes $K$, and so $K \subseteq \Phi_\pi(G)$. We finally have $\Phi_\pi(G) = \Phi_\pi(G) \cap NK = (\Phi_\pi(G) \cap N)K = K$. The lemma is proved.

We will rely upon the following results, which we give as lemmas:

**Lemma 2.5** [7, Theorem 1.4]. *Let $G$ be a soluble group and let $V$ be a faithful finite $G$-module. If $V$ is completely reducible then there exist elements $v_1, v_2, v_3 \in V$ for which $C_G(v_1) \cap C_G(v_2) \cap C_G(v_3) = 1$.*

**Lemma 2.6** [9, Lemma A.16.3]. *Let $G = NH$ be the semidirect product of a normal subgroup $N$ and a subgroup $H$. Then*

(1) *if $n \in N$ then $H \cap H^n = C_H(n)$;*

(2) $\mathrm{Core}_G(H) = C_H(N)$.

**Lemma 2.7** [1, Lemma 21]. *Let $G$ be a soluble group and let $V$ be a finite faithful completely reducible $G$-module. Let $H$ be a subgroup of $G$ such that the semidirect product $VH$ is $S_4$-free. Then $H$ has at least two regular orbits on $V \oplus V$.*

Recall that a group is called *$S_4$-free* if it has no sections isomorphic to the symmetric group of degree 4.

## 3. A Minimal Counterexample

Let $H$ and $K$ be subgroups of $G$, where $K = \mathrm{Core}_G(H)$. Following [1], we say that the triple $(G, H, K)$ is a *$k$-conjugate system* if there exist elements $g_1, g_2, \ldots, g_k \in G$ such that $K = H^{g_1} \cap H^{g_2} \cap \cdots \cap H^{g_k}$.

Let $\pi$ be a set of primes and let $H$ be a $\pi$-prefrattini subgroup of a group $G$. Assume from now on that $G$ is a group of the least order for which the triple $(G, H, \Phi_\pi(G))$ is not a $k$-conjugate system. Then

**3.1.** $\Phi_\pi(G) = 1$. *In particular,* $\Phi(G) = 1$.

Assume first that $\Phi_\pi(G) \neq 1$. Then $H\Phi_\pi(G)/\Phi_\pi(G)$ is a $\pi$-prefrattini subgroup in the group $G/\Phi_\pi(G)$ in view of Lemma 2.2. Since $|G/\Phi_\pi(G)| < |G|$, $(G/\Phi_\pi(G), H\Phi_\pi(G)/\Phi_\pi(G), \Phi_\pi(G/\Phi_\pi(G))$ is a $k$-conjugate system. Moreover, the properties of the subgroup $\Phi_\pi(G) \neq 1$ (Lemma 2.1) imply that $\Phi_\pi(G/\Phi_\pi(G)) = 1$. Therefore, the triple $(G, H, \Phi_\pi(G))$ is a $k$-conjugate system. We get a contradiction. Consequently, assertion 3.1 holds.

**3.2.** *There exist a minimal normal subgroup $N$ and a maximal normal subgroup $M$ such that $G = MN$, $M \cap N = 1$, and the triple $(M, H, \Phi_\pi(M))$ is a $k$-conjugate system.*

Let $N$ be a minimal normal subgroup of $G$. Since $\Phi_\pi(G) = 1$, we have $O_\pi(G) = 1$ and $\Phi(G) = 1$. Therefore, the solubility of $G$ implies that $N$ is a complemented $\pi'$-subgroup of $G$, and hence there exists a maximal subgroup $M$ in $G$ with $G = MN$ and $M \cap N = 1$. By Lemma 2.3, we may assume that $H \subseteq M$. By the choice of $G$, the triple $(M, H, \Phi_\pi(M))$ is a $k$-conjugate system. Consequently, there are elements $m_1, m_2, \ldots, m_k \in M$ for which $\Phi_\pi(M) = H^{m_1} \cap H^{m_2} \cap \cdots \cap H^{m_k}$.

**3.3.** *The subgroup $N$ is a faithful completely reducible $\Phi_\pi(M)$-module over the field $F_p$ of $p$ elements. In particular,* $\mathrm{Core}_{\Phi_\pi(M)N}(\Phi_\pi(M)) = 1$.

Obviously, $N$ is an irreducible $M$-module over $F_p$. By Clifford's Theorem (see [9, Theorem B.7.3]), $N$ is a completely reducible $\Phi_\pi(M)$-module. By Lemma 2.4 and assertion 3.1, we have $C_G(N) = 1$. Therefore, the $\Phi_\pi(M)$-module $N$ is faithful. Hence, $\mathrm{Core}_{\Phi_\pi(M)N}(\Phi_\pi(M)) = 1$.

**3.4.** *The triple $(\Phi_\pi(M)N, \Phi_\pi(M), 1)$ is not a $k$-conjugate system.*

Suppose that the triple $(\Phi_\pi(M)N, \Phi_\pi(M), 1)$ is a $k$-conjugate system. Then, by definition, there exist elements $h_1, h_2, \ldots, h_k$ in $\Phi_\pi(M)N$ such that $\bigcap_{i=1}^k (\Phi_\pi(M))^{h_i} = 1$. Since the element $h_i$ is representable as $h_i = f_i n_i$ for some $f_i \in \Phi_\pi(M)$ and $n_i \in N$ $(i = 1, 2, \ldots, k)$, we have $\bigcap_{i=1}^k (\Phi_\pi(M))^{n_i} = 1$.

Consider the subgroup $D = \bigcap_{i=1}^k H^{m_i n_i}$. By Frattini's Lemma, we have

$$D \subseteq \bigcap_{i=1}^{k} H^{m_i n_i} N = \bigcap_{i=1}^{k} H^{m_i} N = \left( \bigcap_{i=1}^{k} H^{m_i} \right) N = \Phi_\pi(M)N,$$

whence

$$\begin{aligned} D = D \cap \Phi_\pi(M)N &= \bigcap_{i=1}^{k} H^{m_i n_i} \cap \Phi_\pi(M)N = \bigcap_{i=1}^{k} (H^{m_i} \cap \Phi_\pi(M)N)^{n_i} \\ &= \bigcap_{i=1}^{k} \Phi_\pi(M)^{n_i} = 1 = \Phi_\pi(G). \end{aligned}$$

Thus, the triple $(G, H, \Phi_\pi(G))$ is a $k$-conjugate system, which contradicts the assumption.

## 4. Proof of the Theorem

1. Suppose that item 1 of the theorem fails and $G$ is a counterexample of a minimal order. If $H$ is a $\pi$-prefrattini subgroup of $G$ then the triple $(G, H, \Phi_\pi(G))$ is not a 4-conjugate system. Moreover, assertions 3.1–3.4 hold for $k = 4$. In particular, by 3.1 and 3.3, $\Phi_\pi(G) \cap N = 1$ and $N$ is a faithful completely reducible $\Phi_\pi(M)$-module over the field $F_p$ of $p$ elements. In view of Lemma 2.5, there exist elements $n_1, n_2, n_3 \in N$ for which $C_{\Phi_\pi(M)}(n_1) \cap C_{\Phi_\pi(M)}(n_2) \cap C_{\Phi_\pi(M)}(n_3) = 1$. Hence, by Lemma 2.6, we have

$$\Phi_\pi(M) \cap (\Phi_\pi(M))^{n_1} \cap (\Phi_\pi(M))^{n_2} \cap (\Phi_\pi(M))^{n_3} = 1.$$

Then the triple $(\Phi_\pi(M)N, \Phi_\pi(M), 1)$ is a 4-conjugate system, which contradicts assertion 3.4 of the minimal counterexample. Item 1 of the theorem is proved.

2. Suppose that item 2 of the theorem fails and $G$ is a counterexample of the minimal order. If $H$ is a $\pi$-prefrattini subgroup of $G$ then the triple $(G, H, \Phi_\pi(G))$ is not a 3-conjugate system. Moreover, assertions 3.1–3.4 hold for $k = 3$. In particular, $\Phi_\pi(G) \cap N = 1$ and $N$ is a faithful completely reducible $\Phi_\pi(M)$-module over the field $F_p$ of $p$ elements by 3.1 and 3.3. Since the group $G$ is $S_4$-free, the subgroup $\Phi_\pi(M)N$ is also $S_4$-free. Then, by Lemma 2.7, there exist elements $n_1, n_2 \in N$ for which $C_{\Phi_\pi(M)}(n_1) \cap C_{\Phi_\pi(M)}(n_2) = 1$. Hence, by Lemma 2.6 $\Phi_\pi(M) \cap (\Phi_\pi(M))^{n_1} \cap (\Phi_\pi(M))^{n_2} = 1$. Then the triple $(\Phi_\pi(M)N, \Phi_\pi(M), 1)$ is a 3-conjugate system, which contradicts assertion 3.4 of the minimal counterexample. Item 2 of the theorem is proved.

3. As in item 2, it suffices to show that the subgroup $\Phi_\pi(M)N$ is $S_4$-free. Suppose that this fails, i.e., $\Phi_\pi(M)N$ contains a section $A/B$ isomorphic to $S_4$. Since $N$ is a $\pi'$-group, by Lemma 2.1, $O_\pi(M)$ is a Hall $\pi$-subgroup of $\Phi_\pi(M)N$ and, moreover, $\Phi_\pi(M)N/O_\pi(M)N$ is nilpotent. Let $F$ be a Hall $\pi'$-subgroup in $\Phi_\pi(M)N$. Then, by the isomorphism,

$$\begin{aligned}\Phi_\pi(M)N/O_\pi(M)N = FO_\pi(M)N/O_\pi(M)N &\cong F/F \cap O_\pi(M)N \\ &= F/(F \cap O_\pi(M))N = F/N,\end{aligned}$$

we conclude that $F \in \mathfrak{N}^2$. By Hall's Theorem, there is an element $h \in \Phi_\pi(M)N$ such that $F^h \cap A$ is a Hall $\pi'$-subgroup in $A$. Since $A/B$ is isomorphic to $S_4$, we have $A = (F^h \cap A)B$, whence

$$A/B = (F^h \cap A)B/B \cong (F^h \cap A)/(F^h \cap A) \cap B = (F^h \cap A)/(F^h \cap B),$$

i.e., the section $A/B$ belongs to $\mathfrak{N}^2$; a contradiction to $A/B \cong S_4$. Consequently, the subgroup $\Phi_\pi(M)N$ is $S_4$-free. Then, by Lemma 2.7, there exist elements $n_1, n_2 \in N$ for which $C_{\Phi_\pi(M)}(n_1) \cap C_{\Phi_\pi(M)}(n_2) = 1$. By Lemma 2.6, $\Phi_\pi(M) \cap (\Phi_\pi(M))^{n_1} \cap (\Phi_\pi(M))^{n_2} = 1$. Then the triple $(\Phi_\pi(M)N, \Phi_\pi(M), 1)$ is a 3-conjugate system, which contradicts assertion 3.4 of the minimal counterexample. The theorem is proved.

## 5. An Example and Corollaries

Example 5.1. Let $\pi$ be a set of primes. Then, for every saturated formation $\mathfrak{F}$, there exists a group $G$ in which $\pi$-prefrattini subgroups are not $\mathfrak{F}$-prefrattini.

Suppose that there exists a saturated formation $\mathfrak{F}$ for which, in any group $G$, the set of all $\pi$-prefrattini subgroups coincides with the set of all its $\mathfrak{F}$-prefrattini subgroups.

Assume that $\pi(\mathfrak{F}) \cap \pi' \neq \varnothing$. Take $p \in \pi(\mathfrak{F}) \cap \pi'$. Since the formation $\mathfrak{F}$ is saturated, the group $Z_p$ of order $p$ belongs to $\mathfrak{F}$. But then a $\mathfrak{F}$-prefrattini subgroup of $Z_p$ coincides with $Z_p$ and its $\pi$-prefrattini subgroup is identity; a contradiction with the choice of $\mathfrak{F}$. Consequently, $\pi(\mathfrak{F}) \cap \pi' = \varnothing$, and hence $\pi(\mathfrak{F}) \subseteq \pi$.

Suppose that $\mathfrak{F} \subset \mathfrak{S}_\pi$, where $\mathfrak{S}_\pi$ is the formation of all soluble $\pi$-groups. Then there exists a $\pi$-group not belonging to $\mathfrak{F}$. Let $G$ be a group of the least order in $\mathfrak{S}_\pi \setminus \mathfrak{F}$. Then $G$ has a unique minimal normal subgroup $N$ such that $G/N \in \mathfrak{F}$. Since the formation $\mathfrak{F}$ is hereditary, $N$ is not contained in $\Phi(G)$. Then a maximal subgroup of $G$ not containing $N$ is its $\mathfrak{F}$-prefrattini subgroup, whereas a $\pi$-prefrattini subgroup of $G$ coincides with $G$; a contradiction. Consequently, $\mathfrak{F} = \mathfrak{S}_\pi$.

Suppose that $p$ and $q$ are primes; moreover, $q \notin \pi$ and $p \in \pi$. Let $U$ be a faithful irreducible $F_q[Z_p]$-module over the field $F_q$ consisting of $q$ elements. Consider the group $H = [U]Z_p$. Since $O_p(H) = 1$ and $U$ is a unique minimal normal subgroup in $H$, by Theorem 10.3.B in [9], there exists a faithful irreducible $F_p[H]$-module $V$ over the field $F_p$. Consider the group $D = [V]H = [V]([U]Z_p)$. Since $C_D(V) = V$ and $D/V$ is not a $\pi$-group, the minimal normal subgroup $V$ of $D$ is not $\mathfrak{F}$-central in $D$. Therefore, $Z_p$ is a $\mathfrak{F}$-prefrattini subgroup in $D$. At the same time, $[V]Z_p$ is its $\pi$-prefrattini subgroup. We again get a contradiction.

For $\pi = \varnothing$, item 3 of the theorem gives

**Corollary 5.1** [2]. *For every finite soluble group $G$ and any its prefrattini subgroup $H$, there are elements $x$ and $y$ in $G$ such that $H \cap H^x \cap H^y = \Phi(G)$.*

**Corollary 5.2.** *Let $\pi$ be a set of primes. Then the following hold for any finite soluble group $G$ and any its $\pi$-prefrattini subgroup $H$:*

(1) $|H| \leq \sqrt[4]{|G|^3 |\Phi_\pi(G)|}$;

(2) $|H/\Phi_\pi(G)| \leq |G : H|^3$.

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