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GENERALIZED RIGID GROUPS: DEFINITIONS, BASIC PROPERTIES, AND PROBLEMS

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Abstract: We find a natural generalization of the concept of rigid group. The generalized rigid groups are also called r-groups. The terms of the corresponding rigid series of every r-group can be characterized by both $\exists$-formulas and $\forall$-formulas. We find a recursive system of axioms for the class of r-groups of fixed solubility length. We define divisible r-groups and give an appropriate system of axioms. Several fundamental problems are stated.

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1. Introduction

About a decade ago the author defined the concept of rigid (soluble) group. Assume that a group G has the normal series

$$G = G_1 > G_2 > \cdots > G_m > G_{m+1} = 1, \tag{1}$$

with abelian quotients G_i/G_{i+1}. The action of G on G_i by conjugation endows G_i/G_{i+1} with the structure of a (right) module over the group ring $\mathbb{Z}[G/G_i]$. The group G is called *rigid* whenever these modules are torsion-free for all i; i.e., each nonzero element of $\mathbb{Z}[G/G_i]$ acts nontrivially on each nonzero element of G_i/G_{i+1}. The class of rigid groups contains free soluble groups and iterated wreath products of torsion-free abelian groups. This class turned out very appropriate in algebraic geometry over groups and model theory. The author, together with Myasnikov, obtained the following results in this area. Every rigid group is shown to be equationally Noetherian [1], which enabled us to develop algebraic geometry over rigid groups and, in particular, construct dimension theory [2]. Following [3], a rigid group G is called *divisible* whenever the elements of G_i/G_{i+1} for every i are divisible by nonzero elements of $\mathbb{Z}[G/G_i]$. It is established that each rigid group embeds into a divisible one. The divisible rigid groups are characterized as the algebraically closed objects in the class of rigid groups [4, 5]. The coordinate groups of irreducible algebraic sets over a divisible rigid group are described in [6]. We found and proved in [7] a reasonable statement for Hilbert's *Nullstellensatz* in algebraic geometry over rigid groups. We gave a recursive system of axioms for the theory of divisible rigid groups of fixed solubility length, established that this theory is complete in [5] and ω-stable, and described saturated models in [8, 9]. Ovchinnikov described the automorphisms of an arbitrary divisible rigid group in [10].

This article presents a rather natural generalization of the concept of rigid group. The new generalized rigid groups are also called r-groups. It is established that the terms of the corresponding rigid series of an r-group can be characterized by both $\exists$-formulas and $\forall$-formulas. We find a recursive system of axioms for the class of r-groups of fixed solubility length. We define divisible r-groups and give an appropriate system of axioms. Several fundamental problems concerning r-groups are stated. The author is not very optimistic that it would be possible to carry over many results on rigid groups to r-groups; the length 2 soluble case is, however, more promising.

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2. Condition (r1)

Given elements $x, y, x_1, \dots, x_n$ of a group, we usually write

$$x^y = y^{-1}xy, \quad [x, y] = x^{-1}y^{-1}xy, \quad [x_1, x_2, \dots, x_n] = [[\dots[x_1, x_2], \dots,], x_n].$$

Denoting the commutator subgroup of a group G by G' or $[G, G]$, put $G^{(0)} = G$, $G^{(1)} = G'$, and $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$ for $n > 1$.

Say that the *pair* (G, A), where A is a proper abelian normal subgroup of G, *satisfies condition* (r1) whenever all nontrivial elements of A do not commute with any element of $G \setminus A$ or, in other words, A coincides with the centralizer of each nontrivial element of itself.

Say that a *group G satisfies condition* (r1) whenever G has a normal series (1) with abelian quotients such that all pairs $(G/G_{i+1}, G_i/G_{i+1})$ satisfy condition (r1).

Theorem 1. *A group G with condition* (r1) *and the corresponding series* (1) *has the following properties.*

(1) *The solubility length of the group is exactly m.*

(2) *If $g_1 \in G_1 \setminus G_2$, $g_2 \in G_2 \setminus G_3, \dots, 1 \neq g_m \in G_m$, then*

$$G_i = \{x \mid [x, g_i, g_{i+1}, \dots, g_m] = 1\}, \quad i = 1, \dots, m.$$

(3) *In the standard group signature, the terms of* (1) *can be defined by both ∃-formulas and ∀-formulas; in particular, this series consists of automorphically closed subgroups and is uniquely determined by G itself.*

Proof. Observe that if G satisfies (r1) then so do all quotients G/G_{i+1}.

(1) Using induction, we may assume that the solubility length of G/G_m is $m-1$. Then the $(m-1)$th commutator subgroup $G^{(m-1)}$ does not lie in G_m. Take $g \in G^{(m-1)} \setminus G_m$ and $1 \neq a \in G_m$. We have $1 \neq [g, a] \in G^{(m-1)}$ and then $1 \neq [g, [g, a]] \in G^{(m)}$. Hence, the solubility length of G is at least m. Since the normal series in (1) of length m has abelian quotients, this length is at most m.

(2) For $i = m$ everything is clear. Take $i < m$ and, by induction, assume that

$$G_{i+1} = \{x \mid [x, g_{i+1}, g_{i+2}, \dots, g_m] = 1\}.$$

Then $[x, g_i, g_{i+1}, \dots, g_m] = 1$ implies that $[x, g_i] \in G_{i+1}$. Hence, x centralizes g_i modulo G_{i+1}, and so $x \in G_i$. Conversely, if $x \in G_i$ then $[x, g_i, g_{i+1}, \dots, g_m] = 1$.

(3) For $i = 1, \dots, m$ denote the set $\{x_{ij} \mid 1 \leq j \leq 2^{i-1}\}$ of 2^{i-1} variables by X_i. Consider the standard commutator $\delta_i(X_i)$ defining $G^{(i-1)}$ as a verbal subgroup. By definition, $\delta_1(x_{11}) = x_{11}$ and $\delta_{i+1}(X_{i+1}) = [\delta_i(X'_{i+1}), \delta_i(X''_{i+1})]$ for $i \geq 1$, where

$$X'_{i+1} = \{x_{i+1,j} \mid 1 \leq j \leq 2^{i-1}\}, \quad X''_{i+1} = \{x_{i+1,j} \mid 2^{i-1} + 1 \leq j \leq 2^i\}.$$

Since the solubility length of the group G/G_{i+1} is $i + 1$, some value of the word $\delta_i(X_i)$ lies outside G_{i+1}. It is known that this value lies in G_i. Basing on that and claim (2), we may assert that each of the following two formulas characterizes G_i:

$$\begin{aligned} \Phi_i(x) = \exists X_i\ \exists X_{i+1} \dots \exists X_m\ ([\delta_i(X_i), \delta_{i+1}(X_{i+1}), \dots, \delta_m(X_m)] \neq 1 \\ \wedge [x, \delta_i(X_i), \delta_{i+1}(X_{i+1}), \dots, \delta_m(X_m)] = 1), \end{aligned} \tag{2}$$

$$\Psi_i(x) = \forall X_i\ \forall X_{i+1} \dots \forall X_m\ ([x, \delta_i(X_i), \delta_{i+1}(X_{i+1}), \dots, \delta_m(X_m)] = 1). \tag{3}$$

The proof of Theorem 1 is complete. □

Theorem 2. *In the standard group signature, the class of length m soluble groups satisfying condition* (r1) *is defined by some finite system of axioms Λ_m^1.*

PROOF. Use the group word $\delta_i(X_i)$ and the formula $\Phi_i(x)$ defined in the proof of some previous theorem. Firstly, take the axioms $\forall X_{m+1}(\delta_{m+1}(X_{m+1}) = 1)$ and $\exists X_m(\delta_m(X_m) \neq 1)$, which mean that the solubility length of the group under consideration is exactly m. Then add the axiom

$$\begin{aligned}\forall x\ \forall y\ \forall z\ \forall X_m\ (((\delta_m(X_m) \neq 1) \wedge ([\delta_m(X_m), x] = 1)\\ \wedge (x \neq 1) \wedge ([x, y] = 1) \wedge ([x, z] = 1)) \to [y, z] = 1).\end{aligned} \tag{4}$$

This axiom implies the following: The elements commuting with $G^{(m)}$ constitute an abelian subgroup, denoted by G_m; since the subgroup $G^{(m)}$ is normal, so is G_m; and the pair (G, G_m) satisfies condition (r1). Also, we may assert that $\Phi_m(x)$ characterizes G_m in G.

The next axiom results from (4) once we replace m with $m-1$ and every equality of the form $v = 1$, where v is a word in the variables, by $\Phi_m(v)$; accordingly, the inequality $v \neq 1$ is replaced by $\neg\Phi_m(v)$. In other words, this is an analog of axiom (4) for the group G/G_m. Following that, we define the subgroup G_{m-1} as the centralizer of $G^{(m-1)}$ modulo G_m. This subgroup is characterized in G by the formula $\Phi_{m-1}(x)$. It should be clear to the reader how the construction of axioms continues. $\square$

3. Condition (r)

Consider an abelian normal subgroup A of some group G. The action of G on A by conjugation endows A with the structure of a right $\mathbb{Z}G$-module, even a $\mathbb{Z}[G/A]$-module. Denote by $\Theta(A)$ the annihilator of A in $\mathbb{Z}G$ which is a two-sided ideal. Take the ring $R = \mathbb{Z}G/\Theta(A)$, so that we can regard A as an R-module.

Say that the *pair* (G, A), where A is a proper abelian normal subgroup of G, *satisfies condition* (r) whenever G satisfies condition (r1) and the following condition (r2): the module A lacks R-torsion; i.e., if $0 \neq a \in A$ and $0 \neq u \in R$ then $au \neq 0$.

The definition directly implies the following statement.

Lemma. *Suppose that a pair (G, A) satisfies condition* (r). *Then*
(1) *the canonical ring epimorphism $\mathbb{Z}[G/A] \to R$ is injective on G/A;*
(2) *the ring R has no zero divisors.*

Say that a *group G satisfies condition* (r) or *G is an r-group* whenever G has a normal series (1) with abelian quotients such that every pair $(G/G_{i+1}, G_i/G_{i+1})$ satisfies condition (r). Refer to (1) as a *rigid series* for G and write $G_i = \rho_i(G)$ as in the case of rigid groups which satisfy condition (r).

Theorem 3. *In the standard group signature, the class of length m soluble r-groups is defined by some recursive system of axioms Λ_m.*

PROOF. By Theorem 2, a finite system of axioms Λ_m^1 determines length m soluble groups with condition (r1). Theorem 1 indicates the formulas $\Phi_i(x)$ characterizing the subgroups $G_i = \rho_i(G)$ in length m soluble groups with condition (r1). For each k with $1 \leq k \leq m$, a positive integer n, and a tuple $(\alpha_1, \dots \alpha_n)$ of integers, add to Λ_m^1 the axiom

$$\begin{aligned}\forall x\ \forall y\ \forall x_1 \dots \forall x_n\ ((\Phi_k(x) \wedge \neg\Phi_{k+1}(x) \wedge \Phi_k(y) \wedge \neg\Phi_{k+1}(y)\\ \wedge \neg\Phi_{k+1}(x^{\alpha_1 x_1 + \dots + \alpha_n x_n})) \to \neg\Phi_{k+1}(y^{\alpha_1 x_1 + \dots + \alpha_n x_n})),\end{aligned} \tag{5}$$

where $x^{\alpha_1 x_1 + \dots + \alpha_n x_n}$ stands for the product $x_1^{-1} x^{\alpha_1} x_1 \dots x_n^{-1} x^{\alpha_n} x_n$.

It is not difficult to understand that the new axioms do realize condition (r2). $\square$

As [11] shows, the integer group ring of a soluble group without zero divisors is a (right and left) Ore domain. Let us strengthen this result.

Proposition. *Consider a soluble group G and a proper ideal I of the group ring $\mathbb{Z}G$. If the quotient ring $R = \mathbb{Z}G/I$ lacks zero divisors then R is a (right and left) Ore domain.*

PROOF. For definiteness, we prove that R is a right Ore domain. We may assume that G embeds into R. The claim is obvious if G is an abelian group. Assume therefore that the solubility length of G is at least 2. Suppose that a normal subgroup H_0 has smaller solubility length than G does, and H_0 includes the commutator subgroup; i.e., G/H_0 is an abelian group. For our purposes we may assume that G is finitely generated over H_0. Suppose that $G = \langle g, g_1, \dots, g_n, H_0 \rangle$ and $H = \langle g_1, \dots, g_n, H_0 \rangle$. By induction on n, and in the case of absence of the elements g_i, on the solubility length of the group, assume that the subring $R(H)$ generated by H is an Ore domain. Clearly, R is a right $R(H)$-module generated by the powers of g. Take a row (a, b) of nonzero elements of the ring R. We should find a nonzero column $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $(a, b) \begin{pmatrix} x \\ y \end{pmatrix} = 0$. If necessary, we can multiply the row (a, b) on the right by the diagonal matrix $\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ with nonzero entries $d_1, d_2 \in R$ on the main diagonal. Therefore, we assume that

$$a = u_0 + gu_1 + \dots + g^n u_n, \quad b = v_0 + gv_1 + \dots + g^m v_m,$$

where

$$u_i, v_j \in R(H), \quad u_0 \neq 0, \quad v_0 \neq 0, \quad u_n \neq 0, \quad v_m \neq 0, \quad n \geq m, \quad n \geq 1.$$

Induct on $n + m$. Since $R(H)$ is an Ore domain, there is a column $\begin{pmatrix} c \\ d \end{pmatrix}$ of nonzero elements in $R(H)$ such that $(u_n, g^{m-n} v_m g^{n-m}) \begin{pmatrix} c \\ d \end{pmatrix} = 0$. Then the first element of the row $(a, b) \begin{pmatrix} c & 0 \\ g^{n-m} d & 1 \end{pmatrix}$ has smaller degree with respect to g than n, while the second equals b. By induction, there is a nonzero column $\begin{pmatrix} z \\ w \end{pmatrix}$ of elements in R such that

$$(a, b) \begin{pmatrix} c & 0 \\ g^{n-m} d & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = 0.$$

It is clear that the column

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c & 0 \\ g^{n-m} d & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}$$

is nonzero and $(a, b) \begin{pmatrix} x \\ y \end{pmatrix} = 0$. This justifies the claim. $\square$

Given an r-group G with a rigid series (1), consider the tuple of rings $(R_1, \dots, R_m)$, where R_i arises from the pair $(G/G_{i+1}, G_i/G_{i+1})$. By the Lemma and the Proposition, these rings are (right and left) Ore domains. Denote by $(Q_1, \dots, Q_m)$ the corresponding tuple of skew fields of fractions. Refer to G as a *divisible r-group* if each quotient G_i/G_{i+1} is a divisible R_i-module. In this situation we can regard the quotient as a vector space over the skew field Q_i.

Theorem 4. *In the standard group signature, the class of length m soluble divisible r-groups is determined by some recursive system of axioms $\overline{\Lambda}_m$.*

PROOF. Given k with $1 \leq k \leq m$, a positive integer n, and a tuple $(\alpha_1, \dots, \alpha_n)$ of integers, append to Λ_m the axiom

$$\begin{aligned} &\forall x \ \forall x_1 \dots \forall x_n ((\Phi_k(x) \wedge \neg \Phi_{k+1}(x^{\alpha_1 x_1 + \dots + \alpha_n x_n})) \\ &\quad \to \exists y (\Phi_k(y) \wedge \Phi_{k+1}(x^{-1} y^{\alpha_1 x_1 + \dots + \alpha_n x_n}))). \end{aligned} \tag{6}$$

This axiom means that if $u = \alpha_1 x_1 + \dots + \alpha_n x_n$ represents a nonzero element in the ring R_k then every nonzero element xG_{k+1} of the R_k-module G_k/G_{k+1} is divisible by it, just as we want. $\square$

4. Problems

In closing, let us state some natural questions.

Problem 1. *Is every r-group equationally Noetherian?*

Problem 2. *Is the universal theory of a finitely generated metabelian r-group decidable?*

Problem 3. *Does every divisible r-group embed into a divisible r-group of the same solubility length?*

Problem 4. *Does every divisible r-group split into a semidirect product of abelian subgroups isomorphic to appropriate quotients of a rigid series?*

Problem 5. *Is the elementary theory of arbitrary divisible r-group decidable?*

The answers to all these questions for rigid groups are positive, and they actually serve as a basis for studying various aspects of the theory of rigid groups.

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