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HOMOLOGICAL RESOLUTIONS IN PROBLEMS ABOUT SEPARATING CYCLES

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Abstract: We study the homological cycles that separate a set of divisors in a complex-analytic manifold. A generalization of the notion of separating cycle is proposed for the case of a collection of closed sets in an arbitrary real manifold.

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Introduction

The notion of separating cycle appeared in complex analysis in connection with the study of multidimensional residues.

Let X be a complex-analytic manifold of complex dimension n. The local residue (Grothendieck residue, see [1]) of a meromorphic n-form ω with polar divisor F with respect to the system of (effective) divisors $Y_1, \dots, Y_n$, $F = Y_1 + \dots + Y_n$ at an isolated point $a \in Y_1 \cap \dots \cap Y_n$ is represented by the integral

$$\operatorname{res}_f \omega = (2\pi i)^{-n} \int\limits_{\Gamma_a} \omega$$

over the n-cycle $\Gamma_a = \{z \in U_a : |f_i(z)| = \varepsilon_i,\ i = 1, \dots, n\}$, where $f_i(z)$ is a function whose zero set represents the divisor Y_i in a sufficiently small neighborhood U_a of a (the local residue is associated with the ideal in the local germ ring $\mathscr{O}_a$ generated by the function system $f = (f_1, \dots, f_n)$).

If $X = \mathbb{C}$ then the local residue coincides with the usual Cauchy residue for a meromorphic function. Moreover, each *local* cycle Γ_a is a circle $\{|z - a| = \varepsilon\}$ and is homologous to zero in $\mathbb{C}$, and the set of the classes of the cycles Γ_a, $a \in F$, generates the whole homology group $H_1(\mathbb{C} \setminus F)$. This implies that the integral of a meromorphic function over any cycle is calculated with the use of residues.

We can demonstrate that, for a manifold X of dimension $n \geq 2$, the local cycle Γ_a involved in the definition of local residue *separates* the set of divisors $Y = \{Y_1, \dots, Y_n\}$ in the sense that Γ_a is homologous to zero in $X \setminus (Y_1 \cup \dots [k] \dots \cup Y_n)$ for all $k \in \{1, \dots, n\}$. This implies that, for the calculation of an integral of a meromorphic form to be reduced to calculating the local residues, it is necessary that the integration cycle separate a set of polar divisors Y of the given form.

The most complete results on the characterization of separating cycles in complex analysis are presented in the works of Tsikh [2, 3] and Yuzhakov [4, 5]. This research was preceded by a series of results of the works of Picard [6], Fantappiè [7], Martinelli [8], and Sorani [9]. Among the possible applications of separating cycles, we mention the article [10] by Pochekutov in which separating cycles are used to describe the properties of the diagonal of the multiple power series representing a rational function.

Another situation of appearance of separating cycles is related to knot theory. A nontrivial link is called a *Brunnian link* if it becomes trivial after removing its every component. The simplest example of a Brunnian link of three components is given by the familiar Borromean rings. The Borromean rings

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are a nontrivial link of three rings (circles) that are not pairwise connected. In this case, by analogy with the situation discussed above, we must say that, as a one-dimensional cycle, each of the circles separates the set of the two other circles.

Detailed inspection of the given two examples generates the following definition which makes it possible to develop a unified approach to the study of both situations. Let X be a real manifold of dimension d and let $Y = \{Y_1, \dots, Y_m\}$ be a collection of closed subsets in X, where $2 \le m \le d-1$.

DEFINITION 1. We say that a $(d-m)$-dimensional cycle Γ in $X \setminus (Y_1 \cup \dots \cup Y_m)$ *separates the collection* Y if Γ is homologous to zero in $X \setminus (Y_1 \cup \dots [k] \dots \cup Y_m)$ for all $k \in \{1, \dots, m\}$.

Consider the simplest case that a cycle Γ separates a pair of two closed subsets $Y = \{Y_1, Y_2\}$ in a manifold X of dimension $d \ge 3$. Suppose that the $(d-1)$-homology of X is trivial and $Y_1 \cap Y_2 = \varnothing$. In particular, these conditions are fulfilled for the Borromean rings and any other Brunnian links with three components. The Mayer–Vietoris exact sequence

$$\dots \to H_{d-1}(U_1 \cup U_2) \to H_{d-2}(U_1 \cap U_2) \to H_{d-2}(U_1) \oplus H_{d-2}(U_2) \to \cdots$$

is defined, where $U_i = X \setminus Y_i$, $i = 1, 2$. Since the cycle Γ from $X \setminus (Y_1 \cup Y_2) = U_1 \cap U_2$ separates Y, the class of Γ has zero image in $H_{d-2}(U_1) \oplus H_{d-2}(U_2)$. The exactness of the sequence implies that the class of the cycle Γ has an inverse image in $H_{d-1}(U_1 \cup U_2) = H_{d-1}(X \setminus (Y_1 \cap Y_2)) = H_{d-1}(X) \simeq 0$, whence we infer that the separating cycle Γ is homologous to zero. For Brunnian links with three components, this means that, on each topological circle of this link, we can span a "membrane" lying in the complement to the remaining two circles (one can check this directly for the Borromean rings).

For $m > 2$, the application of the Mayer–Vietoris sequence in problems connected with separating cycles becomes somewhat difficult. Such an approach, however, was successfully realized by Tsikh in the study of the cycles separating sets of divisors in a complex-analytic manifold in connection with the theory of multidimensional residues. Another approach to the study of separating cycles, proposed in the present paper, is based on the use of the double complex for the homology of a union. In this case, we use the notion of a resolution for the cycle (and also some other ideas) from Gleason's paper [11]. We give a shorter proof of Tsikh's Theorem of [3] on separating cycles in Stein manifolds. In the local case, we prove a theorem on the characterization of separating cycles for an "overfilled" set that includes one "extra" hypersurface. We prove an assertion about collections of multidimensional spheres which is similar to that on the homological triviality of the components of a Brunnian link with three components.

1. The Double Complex for the Homology of a Union

Let X be a topological space and let $\{\mathscr{U}_j\}_{j \in J}$ be an open covering of X, where J is an ordered index set. Put $\mathscr{U}_{j_0 j_1 \dots j_p} = \mathscr{U}_{j_0} \cap \mathscr{U}_{j_1} \cap \dots \cap \mathscr{U}_{j_p}$. Consider the bigraded group of singular chains $(S_{p,q})$ for a covering $\{\mathscr{U}_j\}$, where

$$S_{p,q} = \bigoplus_{j_0 < j_1 < \dots < j_p} S_q(\mathscr{U}_{j_0 j_1 \dots j_p}), \quad p, q = 0, 1, \dots .$$

The inclusions $\mathscr{U}_{j_0 j_1 \dots j_p} \subset \mathscr{U}_{j_0 j_1 \dots [j_k] \dots j_p}$ induce the Čech boundary operator $\delta : S_{p,\bullet} \to S_{p-1,\bullet}$, $\delta^2 = 0$, defined by the "alternating sum" formula:

$$(\delta\sigma)_{j_0 j_1 \dots j_{p-1}} = \sum_{j \in J} \sigma_{j j_0 \dots j_{p-1}}.$$

Here the alternation of the sum means the agreement that, after interchanging two indices, the chain $\sigma_{j j_0 \dots j_{p-1}}$ is taken with the minus sign. For $p = 0$, the corresponding inclusions $\mathscr{U}_j \subset X$ induce the operator $\varepsilon : S_{0,\bullet} \to S_\bullet(X)$ whose action, by the same "alternating sum" formula, amounts to taking the sum of the chains over all elements of the covering. Moreover, $\varepsilon\delta = 0$. We obtain the Mayer–Vietoris sequence for singular chains (see [12, 13]):

$$0 \longleftarrow S_\bullet(X) \xleftarrow{\varepsilon} S_{0,\bullet} \xleftarrow{\delta} S_{1,\bullet} \xleftarrow{\delta} S_{2,\bullet} \xleftarrow{\delta} \cdots ,$$

which generalizes the familiar short exact sequence

$$0 \longleftarrow S(\mathscr{U}_1 \cup \mathscr{U}_2) \longleftarrow S(\mathscr{U}_1) \oplus S(\mathscr{U}_2) \longleftarrow S(\mathscr{U}_1 \cap \mathscr{U}_2) \longleftarrow 0,$$

obtained for $J = \{1, 2\}$.

Consider the (extended) double complex $\mathscr{S}$ in which the rows are the Mayer–Vietoris exact sequences for the groups of singular chains and the following boundary operator $\partial : S_{\bullet,q} \to S_{\bullet,q-1}$, $(\partial\sigma)_{j_0 j_1 \dots j_p} = \partial\sigma_{j_0 j_1 \dots j_p}$, $\partial^2 = 0$, acts in the columns:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
0 & \longleftarrow & S_n(X) & \xleftarrow{\varepsilon} & S_{0,n} & \xleftarrow{\delta} & S_{1,n} & \xleftarrow{\delta} & \cdots \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
0 & \longleftarrow & S_{n-1}(X) & \xleftarrow{\varepsilon} & S_{0,n-1} & \xleftarrow{\delta} & S_{1,n-1} & \xleftarrow{\delta} & \cdots \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
0 & \longleftarrow & S_0(X) & \xleftarrow{\varepsilon} & S_{0,0} & \xleftarrow{\delta} & S_{1,0} & \xleftarrow{\delta} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

This complex is analogous to the familiar Čech–de Rham double complex for differential forms.

For $J = \{1, 2\}$, the short exact sequences of the rows of the double complex $\mathscr{S}$ induce the Mayer–Vietoris long exact sequence for the homology groups. In the general case, the analog of the Mayer–Vietoris sequence for homology is given by the spectral sequence of the double complex.

Consider the total complex $T\mathscr{S}$ of the dual complex $\mathscr{S}$:

$$(T\mathscr{S})_k = \bigoplus_{p+q=k} S_{p,q},$$

with the boundary operator $D : (T\mathscr{S})_k \to (T\mathscr{S})_{k-1}$ acting by the formula $D = \delta + (-1)^p \partial$. Like in the proof the isomorphism between the de Rham and Čech cohomologies, the exactness of the rows of the complex $\mathscr{S}$ implies that its spectral sequence degenerates at the term E^2 and gives an isomorphism

$$H^D_*(T\mathscr{S}) \simeq H_*(X),$$

which is induced by the chain mapping

$$\varepsilon_T : (T\mathscr{S})_q \to S_q(X), \quad \varepsilon_T(\tau) = \varepsilon(\tau_{0,q}),$$

where $\tau_{0,q}$ is the "higher" component of the chain $\tau \in (T\mathscr{S})_q$ with the components $\tau_{i,q-i} \in S_{i,q-i}$, $i = 0, 1, \dots$.

2. A Homological Resolution of a Cycle

Let X be a real manifold of dimension d and let $Y = \{Y_1, \dots, Y_m\}$ be a collection of closed subsets in X, where $2 \le m \le d - 1$. Put $\mathscr{U}_j = X \setminus Y_j$, $j \in J = \{1, \dots, m\}$. Then $\{\mathscr{U}_j\}_{j \in J}$ is an open covering for $\widetilde{X} = \bigcup_{j \in J} \mathscr{U}_j$. Consider the double complex $\mathscr{S}$ for such a covering and its total complex $T\mathscr{S}$. Following [11], introduce the notion of resolution of a cycle.

Definition 2. We say that a $(d-m)$*-dimensional cycle* Γ *in* $X \setminus (Y_1 \cup \dots \cup Y_m)$ *has resolution* $\xi = \{\xi_i\}_{i=0}^{m-1}$ if the following are fulfilled:

(1) $\xi_i \in S_{m-i-1,d-m+i}$,
(2) $\xi_0 = \Gamma$,
(3) $\delta(\xi_{i-1}) = \partial(\xi_i)$, $i = 1, \dots, m-1$.

A resolution $\xi = \{\xi_i\}_{i=0}^{m-1}$ can be regarded as an element of the group $(T\mathscr{S})_{d-1}$ of the total complex $T\mathscr{S}$. Moreover, the equalities of (3) make it possible to associate with a resolution the D-cycle $\hat{\xi} \in (T\mathscr{S})_{d-1}$ with components $\hat{\xi}_i = \pm\xi_i$, where the pluses and minuses are chosen so that $\delta(\hat{\xi}_{i-1}) = (-1)^{m-i}\partial(\hat{\xi}_i)$, $i = 1, \dots, m-1$. If we additionally put $\hat{\xi}_0 = +\xi_0$ then the remaining signs are defined uniquely. Observe also that, for each resolution of a cycle Γ, the chain $\varepsilon(\xi_{m-1})$ is a cycle. Indeed,

$$\partial\varepsilon(\xi_{m-1}) = \varepsilon\partial(\xi_{m-1}) = \varepsilon\delta(\xi_{m-2}) = 0$$

by the commutation of the operators ∂ and ε and also the exactness of the rows of $\mathscr{S}$:

$$\begin{tikzcd}
\varepsilon(\xi_{m-1}) \arrow[d, "\partial"] & \xi_{m-1} \arrow[l, "\varepsilon"'] \arrow[d, "\partial"] & \\
0 & \delta(\xi_{m-2}) \arrow[l, "\varepsilon"'] & \xi_{m-2} \arrow[l, "\delta"']
\end{tikzcd}$$

Proposition 1. *Suppose that a cycle* $\Gamma \in S_{m-1,d-m}$ *has a resolution* $\xi = \{\xi_i\}_{i=0}^{m-1}$, *where the cycle* $\varepsilon(\xi_{m-1}) \in S_{0,d-1}$ *is homologous to zero. Then* Γ *is also homologous to zero.*

Proof. A resolution of a cycle Γ defines a D-cycle $\hat{\xi}$ of dimension $d-1$. Since $\varepsilon_T(\hat{\xi}) = \varepsilon(\pm\xi_{m-1}) \sim 0$, the isomorphism $H_*^D(T\mathscr{S}) \simeq H_*(X)$ implies that $\hat{\xi} \sim 0$. This means that there is a chain $\eta \in (T\mathscr{S})_d$ with $D(\eta) = \hat{\xi}$. Consider the component η_0 of η that is an element of the group $S_{m-1,d-m+1}$. We have $\hat{\xi}_0 = \delta(0) + (-1)^{m-1}\partial(\eta_0)$, whence $\Gamma = \pm\partial(\eta_0)$, i.e., $\Gamma \sim 0$.

Corollary. *Suppose that the cycles* Γ *and* Γ' *have resolutions* ξ *and* ξ' *respectively. If* $\varepsilon(\xi_{m-1}) \sim \varepsilon(\xi'_{m-1})$ *then* $\Gamma \sim \Gamma'$.

3. Existence of Resolution for a Separating Cycle

Return to the study of separating cycles. Recall that a $(d-m)$-dimensional cycle Γ separates a collection of closed sets $Y = \{Y_1, \dots, Y_m\}$ if $\Gamma \sim 0$ in $X \setminus (Y_1 \cup \dots [k] \dots \cup Y_m)$ for all $k \in \{1, \dots, m\}$. This means that $\Gamma \in S_{m-1,d-m}$ satisfies $\delta\Gamma \sim 0$.

Suppose that $m = 2$, i.e. $Y = \{Y_1, Y_2\}$. Then, for a cycle $\Gamma \in S_{1,d-2}$ separating Y, we can construct a resolution $\xi = \{\xi_i\}_{i=0}^{1}$ as follows: Put $\xi_0 = \Gamma$; since $\delta\xi_0 = \delta\Gamma \sim 0$, there exists a chain $\xi_1 \in S_{0,d-1}$ with $\partial\xi_1 = \delta\xi_0$. We have constructed some resolution. Describe conditions on a manifold X and a collection of subsets Y sufficient for the existence of a resolution for a separating cycle for $m > 2$. Put

$$H_{p,q} = \bigoplus_{j_0 < j_1 < \dots < j_p} H_q(\mathscr{U}_{j_0 j_1 \dots j_p}).$$

Proposition 2. *Let a manifold* X *and a collection of closed subsets* $Y = \{Y_1, \dots, Y_m\}$, $m > 2$, *be such that, for the covering* $\{\mathscr{U}_j\}$, $\mathscr{U}_j = X \setminus Y_j$, *the homology groups* $H_{p,d-p-2}$ *are trivial for* $p = 0, \dots, m-3$. *Then, for every* $(d-m)$*-dimensional cycle* Γ *in* $X \setminus (Y_1 \cup \dots \cup Y_m)$ *separating* Y, *there exists a resolution* $\xi = \{\xi_i\}_{i=0}^{m-1}$.

Proof. Put $\xi_0 = \Gamma$. As in the case of $m = 2$, we have $\delta\xi_0 = \delta\Gamma \sim 0$; hence, there is a chain $\xi_1 \in S_{m-2,d-m+1}$ such that $\partial\xi_1 = \delta\xi_0$. The chain $\delta\xi_1 \in S_{m-3,d-m+1}$ is a cycle since $\partial\delta\xi_1 = \delta\partial\xi_1 = \delta^2\xi_0 = 0$.

But $H_{m-3,d-m+1} \simeq 0$; consequently, there is a chain $\xi_2 \in S_{m-3,d-m+2}$ such that $\partial\xi_2 = \delta\xi_1$ and once again $\delta\xi_2$ is a cycle:

$$\begin{tikzcd}
\ddots \arrow[d, "\partial"] & & & \\
\bullet & \xi_2 \arrow[l, "\delta"'] \arrow[d, "\partial"] & & \\
& \bullet & \xi_1 \arrow[l, "\delta"'] \arrow[d, "\partial"] & \\
& & \bullet & \xi_0 \arrow[l, "\delta"']
\end{tikzcd}$$

Continuing likewise and using the triviality of $H_{p,d-p-2}$, construct a desired sequence of chains $\xi_i \in S_{m-i-1,d-m+i}$, $i = 0, 1, \dots, m-1$, for which $\delta\xi_{i-1} = \partial\xi_i$.

4. Separating Cycles in Stein Manifolds

Consider the case that X is a complex-analytic manifold, $\dim_{\mathbb{C}} X = n$ and $Y = \{Y_1, \dots, Y_m\}$ is a collection of hypersurfaces in X. Assume first that $m = n$. Suppose that a is an isolated point in $Y_1 \cap \dots \cap Y_n$ and $Y_i = \{f_i = 0\}$ in a sufficiently small neighborhood U_a of a. Consider the n-cycle in $X \setminus (Y_1 \cup \dots \cup Y_n)$ of the form

$$\Gamma_a = \{z \in U_a : |f_j(z)| = \varepsilon_j,\ j = 1, \dots, n\}.$$

The cycle Γ_a will be called the *local* cycle at a defined for the collection of hypersurfaces Y. This definition is conditioned by the fact that the homological class of a local cycle Γ_a contains a cycle with support in a however small neighborhood of a. It is shown in [3] that the local cycle Γ_a defines the collection Y. Moreover, it is not hard to construct some resolution for such a local cycle.

Proposition 3. *The local cycle Γ_a admits a resolution $\xi = \{\xi_k\}_{k=0}^{n-1}$ such that the cycle $\varepsilon\xi_{n-1}$ is homologous to a $(2n-1)$-dimensional sphere S_a surrounding a and not containing other points from $Y_1 \cap \dots \cap Y_n$ inside.*

Proof. Let $J = \{1, \dots, n\}$. For each subset $I = \{i_1, \dots, i_k\} \subset J$, $i_1 < \dots < i_k$, consisting of k elements, consider the $(n+k)$-chain

$$\sigma_k(I) = \sigma_k(i_1, \dots, i_k) = \{z \in U_a : |f_i(z)| \le \varepsilon_i,\ i \in I,\ |f_j(z)| = \varepsilon_j,\ j \in J \setminus I\}$$

in $U_{j_1 \dots j_{n-k}}$, where $\{j_1, \dots, j_{n-k}\} = J \setminus I$, $j_1 < \dots < j_{n-k}$. The chain $\sigma_k(I)$ is an $(n+k)$-face of the polyhedron $\Pi = \{z \in U_a : |f_j(z)| \le \varepsilon_j,\ j \in J\}$ whose orientation is defined by the order $r_1, \theta_1, \dots, r_n, \theta_n$ of the parameters r_i and θ_i, where $f_i = r_i e^{i\theta_i}$. The orientation of the faces of $\sigma_k(I)$ is induced by that of the polyhedron. The set of chains $\sigma_k(I)$ defines the collection of chains $\xi_k \in S_{n-k-1,n+k}$, $k = 0, 1, \dots, n-1$, with the components

$$\xi_k(j_1, \dots, j_{n-k}) = \sigma_k(i_1, \dots, i_k) \in S_{n+k}(U_{j_1 \dots j_{n-k}}).$$

In particular, for $k = 0$ we obtain $I = \varnothing$, $\xi_0 = \Gamma_a$. Check that the collection of chains $\xi_0, \xi_1, \dots, \xi_{n-1}$ satisfies the equality $\delta(\xi_{k-1}) = \partial(\xi_k)$, $k = 1, \dots, n-1$. Given an arbitrary tuple $I = \{i_1, \dots, i_k\}$, we have

$$\partial\xi_k(j_1, \dots, j_{n-k}) = \partial\sigma_k(i_1, \dots, i_k) = \sum_{l=1}^{k} (-1)^{i_l + l} \sigma_{k-1}(i_1, \dots [i_l] \dots, i_k).$$

Indeed, the withdrawal of i_l from I corresponds to the withdrawal of the parameter r_l whose number is $i_l + l - 2$ if the parameters are enumerated starting from zero. On the other hand,

$$(\delta\xi_{k-1})(j_1, \dots, j_{n-k}) = \sum_{i=1}^{n} \xi_{k-1}(i, j_1, \dots, j_{n-k}) = \sum_{l=1}^{k} \xi_{k-1}(i_l, j_1, \dots, j_{n-k}).$$

For obtaining an increasing sequence from a sequence of indices $i_l, j_1, \dots, j_{n-k}$, we must carry out $i_l - l$ consecutive permutations of i_l with the succeeding index. Hence,

$$\sum_{l=1}^{k} \xi_{k-1}(i_l, j_1, \dots, j_{n-k}) = \sum_{l=1}^{k} (-1)^{i_l - l} \sigma_{k-1}(i_1, \dots [i_l] \dots, i_k).$$

Since $i_l + l$ and $i_l - l$ differ by an even number, the chains $\partial\xi_k(j_1, \dots, j_{n-k})$ and $(\delta\xi_{k-1})(j_1, \dots, j_{n-k})$ coincide completely. Thus, we have constructed a resolution of $\xi_0 = \Gamma_a$. It remains to observe that $\varepsilon\xi_{n-1} = \partial\Pi$ is homologous to a $(2n-1)$-dimensional sphere S_a surrounding a and not containing other points from $Y_1 \cap \dots \cap Y_n$ inside.

Suppose that X is a Stein manifold, $\dim_{\mathbb{C}} X = n$. By a result of Serre, $H_q(X) \simeq 0$ for $q > n$. Consider a collection of hypersurfaces $Y = \{Y_1, \dots, Y_n\}$ in X. Then the sets $\mathscr{U}_j = X \setminus Y_j$, $j = 1, \dots, n$ (and all their possible intersections) are also Stein manifolds. Consequently, for the covering $\{\mathscr{U}_j\}$, we have $H_{p,q} = 0$ for $q > n$, $p = 0, \dots, n-1$. In particular, $H_{p,2n-p-2} \simeq 0$ for $p = 0, \dots, n-3$; therefore, by Proposition 2, for each cycle Γ in $X \setminus (Y_1 \cup \dots \cup Y_n)$ separating Y, there exists a resolution $\xi = \{\xi_i\}$. Assume that $Z = Y_1 \cap \dots \cap Y_n$ is discrete (consists only of isolated points) and $Z \neq \varnothing$. Since $H_{2n-1}(X) \simeq 0$, the group $H_{2n-1}(\widetilde{X})$, where $\widetilde{X} = \bigcup_{j=1}^{n} \mathscr{U}_j = X \setminus (Y_1 \cap \dots \cap Y_n)$, is generated by the $(2n-1)$-dimensional spheres S_a, $a \in Z$. Hence, the cycle $\varepsilon\xi_{n-1}$ is homologous to a linear combination of such spheres:

$$\varepsilon\xi_{n-1} \sim \sum_{a \in Z} n_a S_a.$$

Put $\Gamma' = \sum_{a \in Z} n_a \Gamma_a$. Denote by $\xi^{(a)}$ the resolution of the local cycle Γ_a mentioned in Proposition 3. Then the chain $\xi' = \sum_{a \in Z} n_a \xi^{(a)} \in (T\mathscr{S})_{2n-1}$ is a resolution for Γ' and $\varepsilon\xi'_{n-1} \sim \sum_{a \in Z} n_a S_a$. The corollary to Proposition 1 gives that Γ and Γ' are homologous; i.e.,

$$\Gamma \sim \sum_{a \in Z} n_a \Gamma_a.$$

If $Z = \varnothing$ then $\varepsilon\xi'_{n-1} \sim 0$, which by Proposition 2 implies that $\Gamma \sim 0$.

If the intersection $Z = Y_1 \cap \dots \cap Y_n$ is nondiscrete then Z is representable as $Z = Z_0 \cup Z_1$, where Z_0 is the discrete part of Z and Z_1 is the analytic subset consisting of irreducible components of dimension ≥ 1. Consider the manifold $X^{(1)} = X \setminus Z_1$ and the collection of hypersurfaces $Y^{(1)} = \{Y_1^{(1)}, \dots, Y_n^{(1)}\}$, where $Y_j^{(1)} = Y_j \cap X^{(1)}$. We have $X^{(1)} \setminus Y_j^{(1)} = X \setminus Y_j$; therefore, the cycle Γ, which separates Y in X, also separates the set $Y^{(1)}$ in $X^{(1)}$. Since again $H_{2n-1}(X^{(1)}) \simeq 0$ and $X^{(1)} \setminus Y_j^{(1)}$ is a Stein manifold and the intersection $Z_0 = Y_1^{(1)} \cap \dots \cap Y_n^{(1)}$ is discrete and the assertion about the representability of a cycle as a linear combination of the local cycles is connected only with the discrete part of the intersection of the hypersurfaces, we obtain the already considered case. Thus, we have obtained a shorter proof of the following theorem of [3]:

Theorem 1 (Tsikh). *Let X be a Stein manifold of dimension n and let $Y = \{Y_1, \dots, Y_n\}$ be a collection of hypersurfaces in X. Then the n-cycle Γ from $X \setminus (Y_1 \cup \dots \cup Y_n)$ separates Y if and only if Γ is homologous to a linear combination of the local cycles Γ_a, $a \in Z_0$, where Z_0 is the discrete part of $Y_1 \cap \dots \cap Y_n$.*

Remark. If the discrete part of $Y_1 \cap \dots \cap Y_n$ is empty then every cycle Γ separating Y is homologous to zero.

Possible generalization of the notion of separating cycle in the case of complex-analytic manifolds is connected with considering collections $Y = \{Y_1, \dots, Y_m\}$ in which the number of hypersurfaces m exceeds the dimension n of X. In this case, following the definition by Yuzhakov, we say that an n-cycle Γ in $X \setminus (Y_1 \cup \dots \cup Y_m)$ *separates* Y if, for all $(n-1)$-subcollections of indices $\{i_1, \dots, i_{n-1}\} \subset \{1, \dots, m\}$, the cycle Γ is homologous to zero in $X \setminus (Y_{i_1} \cup \dots \cup Y_{i_{n-1}})$.

Let $\mathscr{J} = (J_1, \dots, J_n)$ be a partition of $J = \{1, \dots, m\}$ into n nonempty disjoint subsets J_k. To each such partition, there corresponds the collection of n hypersurfaces $Y_{\mathscr{J}} = \{F_1, \dots, F_n\}$, where $F_k = \bigcup_{i \in J_k} Y_i$. With an isolated point $a \in F_1 \cap \dots \cap F_n$, associate the local cycle $\Gamma_{\mathscr{J},a}$ defined for $Y_{\mathscr{J}}$:

$$\Gamma_{\mathscr{J},a} = \left\{ z \in U_a : \left| \prod_{j \in J_k} f_j(z) \right| = \varepsilon_k,\ k = 1, \dots, n \right\}.$$

It was shown in [5] that the local cycle $\Gamma_{\mathscr{J},a}$ separates the collection $Y = \{Y_1, \dots, Y_m\}$, and two conditions are given for Stein manifolds under each of which the separating cycle is a linear combination of the local cycles $\Gamma_{\mathscr{J},a}$. One of these conditions is formulated in the following *local* case: Let $X = U_a$ be a sufficiently small Stein neighborhood of $a \in \mathbb{C}^n$. Consider the *centered* collections of hypersurfaces in U_a, i.e. $Y = \{Y_1, \dots, Y_m\}$ satisfying $a \in Y_1 \cap \dots \cap Y_m$. It is assumed that the intersection of any n hypersurfaces in Y has no isolated points but possibly a. The above condition under which the separating cycle is representable as a linear combination of the local cycles consists in the fact that all hypersurfaces in a centered collection must be in general position, i.e., a must be an isolated point of the intersection of any n hypersurfaces in this collection. As a continuation of the investigations of separating cycles in the local case, prove the following

Theorem 2. *Let $X = U_a$ be a sufficiently small Stein neighborhood of a point $a \in \mathbb{C}^n$ and let $Y = \{Y_1, \dots, Y_{n+1}\}$ be a centered collection of hypersurfaces. Then an n-cycle Γ from $X \setminus (Y_1 \cup \dots \cup Y_{n+1})$ separates Y if and only if Γ is homologous to a linear combination of the local cycles $\Gamma_{\mathscr{J},a}$.*

Thus, under the conditions of Theorem 2, the arbitrary centered collections of hypersurfaces of $n+1$-elements are considered in which the hypersurfaces need not be in general position.

Proof. For the collection of hypersurfaces $Y = \{Y_1, \dots, Y_{n+1}\}$, put $Y^{[k]} = Y_1 \cap \dots [k] \dots \cap Y_{n+1}$, $Y_{[k]} = Y_1 \cup \dots [k] \dots \cup Y_{n+1}$. The union $Y_1 \cup \dots \cup Y_{n+1}$ will also be denoted by Y. Designate as f_i, $i = 1, \dots, n+1$, the functions defining the germs of the hypersurfaces Y_i. Suppose that an n-cycle Γ separates the centered collection $Y = \{Y_1, \dots, Y_{n+1}\}$. Then the cycle Γ also separates any n-subcollection $\{Y_1, \dots [k] \dots, Y_{n+1}\}$. If, moreover, $Y^{[k]} = \{a\}$, i.e. the hypersurfaces of the subcollection are in general position; then, by Theorem 1 we get $\Gamma \sim n_k \gamma_k$ in $U_a \setminus Y_{[k]}$, where γ_k is the local cycle at the point a defined for the collection $\{Y_1, \dots [k] \dots, Y_{n+1}\}$. If $Y^{[k]} \neq \{a\}$ then (by the remark to Theorem 1) $\Gamma \sim 0$ in $U_a \setminus Y_{[k]}$.

Assume first that Y contains at least one subcollection of n hypersurfaces in general position; i.e., there is $k \in \{1, \dots, n+1\}$ for which $Y^{[k]} = \{a\}$. Changing the enumeration of the hypersurfaces if need be, assume that $Y^{[n+1]} = \{a\}$. Let ω be an arbitrary closed differential n-form in $U_a \setminus Y$. Since U_a is a Stein manifold, ω is cohomologous to some holomorphic n-form φ in $U_a \setminus Y$. We can show (see [14]) that, under the assumptions made, φ admits some decomposition $\varphi = \varphi_1 + \dots + \varphi_n$, where φ_k is an n-form holomorphic in $U_a \setminus Y_{[k]}$. Let K be a set of the indices k in $\{1, \dots, n\}$ for which $Y^{[k]} = \{a\}$. Consider the integral of ω over Γ. Assuming that $K \neq \varnothing$ and involving the fact that, in the domain $U_a \setminus Y_{[k]}$ in which φ_k is holomorphic, $\Gamma \sim n_k \gamma_k$ for $k \in K$ and $\Gamma \sim 0$ for $k \notin K$, we infer that

$$\int_\Gamma \omega = \int_\Gamma \varphi = \sum_{k=1}^{n} \int_\Gamma \varphi_k = \sum_{k \in K} n_k \int_{\gamma_k} \varphi_k.$$

Suppose that $k \in K$ and $\{i_1, \dots, i_{n-1}\} = \{1, \dots [k] \dots, n\}$, where $i_1 < \dots < i_{n-1}$. The partition $\mathscr{J}(k) = (J_1, \dots, J_n)$ of the set $\{1, \dots, n+1\}$ into n nonempty disjoint subsets for which $J_1 =$

$\{i_1\}, \dots, J_{n-1} = \{i_{n-1}\}, J_n = \{k, n+1\}$ defines a local cycle $\Gamma_k = \Gamma_{\mathscr{J}(k),a}$ in $U_a \setminus Y$. It was shown in [5] that, for sufficiently small $\varepsilon > 0$, $\delta/\varepsilon > 0$, the cycle

$$\gamma_k = \{z \in U_a : |f_{i_1}(z)| = \delta, \dots, |f_{i_{n-1}}(z)| = \delta, |f_{n+1}(z)| = \varepsilon\}$$

lies in $U_a \setminus Y$ and is homologous there to the cycle

$$\Gamma_k = \{z \in U_a : |f_{i_1}(z)| = \delta, \dots, |f_{i_{n-1}}(z)| = \delta, |f_k(z) \cdot f_{n+1}(z)| = \varepsilon\}.$$

It follows that

$$\int\limits_{\Gamma} \omega = \sum_{k \in K} n_k \int\limits_{\Gamma_k} \varphi_k.$$

Consider the cycle $\Gamma' = \sum_{k \in K} n_k \Gamma_k$ in $U_a \setminus Y$. We have

$$\int\limits_{\Gamma'} \omega = \int\limits_{\Gamma'} \varphi = \sum_{k \in K} n_k \sum_{s=1}^{n} \int\limits_{\Gamma_k} \varphi_s.$$

If $s \neq k$ then $\Gamma_k \sim 0$ in the domain $U_a \setminus Y_{[s]}$ in which φ_s is holomorphic. Indeed, $\Gamma_k \sim \gamma_k = \pm\partial c$, where $c = \{|f_1(z)| = \delta, \dots [k, s] \dots, |f_n(z)| = \delta, |f_s(z)| \leq \delta, |f_{n+1}(z)| = \varepsilon\}$ is a chain in $U_a \setminus Y_{[s]}$. Thus, we obtain

$$\int\limits_{\Gamma'} \omega = \sum_{k \in K} n_k \int\limits_{\Gamma_k} \varphi_k.$$

We showed that $\int_{\Gamma} \omega = \int_{\Gamma'} \omega$ for every closed differential n-form ω in $U_a \setminus Y$. By de Rham's Theorem, this implies that $\Gamma \sim \Gamma'$ in $U_a \setminus Y$, i.e., Γ is homologous to a linear combination of the local cycles Γ_k. If $K = \varnothing$ then $\int_{\Gamma} \omega$ is zero, and so $\Gamma \sim 0$ in $U_a \setminus Y$.

If Y contains no subcollection of n hypersurfaces in general position then the holomorphic n-form φ in $U_a \setminus Y$ admits a decomposition $\varphi = \varphi_1 + \dots + \varphi_n + \varphi_{n+1}$, where φ_k is holomorphic in $U_a \setminus Y_{[k]}$. Since, in this case, $\Gamma \sim 0$ in $U_a \setminus Y_{[k]}$ for all k, we have

$$\int\limits_{\Gamma} \omega = \int\limits_{\Gamma} \varphi = \sum_{k=1}^{n+1} \int\limits_{\Gamma} \varphi_k = 0,$$

whence again $\Gamma \sim 0$ in $U_a \setminus Y$.

For finishing the proof of the theorem, it remains to observe that since each local cycle defined for Y is a separating cycle, any linear combination of the local cycles separates Y.

5. Brunnian Links

Consider a Brunnian link consisting of three pairwise disjoint topological circles S_i, $i = 1, 2, 3$, in $X = \mathbb{R}^3$ (or $X = S^3$). Then, in particular, $S_i \sim 0$ in $X \setminus S_j$, where $j \neq i$, i.e., each circle as a cycle in the complement to the collection of the remaining two circles separates this collection. But then the cycle S_i admits a resolution $\xi = \{\xi_i\}_{i=0}^{1}$; moreover, since $H_2(X \setminus S_j \cap S_k) = H_2(X) = 0$, we also have $\varepsilon\xi_1 \sim 0$. By Proposition 1, it follows that $S_i \sim 0$ in $X \setminus S_j \cup S_k$. We have shown once again that each component of a Brunnian link of three components is homologically trivial in the complement to the remaining components (which does not contradict the nontriviality of the link).

Consider a more general situation. Let $X = S^{2n+1}$ and let $\{Y_1, \dots, Y_{n+2}\}$ be a collection of pairwise disjoint hypersurfaces homeomorphic to S^n in which each of the surfaces separates the others. Fix one of the surfaces Y_i and consider the collection $Y = \{Y_1, \dots [i] \dots, Y_{n+2}\}$. Show that the hypotheses of Proposition 2 are fulfilled for the manifold $X = S^{2n+1}$ and the collection of surfaces Y.

Since, in our situation, $d = 2n + 1$ and $m = n + 1$, we must verify the triviality of the homology groups $H_{0,2n-1}, H_{1,2n-2}, \dots, H_{n-2,n+1}$. The Alexander–Pontryagin duality implies that for $p = 0, \dots, n$, for any set of indices $\{j_0, \dots, j_p\} \subset \{1, \dots [i] \dots, n+2\}$,

$$H_q(S^{2n-1} \setminus (Y_{j_0} \cup \dots \cup Y_{j_p})) \simeq H_{(2n+1)-q-1}(Y_{j_0} \cup \dots \cup Y_{j_p}) \simeq \bigoplus_{k=0}^{p} H_{2n-q}(Y_{j_k}) \simeq \bigoplus_{k=0}^{p} H_{2n-q}(S^n).$$

Since $H_{2n-q}(S^n) \simeq 0$ for $1 \le 2n - q \le n - 1$, we obtain $H_{p,q} \simeq 0$ for $q = n + 1, \dots, 2n - 1$.

Proposition 2 implies the existence of a resolution $\xi = \{\xi_i\}_{i=0}^n$ for the n-cycle Y_i. Since we are considering a collection of pairwise disjoint spheres, we have

$$H_{2n}\left(X \setminus (Y_1 \cap \dots [i] \dots \cap Y_{n+2})\right) = H_{2n}(X) = H_{2n}(S^{2n+1}) \simeq 0;$$

therefore, $\varepsilon\xi_n \sim 0$, whence, by Proposition 1, the sphere Y_i is homologous to zero in $X \setminus (Y_1 \cup \dots [i] \dots \cup Y_{n+2})$. Thus, we have proved the following result, analogous to the above-formulated corollary to the Tsikh Theorem on separating cycles in Stein manifolds:

Theorem 3. *Let X be a sphere S^{2n+1} of dimension $2n+1$, $n \ge 1$, and let Y be a collection of $n+2$ pairwise disjoint surfaces homomorphic to S^n in which each of the surfaces separates the others. Then each sphere in Y is homologically trivial in the complement to the remaining spheres.*

As was shown in [15], there exist infinitely many nonequivalent Brunnian links with any given number of components. It is observed in [16] that the triple Massey product and the higher-order Massey product are successfully applied for studying the invariants of such links. Here we will not discuss the question of the existence of multidimensional links in S^{2n+1} consisting of $n + 2$ topological spheres S^n with the properties of a Brunnian link.

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